19 Relativistic Equations. An Introduction

This chapter is intended as an introduction to relativistic quantum mechanics, with the aim of providing the student with a basic knowledge of the subject and a link to more advanced courses in relativistic quantum theory.

So far we have been dealing with nonrelativistic QM, which works well in the context of Galilean relativity. When the Schrödinger equation is modified to make it consistent with special relativity, the result is the Klein–Gordon equation. Such a modification gives better qualitative, though not quantitative, agreement with experiment.

On the other hand, introducing the spin of the electron into the Schrödinger equation gives the Pauli equation, as we saw in Section 14.2. In this way, some important spin effects are incorporated into Schrödinger's theory.

When special relativity and spin are considered together, one obtains Dirac's equation. This equation applies only to particles with spin 1/2; the relativistic description of particles (or fields) with spin 1 is given by the Proca equation, that of particles of spin 3/2 by the Rarita–Schwinger equation, and so on. Of all these, the most important is undoubtedly Dirac's equation, since it allows us to obtain those corresponding to higher values of spin, as well as describing the behavior of the most interesting particles, namely the electron and the nucleons, the fundamental constituents of matter.

19.1 The Klein–Gordon Equation

We begin by constructing the Klein–Gordon equation for the relativistic description of spinless particles. To do this we start from the observation that Schrödinger's equation can be obtained simply by "quantizing" the relativistic energy expression $E = \mathbf{p}^2/2m + V$, by replacing E by $i\hbar\partial/\partial t$ and $\hat{\boldsymbol{p}}$ by $-i\hbar\boldsymbol{\nabla}$, which leads to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + V\psi. \tag{19.1}$$

Since the operators $\widehat{E}$ and $\widehat{\boldsymbol{p}}$ are the components of the four-vector $-i\hbar(\partial_i, -ic\partial_4)$,[1] the substitution retains its validity when we move to the relativistic description.

[1] The notation used in this chapter is as follows. The Cartesian components of a tensor in space-time are denoted by Greek indices, $x_\mu, A_\mu, F_{\mu\nu}$, and so on, with $\mu = 1, 2, 3, 4$. The spatial components are indicated with a Latin index, $x_i, A_i, F_{ij}, F_{i\lambda}$, and so on. In Minkowski notation, the fourth component of the four-vectors is imaginary; for example,

$$\boldsymbol{x} = (x, y, z, ict), \quad \boldsymbol{A} = \left(A_x, A_y, A_z, iA_0\right), \tag{19.2}$$

$$\boldsymbol{\partial}_\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{i\partial ct}\right) = \left(\boldsymbol{\nabla}, -\frac{i}{c}\frac{\partial}{\partial t}\right). \tag{19.3}$$

Therefore,

$$-i\hbar\,(\partial_i, -ic\partial_4) = \left(-i\hbar\nabla_i, i\hbar c\frac{\partial}{\partial ct}\right) = \left(\widehat{p_i}, \widehat{E}\right). \tag{19.4}$$

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To obtain an equation consistent with relativity, it is sufficient to use the corresponding expression for the energy, restricting ourselves first to the case of a free particle, $V = 0$. Using this procedure, we obtain the Klein–Gordon equation for a free particle from $E^2 = m^2c^4 + \boldsymbol{p}^2c^2$,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{19.5}$$

The first two terms, corresponding to the wave equation, can be grouped as the d'Alambertian of ψ, which gives

$$\Box\psi - \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{19.6}$$

This equation was derived simultaneously and independently by many authors, including Schrödinger (the first), de Broglie, and of course Klein and Gordon.[2]

19.1.1 Basic properties of the Klein–Gordon equation

Let us look at some basic properties of the Klein–Gordon equation. To simplify the writing, we put $p_\mu = -i\hbar\partial_\mu$, where the index μ takes the four values $\mu = 1, 2, 3, 4$, with $x_4 = ict$. Then $\hbar^2\Box = -p_\mu p_\mu$ and the free-particle equation can be written in the form

$$p_\mu p_\mu\psi + m^2c^2\psi = 0. \tag{19.7}$$

With $\varkappa = mc/\hbar$, we get

$$\partial_\mu\partial_\mu\psi - \varkappa^2\psi = 0 \tag{19.8}$$

and its conjugate

$$\partial_\mu\partial_\mu\psi^* - \varkappa^2\psi^* = 0. \tag{19.9}$$

If we multiply these equations by ψ^* and ψ respectively and subtract the results, we get a conservation equation,

$$\partial_\mu\, j_\mu = 0, \tag{19.10}$$

where the current four-vector $j_\mu = (\boldsymbol{j}, ij_0)$ is defined as (A is a constant to be determined)

$$j_\mu = A(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*). \tag{19.11}$$

The fourth component is $j_4 = ij_0$, with

$$j_0 \equiv c\rho = (A/c)(\psi^*\partial\psi/\partial t - \psi\partial\psi^*/\partial t); \tag{19.12}$$

with this identification and in three-dimensional notation, (19.10) takes the usual form of the continuity equation,

[2] Oskar Benjamin Klein (1894–1977) was a Swedish theoretical physicist; Walter Gordon (1893–1932) was a German physicist.

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$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{19.13}$$

In general, an equation of the form

$$\partial_\mu \theta_{\mu\alpha\beta\ldots} = 0 \tag{19.14}$$

is a conservation equation, meaning that the fourth component of the tensor $\theta_{\mu\alpha\beta\ldots}$ is a conserved quantity. In particular, Eq. (19.10) is equivalent to Eq. (19.13) for the conserved density ρ, given by the expression

$$\rho = \frac{A}{c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right). \tag{19.15}$$

Since this expression has no definite sign, we cannot interpret it as the density of particles, although it is customary to give it the meaning of charge density (choosing the constant A accordingly); $\boldsymbol{j}$ is then the electric current density. Note, however, that the time evolution of the system could, in principle, change the sign of ρ.[3] For example, if we write $\psi = R\, e^{iS}$, using (19.15), we obtain

$$\rho = \frac{A}{c^2} R^2 (2i\dot{S}) = -\frac{\hbar e}{mc^2} R^2 \dot{S}, \tag{19.16}$$

and it is clear that its sign depends on that of $\dot{S}$, which can change with the evolution (the appropriate value of A has already been inserted).

To set the coefficient A, we have taken into account that in the nonrelativistic limit Schrödinger's theory has to be recovered. Considering the special case where ψ refers to a stationary state $\psi = e^{-iEt/\hbar}\varphi(\mathbf{x})$ and substituting in (19.11), we get $\rho = -2i\frac{AE}{\hbar c^2}\psi^*\psi$, which in the nonrelativistic limit ($E \to mc^2$) gives $\rho = -2i\frac{Am}{\hbar}\psi^*\psi$. For this quantity to agree with the charge density of the nonrelativistic theory, $\rho = e\psi^*\psi$, we have to set $A = i\hbar e/2m$. From (19.11) we thus obtain the current density

$$j_\mu = \frac{i\hbar e}{2m}(\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*), \tag{19.17}$$

whose spatial components are the same as in the nonrelativistic case, although its fourth component, given by Eq. (19.15), differs from the corresponding nonrelativistic expression.

As a solution to the Klein–Gordon equation for the free particle, we propose

$$\psi = Ae^{-iEt/\hbar + i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \boldsymbol{k} = \boldsymbol{p}/\hbar. \tag{19.18}$$

Introducing this into (19.6) we see that the equation is satisfied if $E = \pm E_0$, with $E_0 = +\sqrt{m^2c^4 + c^2\boldsymbol{p}^2} \equiv \hbar\omega$, The theory thus predicts the coexistence of states with positive and negative energy.[4] To see this in more detail, let us consider the two solutions

$$\psi_+ = A_+ e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \psi_- = A_- e^{i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{19.19}$$

[3] This problem was the main argument that led Schrödinger to abandon this equation for its nonrelativistic, first-order time derivative version.

[4] Models can be built that allow only positive energies, although they are somewhat artificial; see, for example, Dirac (1971).

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where ψ_+ corresponds to $E = \hbar\omega > 0$ and ψ_- to $E = -\hbar\omega < 0$. Substituting these into the expression for the density, we get

$$\rho_+ = +\frac{eE_0}{mc^2}|A_+|^2, \quad \rho_- = -\frac{eE_0}{mc^2}|A_-|^2, \tag{19.20}$$

so that with the present convention the solutions of positive energy correspond to particles with charge of one sign (that of e), while those of negative energy correspond to charges of opposite sign. Incidentally, we note that something similar happens with any additive quantum number. This leads us to conclude that the Klein–Gordon equation contains solutions for both particles and their antiparticles, as we will see in more detail later.

19.1.2 Klein–Gordon equation with minimal coupling

To introduce the electromagnetic field, we use the principle of minimal coupling (see Section 12.7) in its relativistic version; this means that we have to replace the momentum-energy operator $\hat{p}_\mu$ by the corresponding canonical operator, $\hat{p}_\mu - (e/c)\,A_\mu$. This substitution in Eq. (19.7) gives the Klein–Gordon equation for particles subject to the action of an electromagnetic potential,

$$\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right)\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right)\psi + m^2c^2\psi = 0. \tag{19.21}$$

To apply this equation to the atomic case we introduce the following conditions:
(a) In the Lorentz gauge, $\boldsymbol{A} = 0$ and $A_4 = -V = Ze^2/r$.
(b) The atom is in a stationary state, so we can write $\psi(\boldsymbol{x}, t) = e^{-iE't/\hbar}\psi(\boldsymbol{x})$. The total energy is then $E' = E + mc^2$, where mc^2 is the rest energy and $E \ll mc^2$.
Under these conditions, we obtain from Eq. (19.21)

$$\nabla^2\psi + \frac{1}{\hbar^2c^2}\left[\left(E + mc^2 + \frac{Ze^2}{r}\right)^2 - m^2c^4\right]\psi = 0. \tag{19.22}$$

Let us first consider the nonrelativistic approximation that occurs for $\left|E + Ze^2/r\right|^2 \ll mc^2$. The term in square brackets reduces in this case to $2mc^2(E + Ze^2/r)$ to the first order, so that (19.22) takes the approximate form

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(E + \frac{Ze^2}{r}\right)\psi = 0, \tag{19.23}$$

which is exactly the Schrödinger equation. Therefore, we can consider Eq. (19.21) as the relativistic version of Schrödinger's equation.[5]

Returning to the problem of the relativistic atom, we need to solve Eq. (19.22). This can be done exactly by writing

$$\psi = R(r)Y_l^m(\theta, \varphi); \tag{19.24}$$

the resulting radial equation is

$$\left(\nabla_r^2 - \frac{l(l+1) - \alpha^2Z^2}{r^2} + \frac{2B}{r} - A\right)R = 0, \tag{19.25}$$

[5] This transition from the Klein–Gordon equation to the Schrödinger equation was precisely the path that Schrödinger followed to arrive for the first time at the nonrelativistic version of his equation.

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where

$$\nabla_r^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r, \quad A = \frac{m^2c^2}{\hbar^2}\left[1 - \left(1 + \frac{E}{mc^2}\right)^2\right], \quad B = \frac{Zme^2}{\hbar^2}\left(1 + \frac{E}{mc^2}\right), \tag{19.26}$$

and $\alpha = e^2/\hbar c \approx 1/137$. Equation (19.22) has just the form of the radial Schrödinger equation for Kepler's problem, although with different parameters. The comparison with the nonrelativistic results shows that for the boundary conditions

$$R \underset{r\to 0}{\longrightarrow} \text{const} < \infty, \quad R \underset{r\to\infty}{\longrightarrow} 0 \tag{19.27}$$

to be satisfied, the relation between the coefficients A and B must be such that

$$\frac{B}{\sqrt{A}} - \sqrt{(l+\tfrac{1}{2})^2 - Z^2\alpha^2} - \tfrac{1}{2} = k, \quad k = 0, 1, 2, \ldots \tag{19.28}$$

Since both coefficients must be real, the theory only applies to problems that meet the condition $Z\alpha < \frac{1}{2}$, that is, to light or medium atoms. From the relation between A and B and taking into account Eq. (19.22), we obtain for the energy (including the rest energy mc^2),

$$E' = \frac{mc^2}{\sqrt{1 + \dfrac{\alpha^2 Z^2}{\left(k + \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - Z^2\alpha^2}\right)^2}}}. \tag{19.29}$$

Using the old definition $n = k + l + 1$, this expression shows that $E_{nl} = E' - mc^2$ depends on n and l, that is, the degeneracy on l has been broken. For the case $Z\alpha \ll 1$ we get the approximate result

$$E_{nl} = -\frac{R\hbar Z^2}{n^2}\left[1 + \frac{\alpha^2 Z^2}{n^2}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right) + \ldots\right], \tag{19.30}$$

with R the Rydberg constant. The first term coincides with the Schrödinger solution and the second represents the relativistic correction; this term is the one that causes the breakdown of the degeneracy between different values of l. For example, between the 2s and 2p levels of the hydrogen atom there is an energy difference $E_{21} - E_{20}$, which corresponds to the transition frequency

$$\omega = \frac{E_{21} - E_{20}}{\hbar} = \tfrac{1}{6} R\alpha^2. \tag{19.31}$$

This result differs from the experimental value by a factor of 1/3. In other words, the relativistic correction to Schrödinger's theory improves the qualitative properties of the predicted spectrum, but it is not sufficient. We will see later that this discrepancy is due to the fact that it does not take into account the effects of the electron spin on the energy levels, which turn out to be of the same order of magnitude (and opposite sign) as the relativistic effects. The equation that takes into account both the relativistic and the spinorial corrections is the one that can reproduce the empirical energy values; this is Dirac's equation.

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19.2 The Dirac Equation

Dirac's original intention in proposing his equation for the electron was to simultaneously solve several problems that made the Klein–Gordon theory inadequate. One is the unsatisfactory shape of the particle density, another is the need to introduce the electron spin manually, and a third is to avoid the negative energy solutions characteristic of a relativistic theory.

Dirac's aim was to make the theory generate spin by itself, rather than introducing it phenomenologically from outside, in addition to providing a suitable solution to the other problems mentioned. This attitude is reminiscent of Einstein's approach to Newton's theory of gravity in classical mechanics, where the gravitational force is introduced by hand and its form chosen to reproduce the observed results. In both cases it was a matter of principle. However, precisely because it is characteristic of a relativistic theory to predict negative energies, the latter requirement has to be reformulated in an appropriate way, as we will see next.

The unsatisfactory expression (19.15) for the particle density arises from the fact that the Klein–Gordon equation contains a second-order time derivative. One way of avoiding the second derivative and preserving the Lorentz invariance of the theory is to linearize (to "extract the square root of") the formula $E^2 = c^2\boldsymbol{p}^2 + m^2c^4$, that is, to propose that there exists a first-order linear relationship between the operators $\widehat{E}$ and $\widehat{\boldsymbol{p}}$, so that we can write

$$\widehat{E} = c\boldsymbol{\alpha} \cdot \widehat{\boldsymbol{p}} + \beta mc^2, \tag{19.32}$$

while demanding that its square be consistent with the relativistic expression. Applied to the wave function Ψ, this relationship leads to the following equation, with $\boldsymbol{\alpha}$ and β constant,

$$\widehat{E}\Psi = c\boldsymbol{\alpha} \cdot \widehat{\boldsymbol{p}}\Psi + \beta mc^2\Psi. \tag{19.33}$$

As we will see in more detail later, this equation also has negative energy solutions, so to solve the difficulty,[6] Dirac proposed to introduce the notion of positron or, more generally, of *antimatter*. The negative root of the equation $E^2 = c^2\boldsymbol{p}^2 + m^2c^4$, $E = -E_0$ then does not refer to negative energies of the electron – since it always has positive energy $E \geqq mc^2$ – but to particles below the energy level 0. It is therefore proposed that all levels with $E < 0$ are normally occupied by electrons, forming the *Dirac sea*, so that electrons cannot fall into such states due to the exclusion principle. When one of the electrons in the Dirac sea is excited and enters a positive energy state, this electron and the hole appear simultaneously in the sea; the hole is then identified as an antielectron, that is, a positron, since it corresponds to the *absence* of a negative charge, which is observed as a positive effective charge; similarly, the absence of negative energy appears as positive energy, and so on for the other properties of the electron. In this way, the negative energy states correspond to antimatter.

[6] The main difficulty with the existence of negative energy solutions is that particles with positive energy would spontaneously make transitions to states of increasingly negative energy, radiating the excess energy. The purpose of the Dirac sea is to put a limit on such processes.

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In 1932, two years after Dirac's proposal, Anderson[7] experimentally found the positron in plates traced by cosmic rays. The prediction of antimatter was the first major prediction of the Dirac equation. Modern quantum field theory treats particles and antiparticles completely symmetrically, so the notion of a Dirac sea has become obsolete.

Introducing the expressions $\widehat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}$, $\widehat{E} = i\hbar\partial/\partial t$, we can write the Dirac equation explicitly as

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi + \beta mc^2\Psi. \tag{19.34}$$

The four quantities α_i, β have to be chosen so that the relativistic laws are fulfilled. To determine them, we rewrite Eq. (19.33) using a convenient noncovariant provisional notation (exclusive for this calculation), in the form $E = c\alpha_\mu p_\mu$, with $p_0 = m\,c, \alpha_0 = \beta$. Note carefully that α_μ, p_μ defined in this way are *not* four-vectors, since, $p_0 = mc$ is a scalar, not the fourth component of a four-vector. Squaring and taking into account that the components p_μ commute among themselves, we obtain

$$E^2 = c^2\alpha_\mu p_\mu\alpha_\nu p_\nu = c^2 p_\mu p_\nu\alpha_\mu\alpha_\nu = c^2 p_\mu p_\nu\tfrac{1}{2}\left(\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu\right), \tag{19.35}$$

where we have symmetrized by interchanging the indices ν, μ and taking the half sum. Since E^2 must be equal to $c^2\boldsymbol{p}^2 + c^2 p_0^2 = c^2 p_\mu p_\mu$, it follows that we must write

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}. \tag{19.36}$$

This shows that that the α_μ should anticommute, so they cannot be simple numbers, but rather represent numerical matrices. Since the coefficients α_i, β in Eq. (19.34) are matrices, the functions Ψ must have several components to make sense. To propose a first-order equation for the electron, therefore, requires expanding the dimensionality of the space of solutions Ψ.

In order to construct the matrices that satisfy (19.36) and describe the spin, we tentatively consider that the Pauli matrices satisfy a similar equation

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}, \tag{19.37}$$

for $i, j = 1, 2, 3$: but when completing the basis with the unit matrix it turns out that the latter, combined with some $\hat{\sigma}_i$, does not satisfy, but commutes with, Eq. (19.36). Therefore, we cannot identify the α_i with the Pauli matrices and we have to look for a space of higher dimensionality. Since there are two degrees of freedom associated with the spin $\frac{1}{2}$ and two possible signs for the energy, there can be four different states for a given E^2, so a wave function with at least four linearly independent components is required; this means that the matrices α_μ must be of dimension at least 4×4. In the general case, for a particle with spin S, the matrices must be of dimension $N \times N$, with $N = 2(2S + 1) = 4(S + \frac{1}{2})$. The simplest Dirac equation is therefore one of rank 4, corresponding to spin 1/2. Restricting ourselves to this case and choosing $\alpha_0 = \beta$, we rewrite Eq. (19.34) in the (classical) Dirac form,

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi + mc^2\beta\Psi \equiv \widehat{H}\Psi. \tag{19.38}$$

[7] Carl David Anderson (1905–1991) was a US-born physicist, who won the 1936 Nobel Prize for his discovery (observation and identification) of the positron. In 1936 he discovered the muon, a subatomic particle 207 times more massive than the electron (and ten times lighter than the proton).

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Multiplying this equation from the left by β, taking into account that $\beta^2 = \alpha_\mu^2 = \mathbb{I}$, as follows from (19.36), introducing four new matrices γ_μ defined as

$$\gamma_0 = -i\gamma_4 = -i\beta = -i\alpha_0, \quad \gamma_i = -i\beta\alpha_i, \tag{19.39}$$

and dividing by $\hbar c$ gives (matrices γ_α and their relatives are not usually marked as operators, as there is no risk of confusion),

$$(\gamma_0\partial_0 + \gamma_i\partial_i)\,\Psi + \frac{mc}{\hbar}\Psi = 0, \tag{19.40}$$

or since $\gamma_0\partial_0 + \gamma_i\partial_i = \gamma_\mu\partial_\mu$,

$$\gamma_\mu\partial_\mu\Psi + \frac{mc}{\hbar}\Psi = 0. \tag{19.41}$$

This is the most common and convenient form of Dirac's equation for a free particle.[8] Since $mc/\hbar$ is a scalar, $\gamma_\mu\partial_\mu$ must also be a scalar, so γ_μ is a four-vector, and Eq. (19.41) is explicitly covariant under Lorentz transformations. From (19.38) we see that the Dirac Hamiltonian

$$\widehat{H} = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 \tag{19.42}$$

is a matrix of dimension 4×4, so the functions Ψ are *bispinors or spinors of four components*.

Exercise E19.1. The Dirac oscillator

The Dirac Hamiltonian is obtained by "taking the square root" of the Klein–Gordon–Hamiltonian. Use this idea to construct a linear Hamiltonian describing an isotropic 3D harmonic oscillator and show that it is indeed an oscillator with some peculiarities.[9]

Solution. The proposal is to "linearize" the operator (leaving aside the mass term)

$$\boldsymbol{p}^2 + m^2\omega^2\boldsymbol{r}^2 \Rightarrow \left[\alpha\cdot(\boldsymbol{p} + im\omega\boldsymbol{r})\right]\left[\alpha\cdot(\boldsymbol{p} - im\omega\boldsymbol{r})\right], \tag{19.43}$$

using an idea similar to the preceding, but now including a linear coupling term in $\boldsymbol{r}$. It is clear that the terms $\pm im\omega\boldsymbol{r}$ can still be multiplied by any (scalar or pseudoscalar) matrix that anticommutes with the α_i matrices without affecting the factorization. The detailed analysis of this possibility leads to the conclusion that this factor must be taken as the β matrix, as this is the only way to guarantee that the resulting equation has the appropriate Lorentz covariance properties. This can be understood by recalling that the mass term in the free-particle Dirac equation itself, Eq. (19.34), has the form βm, so the m of the interaction terms must be accompanied by the factor β, $\pm im\omega\beta\boldsymbol{r}$. From these considerations we conclude that the proposed equation must be

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi = c\alpha\cdot(\boldsymbol{p} - im\omega\boldsymbol{r})\,\Psi + mc^2\beta\Psi. \tag{19.44}$$

It is easy to verify that this Hamiltonian is self-adjoint. One way to make physical sense of Eq.(19.44) is to consider the term $im\omega\beta\boldsymbol{r}$ as due to the coupling of the particle to an external (e.g., electric) field that grows linearly in the region of interest.

[8] There are variants in the literature which may differ from the present notation by signs or factors such as $\pm i$ in the definition of the fundamental matrices or the metric.

[9] This topic is discussed in detail in Moshinsky and Smirnov (1996).

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To prove that Eq.(19.44) describes an oscillator and to study its properties, we calculate the square of the Hamiltonian, which, after some simplifications, gives (with $\hbar = c = 1$)

$$\begin{aligned}\widehat{H}^2 &= \left[\alpha\cdot(\boldsymbol{p} - im\omega\boldsymbol{r}) + m\beta\right]^2 \\ &= \alpha_i\alpha_j p_i p_j - m^2\omega^2\alpha_i\beta\alpha_j\beta x_i x_j + m^2 \\ &\quad - im\omega\left(\alpha_i\alpha_j\beta p_i x_j + \alpha_i\beta\alpha_j x_i p_j\right),\end{aligned} \tag{19.45}$$

that is,

$$\widehat{H}^2 = \boldsymbol{p}^2 + m^2\omega^2\boldsymbol{r}^2 + m^2 - (4\boldsymbol{L}\cdot\boldsymbol{S} + 3)m\omega\beta, \tag{19.46}$$

where $\boldsymbol{L}$ is the orbital angular momentum and $\boldsymbol{S} = \boldsymbol{\sigma}/2$ is the electron spin. This (squared) Hamiltonian effectively corresponds to that of a HO with spin–orbit coupling, plus irrelevant constants. An interesting exercise, left to the reader, is to show that the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ commutes with $\widehat{H}^2$ and is therefore conserved; separately, neither $\boldsymbol{L}$ nor $\boldsymbol{S}$ is conserved.

19.2.1 The Dirac γ_μ matrices

We now move on to the study of the properties of the Dirac γ_μ matrices. The complete Dirac algebra consists of 16 linearly independent matrices of dimension 4×4; later the matrices γ_5 (one pseudoscalar) and $\sigma_{\mu\nu}$ (the six components of an antisymmetric tensor) are introduced in addition to the unit matrix. The remaining four matrices can be taken as $\gamma_5\gamma_\mu$ (components of a pseudo-four-vector). This set is closed with respect to multiplication and constitutes a Clifford algebra, that is, its elements anticommute. (We recall that a pseudotensor is a tensor whose components change sign when rotated by 2π.)

From (19.39) we see that $\alpha_i = i\beta\gamma_i$. On the other hand, from (19.36) it follows that $\alpha_\mu\alpha_\nu = -\alpha_\nu\alpha_\mu$ for $\mu \neq \nu$; taking here $\mu = i$, $\nu = 0$, with $\alpha_0 = \beta$ we get $\gamma_i\beta = -i\beta\alpha_i\beta = i\beta^2\alpha_i = i\alpha_i$, that is,

$$\gamma_i\beta = -\beta\gamma_i \tag{19.47}$$

or, again with (19.39),

$$\gamma_i\gamma_4 + \gamma_4\gamma_i = 0. \tag{19.48}$$

Taking now $\mu = i, \nu = j$, we obtain

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} = -\beta\gamma_i\beta\gamma_j - \beta\gamma_j\beta\gamma_i = \beta^2[\gamma_i\gamma_j + \gamma_j\gamma_i], \tag{19.49}$$

that is,

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{19.50}$$

Finally, for $\mu = \nu = 0$ we get $\alpha_0\alpha_0 + \alpha_0\alpha_0 = 2 = 2i\gamma_0 i\gamma_0$, or

$$\gamma_4\gamma_4 + \gamma_4\gamma_4 = 2. \tag{19.51}$$

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Equations (19.48–19.51) show that the matrices γ_μ satisfy a relationship similar to (19.36),[10]

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \tag{19.52}$$

An important property of the γ_μ matrices is that they all have zero trace. This follows immediately from Eq. (19.52) with $\nu \neq \mu$, multiplying it on the left by γ_μ and calculating the trace, which gives

$$\text{tr}\ \gamma_\mu^2 \gamma_\nu + \text{tr}\ \gamma_\mu \gamma_\nu \gamma_\mu = 2\text{tr}\ \gamma_\mu^2 \gamma_\nu = 2\delta_{\mu\nu} \text{tr}\ \gamma_\mu = 0 \quad (\mu \neq \nu). \tag{19.53}$$

Since $\gamma_\mu^2 = 1$, it follows that for each μ,

$$\text{tr}\ \gamma_\mu = 0. \tag{19.54}$$

19.2.2 The van der Waerden equation: Helicity operator

The preceding derivation of the Dirac equation, although very popular, is purely formal and it seems to be a matter of luck that it applies to the electron. For this reason, it is convenient to present a second derivation in which the physical content of the postulates is more transparent. To introduce the spin, we will make the substitution $\boldsymbol{p}^2 \to (\boldsymbol{\sigma} \cdot \boldsymbol{p})^2$ successfully used in the nonrelativistic case to take account of the electron spin; see Section 14.2. Now we will use it in a relativistic context to arrive at a theory that contains both spin 1/2 and relativity by construction.

Our starting point is once again the equation $E^2/c^2 - \boldsymbol{p}^2 = m^2c^2$, which with the substitution $\boldsymbol{p}^2 \to (\boldsymbol{\sigma} \cdot \boldsymbol{p})^2$ and the transition to operators becomes

$$\frac{\widehat{E}^2}{c^2} - (\hat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}})(\hat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}}) = m^2c^2 \tag{19.55}$$

or, taking advantage of the fact that $\widehat{E}$ commutes with $\widehat{\boldsymbol{p}}$ and with the Pauli operators,

$$\left(\frac{\widehat{E}}{c} - \hat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}}\right)\left(\frac{\widehat{E}}{c} + \hat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{p}}\right) = m^2c^2. \tag{19.56}$$

Using the quantum expressions for $\widehat{E}$ and $\widehat{\boldsymbol{p}}$ and applying the equation to a spinor ψ (of two components), we obtain a second-order equation in terms of 2×2 matrices,

$$\left(i\hbar\partial_0 + i\hbar\hat{\boldsymbol{\sigma}} \cdot \nabla\right)\left(i\hbar\partial_0 - i\hbar\hat{\boldsymbol{\sigma}} \cdot \nabla\right)\psi = m^2c^2\psi. \tag{19.57}$$

This is known as the *van der Waerden equation*. Because of the way this equation was derived, it clearly describes relativistic particles of spin 1/2. To show that it is just another form of the Dirac equation, we introduce the pair of spinors (of two components) $\phi^{(L)}$ and $\phi^{(R)}$, defined as

$$\phi^{(L)} = \psi, \tag{19.58}$$

$$\phi^{(R)} = \frac{1}{mc}\left(i\hbar\partial_0 - i\hbar\hat{\boldsymbol{\sigma}} \cdot \nabla\right)\phi^{(L)}. \tag{19.59}$$

[10] If two sets of matrices γ_μ and γ'_μ satisfy (19.52), there is always a nonsingular matrix $\hat{S}$ such that $\widehat{S}\gamma_\mu\widehat{S}^{-1} = \gamma'_\mu$. This so-called fundamental Pauli theorem leads to the invariance of Dirac's equation with respect to the change of representation.

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From here and (19.57) it follows that these spinors satisfy the system of equations

$$\begin{aligned}\left(i\hbar\hat{\boldsymbol{\sigma}}\cdot\nabla - i\hbar\partial_0\right)\phi^{(L)} &= -mc\phi^{(R)},\\ \left(i\hbar\hat{\boldsymbol{\sigma}}\cdot\nabla + i\hbar\partial_0\right)\phi^{(R)} &= mc\phi^{(L)}.\end{aligned} \tag{19.60}$$

Each of these equations has two components, so that in total we have four first-order equations for four functions, as is the case with Dirac's equation.

Before proceeding further, it is convenient to make a brief comment to establish the physical meaning of the spinors $\phi^{(R)}$ and $\phi^{(L)}$, which is interesting and useful. We consider the case $m = 0$, which decouples Eqs. (19.60),

$$\left(i\hbar\hat{\boldsymbol{\sigma}}\cdot\nabla - i\hbar\partial_0\right)\phi^{(L)} = 0, \quad \left(i\hbar\hat{\boldsymbol{\sigma}}\cdot\nabla + i\hbar\partial_0\right)\phi^{(R)} = 0. \tag{19.61}$$

Introducing $|\boldsymbol{p}| = E/c$ with $E > 0$ in the first equation, for eigenstates of E and $\boldsymbol{p}$ we get $\left(\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}} + |\boldsymbol{p}|\right)\phi^{(L)} = 0$, that is,

$$\frac{\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}}{|\boldsymbol{p}|}\phi^{(L)} = -\phi^{(L)}, \tag{19.62}$$

and a similar equation for $\phi^{(R)}$, with the sign reversed. The operator

$$\hat{\mathfrak{h}} = \hat{\boldsymbol{\sigma}}\cdot\frac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|}, \tag{19.63}$$

called the *helicity operator,* describes the projection of the spin operator in the direction of the momentum; the negative eigenvalue in Eq. (19.62) indicates that the spin is antiparallel to $\boldsymbol{p}$ and that the helicity of the $\phi^{(L)}$ state is -1. Similarly, the solution for $\phi^{(R)}$ corresponds to the state of spin parallel to $\boldsymbol{p}$ and helicity $+1$. In summary, we see that for $m = 0$ and $S = 1/2$ there are two independent helicity states with values ± 1. For massive particles, the solutions are in general linear combinations of the two helicity states, or what is equivalent, they are not helicity eigenstates.

Returning to Eqs. (19.61) and (19.60) and taking first their difference and then their sum, we get the system of equations

$$-i\hbar\partial_0\Psi_A - i\hbar\hat{\sigma}\cdot\nabla\Psi_B = -mc\Psi_A, \tag{19.64}$$

$$i\hbar\partial_0\Psi_B + i\hbar\hat{\sigma}\cdot\nabla\Psi_A = -mc\Psi_B, \tag{19.65}$$

for the functions

$$\Psi_A = \phi^{(R)} + \phi^{(L)}, \qquad \Psi_B = \phi^{(R)} - \phi^{(L)}. \tag{19.66}$$

These linear combinations are clearly arbitrary, and others could well have been adopted. This observation is important, since this freedom is at the basis for the possibility of constructing different representations of the Dirac matrices, as we will see later. If we rewrite the preceding system in matrix form, we get

$$\begin{pmatrix} -i\hbar\partial_0 & -i\hbar\hat{\sigma}\cdot\nabla \\ i\hbar\hat{\sigma}\cdot\nabla & i\hbar\partial_0 \end{pmatrix}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} = -mc\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}. \tag{19.67}$$

We now construct a four-component bispinor (Dirac spinor) as

$$\Psi = \begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} = \begin{pmatrix}\phi^{(R)} + \phi^{(L)}\\ \phi^{(R)} - \phi^{(L)}\end{pmatrix}, \tag{19.68}$$

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so that Eq. (19.67) becomes

$$\left[-i\hbar\begin{pmatrix}\mathbb{I} & \mathbb{O}\\ \mathbb{O} & -\mathbb{I}\end{pmatrix}\partial_0+\hbar\sum_i\begin{pmatrix}\mathbb{O} & -i\hat{\sigma}_i\\ i\hat{\sigma}_i & \mathbb{O}\end{pmatrix}\partial_i\right]\Psi=-mc\Psi. \tag{19.69}$$

Introducing the 4×4 matrices

$$\gamma_i=\begin{pmatrix}\mathbb{O} & -i\hat{\sigma}_i\\ i\hat{\sigma}_i & \mathbb{O}\end{pmatrix},\quad \gamma_4=\begin{pmatrix}\mathbb{I} & \mathbb{O}\\ \mathbb{O} & -\mathbb{I}\end{pmatrix}, \tag{19.70}$$

and taking into account that $-i\,\partial_0=\partial_4$, we obtain

$$\left(\gamma_\mu\partial_\mu+\frac{mc}{\hbar}\right)\Psi=0, \tag{19.71}$$

which is precisely the Dirac equation, (19.41).

The γ_μ matrices have been written here directly in terms of the Pauli matrices. If instead of $\Psi_A=\phi^{(R)}+\phi^{(L)}$ and $\Psi_B=\phi^{(R)}-\phi^{(L)}$ other linear combinations had been used, a different form for the γ_μ matrices would have been obtained, and thus a different representation of them. The specific representation used here, in addition to satisfying the condition (19.52), is Hermitian, as can be seen from its expression in terms of Pauli matrices. We will come back to this point later.

19.3 Properties of the Dirac Equation. The *Zitterbewegung*

After the Maxwell–Lorentz equations of classical electrodynamics, Dirac's equation was the next major step in the investigation of the properties of the electron. In the following we will introduce the elements needed to construct the conservation equation for the current density. We will then prove that Dirac's equation describes particles with spin 1/2 more generally, and discuss the *zitterbewegung*, the fast oscillation of the particle around its classical (relativistic) trajectory, which Schrödinger predicted in 1930 from his analysis of the solutions of the Dirac equation.

19.3.1 The Dirac adjoint and the continuity equation

Applying to Eq. (19.41) the operator $\gamma_\nu\partial_\nu-mc/\hbar$ from the left, we obtain

$$\left(\gamma_\nu\partial_\nu-\frac{mc}{\hbar}\right)\left(\gamma_\mu\partial_\mu+\frac{mc}{\hbar}\right)\Psi=\left(\gamma_\nu\gamma_\mu\partial_\nu\partial_\mu-\frac{m^2c^2}{\hbar^2}\right)\Psi=0, \tag{19.72}$$

which can be written symmetrically as

$$\left[\tfrac{1}{2}\left(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu\right)\partial_\mu\partial_\nu-\frac{m^2c^2}{\hbar^2}\right]\Psi=0. \tag{19.73}$$

This expression is simplified with the help of (19.52), so that

$$\left(\Box-\frac{m^2c^2}{\hbar^2}\right)\Psi=0. \tag{19.74}$$

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The operator in parentheses acts on each of the four components of the spinor separately, which means that each component of the Dirac equation is a solution of the Klein–Gordon equation. This is not to say, however, that *any* solution of the Klein–Gordon equation is appropriate, since the bispinors are constrained by the Dirac equation.

Given a spinor Ψ, the *Dirac adjoint* is defined as

$$\overline{\Psi} = \Psi^\dagger \gamma_4 \tag{19.75}$$

or explicitly, in the preceding representation of the Dirac matrices,

$$\overline{\Psi} = (\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*). \tag{19.76}$$

Writing Dirac's equation in the form

$$\gamma_i \partial_i \Psi + \gamma_4 \partial_4 \Psi + \frac{m\,c}{\hbar}\Psi = 0 \tag{19.77}$$

and taking its adjoint (with $\partial_4^* = -\partial_4$ since $x_4 = ict$), we get

$$\partial_i \Psi^\dagger \gamma_i - \partial_4 \Psi^\dagger \gamma_4 + \frac{m\,c}{\hbar}\Psi^\dagger = 0. \tag{19.78}$$

To recover from the sign change that occurred in the fourth component, we multiply from the right by γ_4,

$$\partial_i \Psi^\dagger \gamma_i \gamma_4 - \partial_4 \Psi^\dagger \gamma_4 \gamma_4 + \frac{m\,c}{\hbar}\Psi^\dagger \gamma_4 = 0. \tag{19.79}$$

We now take advantage of the fact that γ_4 commutes with itself, but anticommutes with the γ_i; rewriting the result in terms of the Dirac adjoint, we get

$$-\partial_i \Psi^\dagger \gamma_4 \gamma_i - \partial_4 \Psi^\dagger \gamma_4 \gamma_4 + \frac{m\,c}{\hbar}\Psi^\dagger \gamma_4 = -\partial_i \overline{\Psi}\gamma_i - \partial_4 \overline{\Psi}\gamma_4 + \frac{m\,c}{\hbar}\overline{\Psi} = 0, \tag{19.80}$$

which leads to the *adjoint Dirac equation*,

$$-\partial_\mu \overline{\Psi}\gamma_\mu + \frac{m\,c}{\hbar}\overline{\Psi} = 0. \tag{19.81}$$

This result can be used to derive the continuity equation. To do this, we multiply the Dirac equation on the left by $\overline{\Psi}$, its adjoint by Ψ from the right, and take the difference,

$$\overline{\Psi}\gamma_\mu \partial_\mu \Psi + (\partial_\mu \overline{\Psi})\gamma_\mu \Psi = 0; \tag{19.82}$$

this can be written as

$$\partial_\mu (\overline{\Psi}\gamma_\mu \Psi) = 0, \tag{19.83}$$

which has the form of a conservation equation $\partial_\mu j_\mu = 0$ if the current density is chosen to be proportional to $\overline{\Psi}\gamma_\mu \Psi$. For this expression to have the correct nonrelativistic limit, it is usual to write

$$j_\mu = i\,c\overline{\Psi}\gamma_\mu \Psi, \tag{19.84}$$

that is, using $i\gamma_4\gamma_i = \alpha_i$ (see Eq. (19.39)),

$$j_\mu = (c\Psi^\dagger \boldsymbol{\alpha}\Psi, i\,c\Psi^\dagger \Psi), \tag{19.85}$$

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where the first term in the parentheses represents the spatial component $\boldsymbol{j}$ and the last the density ρ multiplied by ic,

$$\rho = \Psi^\dagger \Psi = \psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_3^* \psi_3 + \psi_4^* \psi_4, \tag{19.86}$$

$$\boldsymbol{j} = c \Psi^\dagger \boldsymbol{\alpha} \Psi. \tag{19.87}$$

The problem of the sign of ρ has thus been solved, since the expression obtained for ρ is analogous to the nonrelativistic one and legitimately corresponds to a density of particles independent of their charge, unlike what happens with the Klein–Gordon equation. The expression for $\boldsymbol{j}$, Eq. (19.87), holds a surprise for us, since $\boldsymbol{\alpha} c$ plays the role of the velocity operator for the free particle, and the eigenvalues of the matrices α_i are ± 1 (note from (19.5) that $\alpha^2 = \mathbb{I}$). We will return to this point later to study its meaning; in the meantime, note that the expression for $\boldsymbol{j}$ is a scalar in spin space.

19.3.2 The spin of the electron

From the preceding derivations – in particular the van der Waerden derivation – we know that Dirac's equation describes particles with spin 1/2, but it is instructive to prove this directly. For simplicity, we will limit ourselves to the proof for a free particle, which can be generalized to the case of a central potential. Consider the Dirac Hamiltonian for a free particle,

$$\widehat{H} = c\, \alpha \cdot \boldsymbol{p} + \beta m c^2, \tag{19.88}$$

and the orbital angular momentum operator,

$$\widehat{L}_3 = -i \hbar \epsilon_{3ij} x_i \partial_j. \tag{19.89}$$

The commutator

$$\left[\widehat{H}, \widehat{L}_3\right] = \left[c\widehat{\alpha} \cdot \hat{\boldsymbol{p}}, \widehat{L}_3\right] = -i \hbar c\, (\alpha_1 \widehat{p}_2 - \alpha_2 \widehat{p}_1) \tag{19.90}$$

is nonzero and therefore $\widehat{L}_3$ is not an integral of the motion. Since in this case there must be an angle operator that is conserved, we need an operator that completes $\widehat{L}_3$ and gives the conserved quantity. By applying σ_3 to the Hamiltonian, one obtains after some algebra (which we do not reproduce here)

$$\left[\widehat{H}, \sigma_3\right] = 2\, i\, c\, (\alpha_1 \widehat{p}_2 - \alpha_2 \widehat{p}_1), \tag{19.91}$$

which combined with Eq. (19.90) gives

$$\left[\widehat{H}, \widehat{L}_3 + \tfrac{1}{2} \hbar \sigma_3\right] = 0. \tag{19.92}$$

Hence the quantity (be careful not to confuse j_i with J_i)

$$J_3 = \widehat{L}_3 + \widehat{S}_3 = \widehat{L}_3 + \tfrac{1}{2} \hbar \sigma_3, \tag{19.93}$$

with

$$\widehat{S}_3 = \tfrac{1}{2} \hbar \sigma_3, \tag{19.94}$$

is an integral of motion,

$$\left[\widehat{H}, \widehat{J}_3\right] = 0. \tag{19.95}$$

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We conclude that the spin operator of the particle (the quantity added to $\boldsymbol{L}$ to complete the angular momentum) is given by $\boldsymbol{S} = (\hbar/2)\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a Dirac matrix.

Sometimes it is convenient to use spin tensors, which are operators built from the matrices γ_μ that describe the spin of the relativistic electron. It is common to write these in the form

$$\sigma_{\mu\nu} = -\tfrac{1}{2} i(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \tag{19.96}$$

or else

$$\sigma_{\mu\nu} = -i\gamma_\mu \gamma_\nu, \quad \mu \neq \nu, \tag{19.97}$$

$$= 0, \qquad \mu = \nu. \tag{19.98}$$

The spatial components of these tensors are identified with the spin components; the meaning of the other components will be seen later.

19.3.3 *The *zitterbewegung*

We will now analyze some properties of the velocity vector $\dot{x}_k$ for a free particle in order to better understand the meaning of Eq. (19.87). We begin by noting that the commutator of the free-particle Dirac Hamiltonian $\widehat{H}$ with $\widehat{p}_k$ vanishes, hence

$$\dot{\widehat{p}}_k = \frac{i}{\hbar}\left[\widehat{H}, \widehat{p}_k\right] = 0. \tag{19.99}$$

This means that $\widehat{p}_k$ is an integral of the motion for the free particle. However, things are more complicated for the velocity, as follows by writing Heisenberg's equation for x_k,

$$\dot{\widehat{x}}_k = \frac{i}{\hbar}\left[\widehat{H}, \widehat{x}_k\right] = \frac{ic}{\hbar}\alpha_i \left[\widehat{p}_i, \widehat{x}_k\right] = c\alpha_i \delta_{ik} = c\alpha_k. \tag{19.100}$$

This result confirms that $\boldsymbol{\alpha} c$ is the velocity operator,

$$\dot{\widehat{\boldsymbol{x}}} = c\,\boldsymbol{\alpha}, \tag{19.101}$$

for any free particle. Combining with Eq. (19.87) we see that we can write $\boldsymbol{j} = \Psi^\dagger \dot{\widehat{\boldsymbol{x}}} \Psi$. Since the eigenvalues of α_i are ± 1, the free Dirac particle always moves at speed c. In order to understand how this is possible, Schrödinger introduced the concept of *zitterbewegung*, which refers to a very fast (at speed c) vibrating or dancing motion of the electron around its trajectory, and which we translate as *tremor*. Let us see how this idea is justified and how the motion is described.

While the components of the momentum are constants of motion, the $\dot{x}_k$ are not (they are given by operators). This suggests that p_k can be regarded as the local average (over very short times) of $m\dot{x}_k$; in other words, that $\dot{x}_k$ contains more detailed information about the instantaneous motion than p_k, as we will confirm a little later (see Eq. (19.103)). To make the difference between the operators p_k and $m\dot{x}_k$ explicit, we again use Heisenberg's equation for α_k and write

$$\dot{\alpha}_k = \frac{1}{c}\ddot{x}_k = \frac{i}{\hbar}\left[\widehat{H}, \alpha_k\right] = \frac{2i\,c}{\hbar}\widehat{p}_k - \frac{2i}{\hbar}\alpha_k \widehat{H}. \tag{19.102}$$

From here and from $\left[\alpha_i, \alpha_j\right] \neq 0$, with $\alpha_i \sim \dot{x}_i$, we conclude that the velocity components of the free particle do not commute with each other and are not conserved,

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that is, they fluctuate. To study the changes in velocity we integrate Eq. (19.102) for an eigenstate of H and p_k, which allows us to treat these quantities as constant; the solution is

$$\dot{x}_k = c\,\alpha_k = \frac{c^2 p_k}{E} + \left(\dot{x}_k(0) - \frac{c^2 p_k}{E}\right) e^{-2iEt/\hbar} \tag{19.103}$$

and its integral over time gives

$$x_k = x_k(0) + \frac{c^2 p_k}{E} t + \frac{i\hbar c}{2E}\left(\frac{\dot{x}_k(0)}{c} - \frac{c p_k}{E}\right) e^{-2iEt/\hbar}. \tag{19.104}$$

The first two terms in this equation represent the classical solution for the relativistic free particle and the third is an additional term that oscillates in time. The amplitude of these oscillations around the classical trajectory is of the order of $\Delta x \sim \hbar c/E$ and therefore depends on the energy. These oscillations constitute the *zitterbewegung*.[11] For a particle that is not moving very fast (apart from *zitterbewegung*), $E \approx mc^2$ and so $\Delta x \sim \hbar/mc = \lambda_C$, the Compton wavelength (which for the electron is $\sim 10^{-12}\,\mathrm{m}$). At such energies the oscillation frequency of the *zitterbewegung* is $2E/\hbar \sim 2mc^2/\hbar \sim 10^{21}\,\mathrm{s}^{-1}$.

If in Eq. (19.103) we average over some cycles of the tremor (as happens in practice when measuring velocity or position), the term in parentheses vanishes, but the classical term is not altered, and we obtain

$$\overline{v_k} = c\overline{\alpha_k} = \frac{c^2}{E} p_k. \tag{19.105}$$

This result confirms that the relation between $\overline{v_k}$ and p_k is the relativistic one ($m\overline{v_k}$ is practically equal to p_k for $E \simeq m_0 c^2$). Note, however, that for positive energy states ($E \geq m_0 c^2 > 0$), $\langle v_k \rangle$ and p_k have the same sense, but for $E \leq m_0 c^2$ ($E < 0$), these quantities have opposite senses. From a classical point of view, with $p = mv$ (which corresponds to the limit $E = m_0 c^2$), the last case implies a negative mass. Another important observation is that Δx decreases with energy, while the frequency of the oscillation increases with it, which means that for ultrarelativistic particles the oscillation occurs at a very high frequency and negligible amplitude. In other words, the ultrarelativistic electron behaves as a point particle, with the effective structure produced by the tremor disappearing.

19.4 The Free Particle

We now turn to study the solution of the Dirac equation for a free particle. We propose to express the solution of the corresponding equation

$$(\gamma_\mu \partial_\mu + \frac{mc}{\hbar})\Psi = 0 \tag{19.106}$$

[11] The *zitterbewegung* is not an exclusive result of the Dirac equation, but rather a property of relativistic (not necessarily quantum) dynamics when restricted only by the constraints imposed by the Poincaré invariance group. It should therefore be seen as a relativistic prediction for spin particles, regardless of the details of the description.

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in terms of plane waves in the form $\Psi = e^{-i\frac{E}{\hbar}t}\psi(\boldsymbol{r})$, with $\psi(\boldsymbol{r})$ a bispinor. We substitute Eq. (19.67) into the Dirac equation written in terms of the spinors Ψ_A, Ψ_B and cancel the common time factor; this gives the pair of coupled equations

$$c\sigma' \cdot \boldsymbol{p}\Psi_B = (E - mc^2)\Psi_A, \tag{19.107}$$

$$c\sigma' \cdot \boldsymbol{p}\Psi_A = (E + m\,c^2)\Psi_B. \tag{19.108}$$

In the reference system where the particle is at rest ($p = 0$) these equations reduce to

$$(E - mc^2)\Psi_A = 0, \tag{19.109}$$

$$(E + mc^2)\Psi_B = 0. \tag{19.110}$$

If $E > 0$, the first equation holds with $E = mc^2$ and any Ψ_A, but it follows from the second equation that Ψ_B must be zero; if $E < 0$, Ψ_A must be equal to zero and $E = -mc^2$. This allows us to identify Ψ_A as the spinor describing positive energy particles and Ψ_B as the corresponding spinor for negative energy states; the Dirac bispinors are therefore

$$\begin{pmatrix} \Psi_A \\ 0 \end{pmatrix}, \quad E > 0, \qquad \begin{pmatrix} 0 \\ \Psi_B \end{pmatrix}, \quad E < 0. \tag{19.111}$$

To uniquely determine the bispinors $\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$ for $p \neq 0$, it is usual to introduce some additional condition. The simplest is to require that they be eigenfunctions of σ_3, which commutes with $\widehat{H}$ for the particle at rest. According to Eq. (19.91), the same consideration holds for any value of p_3 but with $p_1 = p_2 = 0$ and hence for the eigenstates of the helicity operator, which becomes $\mathfrak{h} = \sigma_3$ according to (19.63). Therefore $\Psi_A^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to the eigenvalue $s_3 = 1$ and $\Psi_A^{(-)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $s_3 = -1$. The eigenfunctions Ψ_B for negative energy are constructed analogously. We thus obtain the following eigenfunctions of $\widehat{H}$ and σ_3 for $p = 0$,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-im\,c^2t/\hbar}, (E > 0, s_3 = 1); \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-im\,c^2t/\hbar}, (E > 0, s_3 = -1); \tag{19.112}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{im\,c^2t/\hbar}, (E < 0, s_3 = 1); \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{im\,c^2t/\hbar}, (E < 0, s_3 = -1). \tag{19.113}$$

Since in experiments with electrons we have $E > 0$, the first pair of solutions is generally used.

To study the particle moving in the laboratory system we can either apply a Lorentz transformation to (19.112) or solve Eqs. (19.107) and (19.108) directly. For this we propose a solution of the type

$$\Psi = \begin{pmatrix} u_A(\boldsymbol{p}) \\ u_B(\boldsymbol{p}) \end{pmatrix} e^{i(-Et + \boldsymbol{p}\cdot\boldsymbol{x})/\hbar}, \tag{19.114}$$

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where u_A and u_B are spinors. From (19.107) and (19.108) we get

$$u_A(\boldsymbol{p}) = \frac{c}{E - mc^2}\sigma' \cdot \boldsymbol{p} u_B(\boldsymbol{p}), \tag{19.115}$$

$$u_B(\boldsymbol{p}) = \frac{c}{E + m\,c^2}\sigma' \cdot \boldsymbol{p} u_A(\boldsymbol{p}). \tag{19.116}$$

Combining these two equations we get a third one that is satisfied by either u_A or u_B, which indicates that there are different solutions whose selection depends on additional conditions of the problem. It is often proposed to take u_A simply as $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which agrees with the solution for the free-particle eigenstate of σ_3'; then the second solution must be

$$u_B(\boldsymbol{p}) = \frac{c}{E + mc^2}\boldsymbol{\sigma}' \cdot \boldsymbol{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{19.117}$$

and the components of $u_B = \begin{pmatrix} u_B^{(1)}(\boldsymbol{p}) \\ u_B^{(2)}(\boldsymbol{p}) \end{pmatrix}$ become

$$u_B^{(1)} = \frac{c}{E + mc^2} p_3, \quad u_B^{(2)} = \frac{c}{E + mc^2}(p_1 + ip_2). \tag{19.118}$$

Since

$$\boldsymbol{\sigma}' \cdot \boldsymbol{p} = \sigma_1' p_1 + \sigma_2' p_2 + \sigma_3' p_3 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}, \tag{19.119}$$

this solution corresponds to $E > 0$ and positive spin projection along the z-axis; for the negative spin projection we take $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that u_A is large compared to u_B for nonrelativistic particle velocities and positive energies (when $E + mc^2 \approx 2mc^2$). The two solutions with negative energy are obtained in an analogous way; in this case u_A becomes small compared to u_B. The final set of solutions is

$$u_1(\boldsymbol{p}) = N \begin{pmatrix} 1 \\ 0 \\ +\frac{cp_3}{E+mc^2} \\ +\frac{c(p_1+ip_2)}{E+mc^2} \end{pmatrix}, \quad u_2(\boldsymbol{p}) = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_1-ip_2)}{E+mc^2} \\ -\frac{cp_3}{E+mc^2} \end{pmatrix}, \tag{19.120}$$

$$u_3(\boldsymbol{p}) = N \begin{pmatrix} -\frac{cp_3}{|E|+mc^2} \\ -\frac{c(p_1+ip_2)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u_4(\boldsymbol{p}) = N \begin{pmatrix} -\frac{c(p_1-ip_2)}{|E|+mc^2} \\ \frac{cp_3}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix}. \tag{19.121}$$

These spinors are solutions of the equation

$$(i\gamma_\mu p_\mu + m\,c)u_r(\boldsymbol{p}) = 0, \quad r = 1, 2, 3, 4, \tag{19.122}$$

which is the Fourier transform of the Dirac equation for the free particle. Each u_r is a column of four components, which in general is not an eigenfunction of σ_3 except when $p_1 = p_2 = 0$, as we have already noted.

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From the previous discussion we conclude that there are only two cases where solutions of the free-particle Dirac equation exist which are eigenfunctions of $\boldsymbol{\sigma} \cdot \widehat{\mathbf{n}}$, namely

(i) when $p = 0$,

(ii) when $\widehat{\mathbf{n}} = \boldsymbol{p}/|\boldsymbol{p}|$.

These solutions are indeed eigenfunctions of the helicity operator. To verify this we start from the relation

$$\left[\widehat{H}, \boldsymbol{a} \cdot \boldsymbol{\sigma}\right] = -2ic\boldsymbol{\alpha} \cdot (\boldsymbol{p} \times \boldsymbol{a}) \tag{19.123}$$

and note that when $p = 0$ or when $\boldsymbol{a}$ is parallel to $\boldsymbol{p}$, the commutator vanishes (see problem P19.5).

To normalize the solution of Dirac's equation, we write

$$u^{(r)^\dagger} u^{(r)} = C \tag{19.124}$$

and fix the constant C with an appropriate criterion. Two commonly used are

(a) Normalization to unity,

$$C = 1, \quad N = \sqrt{\frac{|E| + m\,c^2}{2|E|}}, \tag{19.125}$$

(b) Lorentz-invariant normalization,

$$C = \frac{|E|}{m\,c^2}, \quad N = \sqrt{\frac{|E| + m\,c^2}{2\,m\,c^2}}. \tag{19.126}$$

Exercise E19.2. The free particle with positive energy

Construct the solutions of the Dirac equation for a free particle with positive energy and helicity states ± 1.

Solution. We propose a solution of the form (in units of $\hbar = 1$)

$$\Psi = u(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)}, \tag{19.127}$$

with $E > 0$. The helicity operator is

$$\mathfrak{h} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p} = \frac{1}{p} \sum_k \sigma_k p_k. \tag{19.128}$$

Using the standard representation of the Pauli matrices σ_k, the eigenvalue equation for the helicity applied to the spinor Ψ, $\mathfrak{h}\Psi = h\Psi$, $h = \pm 1$, is decomposed into equal systems of two equations, one for u_1, u_2 and a similar one for u_3, u_4, which means that the small spinor (with components u_3, u_4) is proportional to the large one (with components u_1, u_2). For the latter we obtain

$$p_3 u_1 + (p_1 - ip_2)\, u_2 = hpu_1, \tag{19.129}$$

$$(p_1 + ip_2)\, u_1 - p_3 u_2 = hpu_2. \tag{19.130}$$

It is convenient to simplify the expressions that follow by writing the momentum components in terms of the angles that determine their direction,

$$p_1 \pm ip_2 = p \sin\theta e^{\pm i\varphi}, \quad p_3 = p\cos\theta. \tag{19.131}$$

Equations (19.129) and (19.130) then reduce to

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$$u_1 \cos\theta + u_2 \sin\theta e^{-i\varphi} = h u_1, \quad u_1 \sin\theta e^{i\varphi} - u_2 \cos\theta = h u_2, \tag{19.132}$$

which gives

(a) for $h = +1$, $\quad u_{2,4} = \tan\frac{1}{2}\theta e^{i\varphi} u_{1,3}$,
(b) for $h = -1$, $\quad u_{2,4} = -\cot\frac{1}{2}\theta e^{i\varphi} u_{1,3}$.

To find the components $u_{1,3}$ we substitute Ψ and Eqs. (19.131) into the Dirac equation, giving the system

$$\begin{aligned} u_4 \sin\theta e^{-i\varphi} + u_3 \cos\theta - \varkappa u_1 &= 0, \\ u_3 \sin\theta e^{+i\varphi} - u_4 \cos\theta - \varkappa u_2 &= 0, \\ -u_2 \sin\theta e^{-i\varphi} - u_1 \cos\theta + \frac{1}{\varkappa} u_3 &= 0, \\ -u_1 \sin\theta e^{+i\varphi} + u_2 \cos\theta + \frac{1}{\varkappa} u_4 &= 0, \end{aligned} \tag{19.133}$$

where

$$\varkappa = \frac{cp}{E + mc^2}. \tag{19.134}$$

By inserting Eqs. (19.132) here, the system is reduced to $u_3 = h\varkappa u_1$. The still missing relation between the amplitudes is determined by the normalization, which is given by Eq. (19.124), $u^\dagger(\boldsymbol{p})u(\boldsymbol{p}) = 1$. (There are a number of uses here.) This gives

(a) for $h = +1$,

$$u_+(\boldsymbol{p}) = \frac{1}{\sqrt{1+\varkappa^2}} \begin{pmatrix} \cos\frac{1}{2}\theta e^{-i\varphi/2} \\ \sin\frac{1}{2}\theta e^{+i\varphi/2} \\ \varkappa\cos\frac{1}{2}\theta e^{-i\varphi/2} \\ \varkappa\sin\frac{1}{2}\theta e^{+i\varphi/2} \end{pmatrix},$$

(b) for $h = -1$,

$$u_-(\boldsymbol{p}) = \frac{1}{\sqrt{1+\varkappa^2}} \begin{pmatrix} \sin\frac{1}{2}\theta e^{-i\varphi/2} \\ -\cos\frac{1}{2}\theta e^{+i\varphi/2} \\ -\varkappa\sin\frac{1}{2}\theta e^{-i\varphi/2} \\ \varkappa\cos\frac{1}{2}\theta e^{+i\varphi/2} \end{pmatrix}.$$

The structure of the solution confirms that it corresponds to positive energy, that is, to electrons.

19.5 *The Dirac Particle in an External Electromagnetic Field

We now turn to the study of Dirac's equation for the case where the particle is under the action of an external electromagnetic field. By introducing minimal coupling, we obtain Dirac's equation in the presence of an electromagnetic field,

$$\gamma_\nu \left(\partial_\mu - \frac{ie}{\hbar c} A_\mu \right) \Psi + \frac{mc}{\hbar} \Psi = 0. \tag{19.135}$$

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To transform this equation we need a relativistic generalization of Eq. (14.10),

$$(\widehat{\boldsymbol{\sigma}}\cdot\widehat{\boldsymbol{A}})(\widehat{\boldsymbol{\sigma}}\cdot\widehat{\boldsymbol{B}})=\widehat{\boldsymbol{A}}\cdot\widehat{\boldsymbol{B}}+i\widehat{\boldsymbol{\sigma}}\cdot(\widehat{\boldsymbol{A}}\times\widehat{\boldsymbol{B}}),\qquad\left[\widehat{\boldsymbol{A}},\widehat{\boldsymbol{\sigma}}\right]=0, \tag{19.136}$$

which we obtain as follows. We rewrite Eq. (19.96) in the form $\gamma_\mu\gamma_\nu=\gamma_\nu\gamma_\mu+2i\sigma_{\mu\nu}$, and combine with $\gamma_\mu\gamma_\nu=-\gamma_\nu\gamma_\mu+2\delta_{\mu\nu}$ to obtain

$$\gamma_\mu\gamma_\nu=\delta_{\mu\nu}+i\sigma_{\mu\nu}. \tag{19.137}$$

Now let A_μ be an operator that commutes with γ_μ; by multiplying (19.137) with $A_\mu B_\nu$ on the right, we get

$$(\gamma_\mu A_\mu)(\gamma_\nu B_\nu)=A_\mu B_\mu+i\sigma_{\mu\nu}A_\mu B_\nu. \tag{19.138}$$

On the other hand, if we exchange the indices, we see that

$$\sigma_{\mu\nu}A_\mu B_\nu=\sigma_{\nu\mu}A_\nu B_\mu=-\sigma_{\mu\nu}A_\nu B_\mu, \tag{19.139}$$

due to the antisymmetry of $\sigma_{\mu\nu}$. Combining the two expressions, we get

$$\sigma_{\mu\nu}A_\mu B_\nu=\tfrac{1}{2}\sigma_{\mu\nu}(A_\mu B_\nu-A_\nu B_\mu), \tag{19.140}$$

so that Eq. (19.138) can be written in the more convenient form,

$$(\gamma_\mu A_\mu)(\gamma_\nu B_\nu)=A_\mu B_\nu+\tfrac{1}{2}i\sigma_{\mu\nu}(A_\mu B_\nu-A_\nu B_\mu). \tag{19.141}$$

This is the desired generalization. In particular, for $A=B$ we get

$$(\gamma_\mu A_\mu)^2=A_\mu A_\mu+\tfrac{1}{2}i\sigma_{\mu\nu}\left[A_\mu,A_\nu\right]. \tag{19.142}$$

Returning to our task, we apply from the left to Eq. (19.135) the operator $(\gamma_\nu B_\nu-mc/\hbar)$, with $B_\mu=\partial_\mu-(ie/\hbar c)\,A_\mu$,

$$\left(\gamma_\nu B_\nu-\frac{mc}{\hbar}\right)\left(\gamma_\mu B_\mu+\frac{mc}{\hbar}\right)\Psi=\left[(\gamma_\nu B_\nu)(\gamma_\mu B_\mu)-\frac{m^2c^2}{\hbar^2}\right]\Psi=0. \tag{19.143}$$

We now apply (19.142), substituting B_μ for its original expression; the result is an equation that contains the Klein–Gordon equation plus additional terms describing the coupling between the particle's spin tensor and the external field,

$$\left(\partial_\mu-\frac{ie}{\hbar c}A_\mu\right)^2\Psi-\frac{m^2c^2}{\hbar^2}\Psi+\tfrac{1}{2}i\sigma_{\mu\nu}\left[\partial_\mu-\frac{ie}{\hbar c}A_\mu,\partial_\nu-\frac{ie}{\hbar c}A_\nu\right]\Psi=0. \tag{19.144}$$

The commutator that appears in this expression can be rewritten as follows,

$$\begin{aligned}[,]&=-\frac{ie}{\hbar c}(\partial_\mu A_\nu+A_\mu\partial_\nu-\partial_\nu A_\mu-A_\nu\partial_\mu)\\&=-\frac{ie}{\hbar c}\left[(\partial_\mu A_\nu)-(\partial_\nu A_\mu)\right]=-\frac{ie}{\hbar c}F_{\mu\nu},\end{aligned} \tag{19.145}$$

where

$$F_{\mu\nu}=\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu} \tag{19.146}$$

is the electromagnetic tensor (remember that $F_{ij}=\varepsilon_{ijk}B_k$, $F_{4i}=iE_i$). Substituting and multiplying by $(-i\hbar)^2$, the preceding equation takes the form

$$(p_\mu-\frac{e}{c}A_\mu)^2\Psi+m^2c^2\Psi-\frac{\hbar e}{2c}\sigma_{\mu\nu}F_{\mu\nu}\Psi=0. \tag{19.147}$$

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19.5.1 Electric and magnetic moments of the electron

The preceding result shows that the Dirac equation for a particle subject to the action of an electromagnetic field is the relativistic generalization of the corresponding Pauli equation (see Section 14.2.1), which allows us to write

$$\frac{\hbar e}{4mc}\sigma_{\mu\nu}F_{\mu\nu} = -\frac{i\hbar e}{4mc}\gamma_\mu\gamma_\nu F_{\mu\nu} = -\frac{i\hbar e}{2mc}(\gamma_i\gamma_j F_{ij} + \gamma_i\gamma_4 F_{i4})$$
$$= \frac{\hbar e}{2mc}(\sigma_k B_k + i\gamma_i\gamma_4 i E_i), \tag{19.148}$$

or taking into account that $\gamma_i\gamma_4 = \gamma_i\rho_3 = \rho_2\sigma_i\rho_3 = \rho_2\rho_3\sigma_i = i\rho_1\sigma_i = i\alpha_i$,

$$\frac{\hbar e}{4mc}\sigma_{\mu\nu}F_{\mu\nu} = \frac{\hbar e}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{B} - i\boldsymbol{\alpha}\cdot\boldsymbol{E}) = \boldsymbol{\mu}_{\text{mag}}\cdot\boldsymbol{B} + \boldsymbol{\mu}_{\text{elect}}\cdot\boldsymbol{E}, \tag{19.149}$$

where

$$\boldsymbol{\mu}_{\text{mag}} = -(\hbar e_0/2mc)\,\boldsymbol{\sigma} \tag{19.150}$$

and

$$\boldsymbol{\mu}_{\text{elect}} = i\,(\hbar e_0/2mc)\,\boldsymbol{\alpha} \tag{19.151}$$

are the electric and magnetic Dirac moments of the electron, respectively.

It is particularly interesting to check that Dirac's equation predicts the correct value of the intrinsic magnetic moment, including the factor $g = 2$ that we had to insert by hand in nonrelativistic theory. It also predicts an electric moment of the electron proportional to the velocity operator. Since the eigenvalues of σ are of the order of unity, while those of α are of the order of $|\dot{x}|/c$, in the nonrelativistic limit one has

$$|\langle\mu_{\text{mag}}\rangle| \sim \frac{\hbar e_0}{2mc} >> \left|\frac{\langle\dot{x}\rangle}{c}\right| \frac{\hbar e_0}{2\,m\,c} \sim |\langle\mu_{\text{elect}}\rangle|, \tag{19.152}$$

and the interaction term reduces to the Pauli expression in the low-velocity limit, as expected.

Up to now, the properties of the Dirac electron have been obtained in a natural way, without the need to introduce ad hoc assumptions, so that Dirac's equation is considered to be an exact equation for the electron. However, in a second-quantization theory, the electron also couples to the fluctuations of the radiation field (which in fact always exist) and through them to itself, that is there is a self-interaction of the particle that contributes to modify the parameters that characterize it (mass, charge, etc.). One of the effects of self-interaction is the appearance of the anomalous magnetic moment, a contribution that is added to the proper magnetic moment predicted by Dirac's theory. As a result, the total magnetic moment of the electron is not that given by Dirac's theory for a particle, but carries a correction given by (see the explanation in Section 14.4)

$$\mu = \mu_0\left(1 + \frac{\alpha}{2\pi} + \ldots\right) \tag{19.153}$$

with $\alpha \approx 1/137$. This is an important result of QED, although because of the small size of the correction it is common to start from Dirac's equation to describe the electron and add the anomalous moment as a correction if necessary. In the case of nucleons, on the other hand, although the anomalous magnetic moment is of the order of the

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electronic one, the normal moment is much smaller (of the order of a thousandth of the electronic one, since $\mu \sim m^{-1}$), so in this case the anomalous magnetic moment cannot be introduced as a mere correction.

19.6 Solution of Dirac's Equation for the Hydrogen-Like Atom

In Section 19.2, the first-order Dirac equation for a four-component bi-spinor was transformed into a pair of first-order equations, each for a two-component spinor. A further separation of the angular and radial variables leads to a second-order equation corresponding to the Klein–Gordon equation plus additional terms representing corrections introduced by the coupling of the particle's spin to the external field. Finally, the angular and radial functions can be factored out.

In the stationary case this transformation leads to the radial equation for the hydrogen-like atom,

$$\left(\nabla_r^2 + \frac{\Lambda_D}{r^2} + \frac{2c_E}{r} - d_E\right) R = 0, \tag{19.154}$$

with

$$\nabla_r^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r, \ \Lambda_D = l(l+1) - \alpha^2 Z^2, \tag{19.155}$$

$$d_E = \frac{m^2c^2}{\hbar^2}\left(1 - \mathcal{E}^2\right), \ c_E = \frac{Zme^2}{\hbar^2}\mathcal{E}, \tag{19.156}$$

where

$$\mathcal{E} = \frac{mc^2 + E}{mc^2} \tag{19.157}$$

is the energy of the atom, including the rest energy, expressed in units of the rest energy, and $\alpha = e^2/\hbar c$.

19.6.1 Exact solution of the central-force problem

For the case of the Coulomb potential, Dirac's equation can be solved exactly. The result for the energy levels (including the proper energy), is given in Exercise 19.33,

$$E_{nj} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2 Z^2}{\left[n - j - \frac{1}{2} + \sqrt{k^2 - \alpha^2 Z^2}\right]^2}}}. \tag{19.158}$$

where $n = k + l + 1$. It is interesting to compare this expression with the one obtained from the Klein–Gordon equation (19.29) with $k = k_{KG}$, in which j takes the place of k due to the absence of electron spin; with $k = k_D$ the correspondence is $k_{KG} + \frac{1}{2} \to k_D + \frac{1}{2} + (l - j)$). An expansion of the previous result in powers of $Z\alpha$ gives

$$E_{nj} = mc^2\left[1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\right) + \ldots\right]. \tag{19.159}$$

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The first term after the proper mass represents the Schrödinger energy; the next represents the relativistic and spinorial correction to order α^4, which follows from Dirac's equation; this correction differs from (19.30) derived from the Klein–Gordon equation by substituting j for l. It is clear that this approximate expression is valid only when $k^2 > Z^2\alpha^2$, that is, for relatively light atoms, or highly excited states of heavy atoms. The solution (19.159) predicts the breaking of the degeneracy with respect to j, but not with respect to l.

Exercise E19.3. Solution of the radial Dirac equation for the H atom using the algebraic consistency method

Construct the exact energy eigenvalues for the Dirac H atom by means of the algebraic procedure developed in Section A.1.

Solution. The method of algebraic consistency conditions can be used to solve the H-atom eigenvalue problem without the need to know the explicit solution of the Dirac equation, as follows.

The consistency conditions hold whenever the operators S and A can be written in the form of Eqs. (A.94) and (A.95), where B_{n-} represents the Hamiltonian operator expressed in the appropriate dynamical variable.

In terms of $u = rR$, Eq. (19.154) acquires the same structure as the radial Schrödinger equation for the hydrogen-like atom, (A.99). They both correspond to the operator B_{1-}, although with a different set of parameters, so (19.154) is equivalent to

$$\left(-q\frac{d^2}{dq^2} + \frac{\Lambda_D}{q} + q\right)u = B_{1-}u = \frac{2c_E}{d_E^{1/2}}u, \tag{19.160}$$

$$\Lambda_D = l\,(l+1) - \alpha^2, \qquad q = d_E^{1/2}r. \tag{19.161}$$

Thus, the solution for the Schrödinger atom can be transferred to the Dirac atom,

$$b_k = b_0 + 2k, \tag{19.162}$$

where b_0 is given by $b_0 = 1 + \sqrt{1 + 4\Lambda_D}$ according to Eq. (A.98). The minus sign is excluded, since $b_0 \geq 0$ for any l. From the preceding equations, after some algebra, we obtain

$$\mathcal{E}_k = \left[1 + \frac{4\alpha^2}{(b_0 + 2k)^2}\right]^{-1/2}, \tag{19.163}$$

which is the exact result for the energy of the Dirac H atom in units of mc^2.

19.7 Approximate Versions of Dirac's Equation

We will now study the corrections to the Schrödinger (or rather to the Pauli) equation predicted by Dirac's equation. An expansion up to the second order in v/c allows us

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to derive the terms coming from both relativity and spin, which can then be treated as perturbations to the Pauli equation, as was done in Section 14.4.

Consider an electron in an external electromagnetic field; we write the wave function as $\Psi = e^{-iEt/\hbar}\Psi(\boldsymbol{r})$ and substitute into Eq. (19.135) to obtain the following system of equations for the spinors (see Eqs. (19.107) and (19.108) with $p_\mu \to p_\mu - eA_\mu/c$),

$$c\boldsymbol{\sigma}' \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})\Psi_B = (E - eA_0 - mc^2)\Psi_A, \tag{19.164}$$

$$c\boldsymbol{\sigma}' \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})\Psi_A = (E - eA_0 + mc^2)\Psi_B. \tag{19.165}$$

Solving for Ψ_B from the second equation and inserting the solution into the first gives a second-order equation of the van der Waerden type,

$$\boldsymbol{\sigma}' \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})\frac{c^2}{E - eA_0 + m\,c^2}\boldsymbol{\sigma}' \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})\Psi_A = (E - eA_0 - m\,c^2)\Psi_A. \tag{19.166}$$

This is an exact Dirac's equation, and it is Lorentz invariant, although not explicitly covariant. In the nonrelativistic approximation we can consider $E \approx mc^2$ and expand up to the first order in $|v/c|$; with $E_0 = E - mc^2$ and $|E_0 - eA_0| \ll \mathrm{mc}^2$, where E is the total energy, we get

$$\frac{c^2}{E - eA_0 + mc^2} = \frac{1}{2m}\frac{2mc^2}{2mc^2 + E_0 - eA_0} = \frac{1}{2m}\left(1 - \frac{E_0 - eA_0}{2mc^2} + \ldots\right). \tag{19.167}$$

Keeping for the moment only the first term 1/2m, (19.166) reduces to

$$\frac{1}{2\,m}\left[\boldsymbol{\sigma}' \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})\right]^2 \Psi_A = (E_0 - eA_0)\Psi_A, \tag{19.168}$$

which is the Pauli equation (14.25) for the stationary case, with $V = eA_0$. The Pauli equation can then be seen as the lowest-order nonrelativistic approximation of the Dirac equation. If, in addition, the spin is suppressed, the previous equation reduces to that of Schrödinger. Note that this result endorses the substitution of $\boldsymbol{p}^2$ by $(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2$ used in Section 14.2 as a method of introducing spin into Schrödinger's theory.

After some algebra carried out to second order, the Dirac Hamiltonian reduces to

$$H = \frac{\boldsymbol{p}^2}{2m} + V - \frac{\boldsymbol{p}^4}{8m^3c^2} - \frac{e\hbar^2}{8m^2\,c^2}(\nabla \cdot \boldsymbol{E}) - \frac{\hbar e}{4m^2c^2}\boldsymbol{\sigma} \cdot \boldsymbol{E} \times \boldsymbol{p}. \tag{19.169}$$

We see that to this order, Dirac's theory predicts three correction terms to Schrödinger's theory, namely,

(i) the relativistic correction to the kinetic energy, $\widehat{H}_1 = -\boldsymbol{p}^4/8m^3c^2$,

(ii) the Darwin term due to the *zitterbewegung* of the electron,

$$\widehat{H}_2 = -\frac{e\hbar^2}{8m^2c^2}(\nabla \cdot \boldsymbol{E}) = \frac{\hbar^2}{8m^2c^2}\nabla^2 V, \tag{19.170}$$

(iii) the term due to the coupling between orbital motion and spin,

$$\widehat{H}_3 = -\frac{\hbar e}{4m^2\,c^2}\boldsymbol{\sigma} \cdot \boldsymbol{E} \times \boldsymbol{p} = \frac{\hbar}{4m^2\,c^2}\boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} V) \times \boldsymbol{p}. \tag{19.171}$$

Note that $\widehat{H}_2$ has appeared here as a result of the coupling of the electron's electric moment to the electric field, generated by the orbital motion. This makes it possible to

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link the origin of the electron's electric moment to the *zitterbewegung*. Sometimes this correspondence is carried further by considering that even the magnetic moment, and hence the electron spin, is due to the *zitterbewegung*. This possibility has been seriously considered by several authors over the years, but no consensus has been reached.

Problems

P19.1 Solve the Klein–Gordon equation for an isotropic attractive potential of depth V_0 and radius a; find the continuity conditions at $r = a$. Obtain the minimum value of V_0 needed to bind a particle of mass m.

P19.2 Use the method suggested in Problem 19.7.1 and the approximate equivalence

$$\frac{2m}{\hbar^2}\left[E - V(r)\right] \to \frac{1}{\hbar^2 c^2}\left[(E - V(\boldsymbol{r}))^2 - m^2c^4\right], \tag{19.172}$$

to obtain the Klein–Gordon solution for the spherical uniform well from the solution of the corresponding Schrödinger problem.

P19.3 Show that the expectation values of E^2 and $\boldsymbol{p}^2$ for a general packet solution of the Klein–Gordon equation, satisfy the equation $\langle E^2\rangle = c^2\langle \boldsymbol{p}^2\rangle + m^2c^4$.

P19.4 Show explicitly that the matrices γ_μ and α_μ satisfy the appropriate anticommutation rules, Eqs. (19.36) and (19.52).

P19.5 In applications, the matrix γ_5, defined as the product of the four matrices γ_μ,

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \tag{19.173}$$

is frequently used. Prove that $\hat{\gamma}_5$ is Hermitian, that it anticommutes with the matrices γ_μ,

$$\gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0 \quad (\mu = 1,2,3,4), \tag{19.174}$$

and that

$$\gamma_5 = \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}, \quad \gamma_5^2 = 1. \tag{19.175}$$

P19.6 Solve Problem 19.7.1, this time using Dirac's equation. Compare and discuss your results.

P19.7 Prove that $\left[\widehat{H}, \boldsymbol{a}\cdot\boldsymbol{\sigma}\right] = -2ic\boldsymbol{\alpha}\cdot(\boldsymbol{p}\times\boldsymbol{a})$ for $\boldsymbol{a}$ a fixed vector and $\widehat{H}$ the free-particle Dirac Hamiltonian.

P19.8 Consider the sign operator for the free-particle Dirac equation given by the equation

$$\hat{\Lambda} = \frac{\widehat{H}_D}{E_+} = \frac{c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2}{E_+}, \quad E_+ = +c\sqrt{\boldsymbol{p}^2 + m^2c^2}. \tag{19.176}$$

Show that this operator is Hermitian and unitary, with eigenvalues $\lambda = \pm 1$, and that these eigenvalues are integrals of motion for the free particle. Construct with its help the projectors $\hat{\Pi}^{\pm} = \frac{1}{2}(1 \pm \hat{\Lambda})$ and show that these operators project the positive and negative energy states, respectively, for the free particle.

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P19.9 Given the Dirac Hamiltonian with minimal coupling (in units $c = \hbar = 1$)

$$\widehat{H}_D = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta m + e\phi, \tag{19.177}$$

where $\boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$, prove that

$$\dot{\boldsymbol{r}} = \boldsymbol{\alpha}, \quad \ddot{\boldsymbol{r}} = \dot{\boldsymbol{\alpha}} = 2i\left[\boldsymbol{\pi} - \boldsymbol{\alpha}(\widehat{H}_D - e\phi)\right], \tag{19.178}$$

$$\dot{\boldsymbol{\pi}} = e\boldsymbol{E} + e\boldsymbol{\alpha} \times \boldsymbol{B}. \tag{19.179}$$

Note that the Lorentz force is expressed in terms of $\boldsymbol{\alpha}$.

P19.10 Prove the equation

$$(\boldsymbol{\alpha} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{J}) = \hat{\rho}_1(\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{J}) = \rho_1 \boldsymbol{p} \cdot \boldsymbol{J} + i\boldsymbol{\alpha} \cdot (\boldsymbol{p} \times \boldsymbol{J}). \tag{19.180}$$

P19.11 The selection rules for dipole transitions that apply to Dirac's theory are $\Delta l = \pm 1$, $\Delta j = 0, \pm 1$. Determine the frequencies of the transitions allowed between the states with $n = 2$ and $n = 3$ for the H atom in Dirac's theory, and compare with the results predicted by Schrödinger's theory.

Bibliographical Notes

Dirac's classic book (2023), first published in 1930, is still a standard reference.

The idea that the electron spin is related to the *zitterbewegung* has been proposed independently by many authors, starting with Schrödinger (1935). This has been an ongoing topic of research over the years; see, for example, Dirac (2023), Huang (1952), Bhabha and Corben (1941), Barut and Zhangi (1984), Hestenes (1990), Rodrigues et al. (1993), G. Salesi (1993). The last word on this subject has not yet been said, however.

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