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P-localizing group extensions with a finite kernel

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Abstract

Assume P is a family of primes, and let $()_P$ represent the P-localization functor. If $1 \to N \xrightarrow{\iota} G \xrightarrow{\epsilon} Q \to 1$ is an exact sequence of groups with N finite, we prove that the sequence $N_P \xrightarrow{\iota_P} G_P \xrightarrow{\epsilon_P} Q_P \to 1$ is exact. Moreover, we provide an explicit description of Ker ι_P when Q belongs to a specific class of groups defined by a cohomological property. This class contains all nilpotent groups, all free groups and all P-local groups, as well as certain extensions formed from these three types of groups. In conclusion, we discuss the implications of our results for the study of finite-by-nilpotent groups.

1. *Introduction*

Let P be a family of primes. A group G is said to be *P-local* if the function $g \mapsto g^n$ from G to G is a bijection for every integer n whose prime divisors lie outside of P. As shown in [**17**], there is a functor, called the P-localization functor, that associates to every group G a unique P-local group G_P in a universal way. This functor has been studied in great detail for nilpotent groups; see, in particular, the work of P. Hilton [**9**, **10**, **12**]. However, with some notable exceptions (see [**1**, **4**, **6**, **7**, **18**]), its properties for other types of groups have proved to be more elusive.

One of the keys to understanding the P-localization functor is to determine its effect on short exact sequences. This approach has been adopted in much of the literature on P-localization. For instance, in [**9**] Hilton shows that P-localization preserves exactness when applied to short exact sequences of nilpotent groups. Moreover, in [**4**] C. Casacuberta and M. Castellet examine the P-localization of a group extension

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with nilpotent kernel and torsion quotient. Further investigations in this vein may be found in [**15**] and [**16**], which treat P-localizations of split extensions.

In this paper we obtain some new results on P-localizing short exact sequences. Our focus is on group extensions of the form $1 \to N \xrightarrow{\iota} G \xrightarrow{\epsilon} Q \to 1$ with N finite. In this case, we prove that the sequence

$$N_P \xrightarrow{\iota_P} G_P \xrightarrow{\epsilon_P} Q_P \longrightarrow 1 \tag{1·1}$$

is exact. Furthermore, in our main theorem, we provide an explicit description of $\operatorname{Ker} \iota_P$ when Q belongs to a special class of groups, which we denote by $\mathfrak{U}_P$. The class $\mathfrak{U}_P$ consists of all groups Q such that, for any finite Q-module A with both the order of A and the order of the image of Q in $\operatorname{Aut}(A)$ divisible only by primes in P, the map $H^n(Q_P, A) \to H^n(Q, A)$ induced by P-localizing Q is an isomorphism for all $n \geqslant 0$. As observed in [**5**, **16**], the class of nilpotent groups is contained in $\mathfrak{U}_P$. In this paper we show that $\mathfrak{U}_P$ also contains every free group. In addition, we provide some other examples of groups in $\mathfrak{U}_P$. For instance, $\mathfrak{U}_P$ contains any finite direct product in which each factor is either free or nilpotent.

Our description of $\operatorname{Ker} \iota_P$ in (1·1) for Q in the class $\mathfrak{U}_P$ employs the left action of G on N_P that is induced by the action of G on N by conjugation. We establish that $\operatorname{Ker} \iota_P$ is the subgroup of N_P generated by all the elements of the form $(g \cdot a)a^{-1}$ such that $a \in N_P$ and g is an element of G whose image in $\operatorname{Aut}(N)$ has order relatively prime to all the primes in P. Moreover, to facilitate the calculation of G_P from $\operatorname{Ker} \iota_P$, N_P and Q_P, we observe that if H is an extension of $N_P/\operatorname{Ker} \iota_P$ by Q_P such that there is a homomorphism $\phi: G \to H$ that makes the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 1 & \longrightarrow & N_P/\operatorname{Ker} \iota_P & \longrightarrow & H & \longrightarrow & Q_P & \longrightarrow & 1 \end{array}$$

commute, then $H \cong G_P$.

The plan of the article is as follows. In Sections 2 and 3 we establish some preliminary results that are relevant to our theorems on extensions. Section 2 is devoted to a discussion of P-local group actions. This notion, which is fundamental to understanding the structure of P-local group extensions, was first introduced by A. Garcia Rodicio [**7**]. Our most important contributions here are to identify a large class of group actions that are P-local (Proposition 2·4) and to characterize all the P-local actions on finite groups (Proposition 2·6). In Section 3 we investigate the class of groups $\mathfrak{U}_P$, showing how to construct examples of groups in this class.

The heart of the paper is Section 4. Here we prove our results on P-localizing extensions: Theorem 4·3 establishes the exactness of (1·1), and Theorem 4·8 explicitly describes $\operatorname{Ker} \iota_P$ in (1·1) when Q is in $\mathfrak{U}_P$. Also, we provide examples that establish the necessity of the hypotheses in both of these theorems– the finiteness of N in Theorem 4·3 as well as the condition on Q in Theorem 4·8.

In Section 5 we consider an important case of Theorem 4·8, namely, where Q is nilpotent. Our result for this case implies that the class of finite-by-nilpotent groups is closed under P-localization. Moreover, the theorem enables us to generalize some of the familiar properties of P-localization for nilpotent groups to this larger class (see Propositions 5·1 and 5·2). To conclude the article, we discuss the connection between

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our results on finite-by-nilpotent groups and the work done by A. Descheemaeker and W. Malfait [**6**] on the P-localization of groups that are nilpotent-by-torsion.

Notation and terminology. We will always assume that G is a group. For any $g,\ h \in G$, we define $[g,h] = ghg^{-1}h^{-1}$. The center of G is denoted by $Z(G)$. If H is a subgroup of G, then $N_G(H)$ is the normalizer of H in G.

Throughout the paper, P will represent a set of positive primes and P' the set of all positive primes outside of P. We will use the term *P-number* to refer to a positive integer all of whose positive prime divisors come from P. A group in which every element has a finite order which is a P-number is a *P-group*.

The *P-localization functor* $()_P$ is a functor from the category of groups to the category of P-local groups such that there is a natural transformation l_P from the identity functor to $()_P$ with the following universal property: for any group homomorphism $\phi: G \to H$ with H P-local, there exists a unique map $\psi: G_P \to H$ with $\psi l_P = \phi$. Any map from G to a P-local group obeying the same universal property as $l_P: G \to G_P$ is said to *P-localize* G.

For G finite, it is easy to see that $l_P: G \to G_P$ is surjective and Ker l_P is the subgroup $T_{P'}G$ of G generated by all the Sylow q-subgroups for all primes $q \in P'$.

We will refer to the basic properties of P-localization for nilpotent groups which are established in [**9**]. In particular, if G is nilpotent, then the kernel of $l_P: G \to G_P$ is the set of elements whose orders are P'-numbers. Moreover, for G nilpotent, the group homomorphism $\phi: G \to H$ P-localizes if and only if the following three properties hold: (i) H is P-local; (ii) Ker ϕ is a P'-group; (iii) for every $h \in H$, there is a P'-number n such that $h^n \in \text{Im } \phi$.

The term *G-group* will always refer to a left G-group. If N is a G-group, then the function $G \times N \to N$ will be denoted by $(g,a) \mapsto g \cdot a$. When N is a normal subgroup of G, we will assume that the action of G on N arises from conjugation; that is, $g \cdot a = gag^{-1}$ for every $a \in N$ and $g \in G$. If M is a subgroup of the G-group N, then we define

$$C_G(M) = \{g \in G : g \cdot a = a \quad \text{for all } a \in M\}.$$

For the proofs of our main results in Section 4 we require some terminology and notation regarding group extensions. Any extension $1 \to N \to G \to Q \to 1$ gives rise to a natural homomorphism $\chi: Q \to \text{Out}(N)$ called the *coupling* of the extension. The set of all extensions of N by Q with a given coupling $\chi: Q \to \text{Out}(N)$ will be denoted by $\mathcal{E}(Q,N,\chi)$. An extension $1 \to N \to G \to Q \to 1$ also gives rise to a left action of Q on $Z(N)$; we will regard $Z(N)$ as a left Q-module under this action.

2. *P-local actions*

The following concept will play an important role in this paper. It also appears in [**5**, **7**, **15**, **16**]; moreover, it is prefigured in the much earlier work of G. Baumslag [**2**] on the existence and uniqueness of roots in wreath products.

Definition. Let N be a G-group. The action of G on N is *P-local* if the function $a \mapsto a(g \cdot a) \cdots (g^{n-1} \cdot a)$ from N to N is a bijection for every $g \in G$ and P'-number n.

The relevance of the above notion to group extensions is elucidated by the following proposition; similar observations can be found in [**15**, **16**].

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PROPOSITION 2·1. *Assume N is a normal subgroup of G.*

(i) *If G is a P-local group, then G/N is a P-local group if and only if the action of G on N is P-local.*

(ii) *If G/N is a P-local group and the action of G on N is P-local, then G is a P-local group.*

Proof. The results follow quite easily from the fact that

$$(ag)^n = a(g \cdot a) \cdots (g^{n-1} \cdot a)g^n$$

for all $g \in G$ and $a \in N$.

Our principal goal in this section is to identify a large class of G-groups that are equipped with P-local actions. In this endeavor, the following result will be useful.

LEMMA 2·2 ([**1**, theorem 11·5]). *Let G be a group with normal subgroup N. If N is a P-local group and G/N is a P-group, then G is a P-local group.*

Lemma 2·2 yields:

LEMMA 2·3. *Assume N is a G-group. Let $\omega: G \to \mathrm{Aut}(N)$ be the homomorphism arising from the action of G on N. Then G acts on N P-locally if the following two statements hold*:

(i) *N is P-local*;

(ii) *$\mathrm{Im}\ \omega$ is a P-group.*

Proof. Let H be the semidirect product of N and $\mathrm{Im}\ \omega$ with respect to the obvious action. Then H is P-local by Lemma 2·2. Hence, by Proposition 2·1, the action of $\mathrm{Im}\ \omega$ on N is P-local, which means that the action of G on N is P-local.

We can extend the above result as follows.

PROPOSITION 2·4. *Let N be a G-group and $\omega: G \to \mathrm{Aut}(N)$ the homomorphism arising from the action of G on N. Assume*:

(i) *N is P-local, and*

(ii) *for each $\alpha \in \mathrm{Im}\ \omega$, there is a P-number n such that $\alpha^n \in \mathrm{Inn}(N)$.*

Then the action of G on N is P-local.

To establish the above proposition, we require the following lemma, whose straightforward proof we omit.

LEMMA 2·5. *Assume N is a G-group. If the actions of G on $Z(N)$ and $N/Z(N)$ are both P-local, then G also acts P-locally on N.*

Proof of Proposition 2·4. Let H be the semidirect product of N and G. Consider the extension $1 \to N/Z(N) \to H/C_H(N) \to H/NC_H(N) \to 1$. Since N is P-local, $N/Z(N)$ is also P-local. Furthermore, $H/NC_H(N)$ is a P-group. Thus $H/C_H(N)$ is P-local by Lemma 2·2. Therefore, the action of H on $N/Z(N)$ is P-local, which means that G acts P-locally on $N/Z(N)$. Moreover, the action of G on $Z(N)$ is P-local by Lemma 2·3. An appeal to Lemma 2·5, then, completes the proof.

We now turn to P-local actions on finite groups.

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PROPOSITION 2·6. *Assume N is a finite G-group. Then the action of G on N is P-local if and only if the following two conditions hold*:

(i) *N is a P-group*;

(ii) *the image of G in* Aut(N) *is a P-group.*

Proof.

($\Longrightarrow$) Statement (i) follows from the fact that N is a P-local group. To show (ii), let $g \in G$ and n a P'-number such that $g^n \cdot a = a$ for all $a \in N$. If $a \in N$ and $b = a[g \cdot (a^{-1})]$, then $b(g \cdot b) \cdots (g^{n-1} \cdot b) = 1$, which implies that $b = 1$. Hence we have that $g \cdot a = a$ for all $a \in N$. Therefore, statement (ii) holds.

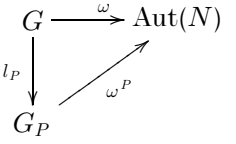
($\Longleftarrow$) This follows immediately from Lemma 2·3.

The above proposition provides us with the following property.

COROLLARY 2·7. *If N is a finite group on which G acts P-locally, then G acts P-locally on every G-subgroup and every G-quotient group of N.*

Next we establish that any P-local action of G on a finite group N induces an action of G_P on N.

LEMMA 2·8. *Assume G acts P-locally on a finite group N. Let $\omega: G \to \mathrm{Aut}(N)$ be the homomorphism arising from the action of G on N. Then there exists a unique map $\omega^P: G_P \to \mathrm{Aut}(N)$ such that the diagram*



commutes. Moreover, Im ω^P = Im ω.

Proof. By Proposition 2·6, Im ω is a P-group. An appeal to the "existence part"of the universal property of P-localization provides us with a map ω^P that makes the diagram commute and whose image is the same as Im ω. The uniqueness of ω^P follows from the "uniqueness part" of the universal property of P-localization and the fact that any finite homomorphic image of a P-local group must be a P-group.

Given a finite G-group N, we now consider the maximal G-quotient of N on which G acts P-locally.

Definition. Assume that N is a finite G-group. We define $\Gamma^G_{P'}N$ to be the subgroup of N generated by all the elements of the form $(g \cdot a)a^{-1}$ such that $a \in N$, $g \in G$, and the order of the image of g in Aut(N) is a P'-number. Let $T^G_{P'}N$ be the subgroup of N generated by $\Gamma^G_{P'}N$ and the subgroup $T_{P'}N$.

It is easy to see that $\Gamma^G_{P'}N$ and $T^G_{P'}N$ are both G-subgroups of N. The G-subgroup $T^G_{P'}N$ possesses the following two properties.

LEMMA 2·9. *Assume N is a finite G-group. Then the G-subgroup $T^G_{P'}N$ is the minimal G-subgroup of N such that the action of G on $N/T^G_{P'}N$ is P-local.*

Proof. This follows readily from Proposition 2·6.

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LEMMA 2·10. *If N is a finite normal subgroup of G, then, for any group homomorphism $\phi: G \to H$ where H is P-local, $T^G_{P'}N$ is contained in* Ker ϕ.

Proof. That $T_{P'}N \subseteq \text{Ker } \phi$ is plain. To show that, in addition, $\Gamma^G_{P'}N \subseteq \text{Ker } \phi$, let $a \in N$ and $g \in G$ such that the order n of the image of g in Aut(N) is a P'-number. Then $[\phi(a)\phi(g^{-1})\phi(a^{-1})]^n = [\phi(g^{-1})]^n$, so that $\phi(a)\phi(g^{-1})\phi(a^{-1}) = \phi(g^{-1})$. Hence $(g \cdot a)a^{-1} \in \text{Ker } \phi$.

3. *The class* $\mathfrak{U}_P$

In this section we discuss a class of groups which will be referred to prominently in our main theorem (Theorem 4·8). We begin by defining this class; then we point out some examples of groups that are contained in the class.

Definition. The class $\mathfrak{U}_P$ is the class of all groups G such that, for any finite abelian group A on which G acts P-locally, $l_P: G \to G_P$ induces an isomorphism $H^n(G_P, A) \to H^n(G, A)$ for all $n \geqslant 0$.

Below we establish that both nilpotent groups and free groups are in $\mathfrak{U}_P$.

PROPOSITION 3·1. *All nilpotent groups are in* $\mathfrak{U}_P$.

Proof. Assume G is nilpotent and A is a finite abelian group on which G acts P-locally. Let $N = C_G(A)$ and $Q = G/N$. Then Q is a P-group by Proposition 2·6. Thus, by [**4**, corollary 2·5], we have an exact sequence $1 \to N_P \to G_P \to Q \to 1$. We will employ the Lyndon–Hochschild–Serre (LHS) cohomology spectral sequences associated with the group extensions $1 \to N \to G \to Q \to 1$ and $1 \to N_P \to G_P \to Q \to 1$. By [**12**, Satz 5·29], the map $H^q(N_P, A) \to H^q(N, A)$ induced by $l_P: G \to G_P$ is an isomorphism for all $q \geqslant 0$. Thus the map $H^p(Q, H^q(N_P, A)) \to H^p(Q, H^q(N, A))$ induced by $l_P: G \to G_P$ is an isomorphism for all $p,\ q \geqslant 0$. This means that the maps induced between all of the corresponding terms of the two spectral sequences are isomorphisms. Therefore, the map $H^n(G_P, A) \to H^n(G, A)$ is an isomorphism for all $n \geqslant 0$ by the convergence of the spectral sequences.

Remark. The above result is also proved by A. Reynol Filho in [**16**, theorem 3·5]. His proof is different from ours, although it, too, appeals to the LHS spectral sequence. In [**5**] Casacuberta and G. Peschke generalize Reynol's argument to show that if G is nilpotent, then, for any abelian group A on which G_p acts P-locally, $H^n(G_P, A) \cong H^n(G, A)$ for each $n \geqslant 0$.

PROPOSITION 3·2. *All free groups are in* $\mathfrak{U}_P$.

Proof. We can argue exactly as in the proof of Proposition 3·1; however, in lieu of [**12**, Satz 5·29], we invoke the following property of free groups, gleaned from [**7**, teorema 1·6·8] and [**7**, teorema 2·3·3].

LEMMA 3·3 (Garcia). *Let F be a free group. Assume A is a P-local abelian group upon which F_P acts trivially. Then the map $H^n(F_P, A) \to H^n(F, A)$ induced by $l_P: F \to F_P$ is an isomorphism for all $n \geqslant 0$.*

Next we show that $\mathfrak{U}_P$ is closed under certain types of extensions by P-local groups.

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PROPOSITION 3·4. *Assume* $1 \to N \to G \to Q \to 1$ *is a group extension with coupling* $\chi: Q \to \mathrm{Out}(N)$. *Then* G *is in* $\mathfrak{U}_P$ *if each of the following statements is true*:

(i) Q *is* P*-local*;

(ii) N *is in* $\mathfrak{U}_P$;

(iii) $\mathrm{Im}\chi$ *is a* P*-group.*

Proof. Form the *relative* P*-localization* (see [**11**]) $1 \to N_P \to G_{(P)} \to Q \to 1$. By Propositions 2·4 and 2·1, $G_{(P)}$ is P-local. Thus, by [**4**, proposition 1·3], $G_{(P)} \cong G_P$. The result now follows by a spectral sequence argument similar to that employed in the proof of Proposition 3·1.

In the following proposition, we show that $\mathfrak{U}_P$ is closed under direct products with FP_∞-groups in $\mathfrak{U}_P$. Note that G is of type FP_∞ if the integers admit a projective resolution as a trivial G-module that is finitely generated in every dimension. In view of Proposition 3·4 and [**3**, example 2·6], any finitely generated group that has either a nilpotent or a free subgroup whose index is a finite P-number is an example of a FP_∞-group that is also in $\mathfrak{U}_P$.

PROPOSITION 3·5. *If* G *is in* $\mathfrak{U}_P$ *and* H *is a* FP_∞*-group in* $\mathfrak{U}_P$, *then* $H \times G$ *is in* $\mathfrak{U}_P$.

Proof. First we remark that $(H \times G)_P = H_P \times G_P$. Now let A be a finite group on which $H \times G$ acts P-locally. To establish that $H^n(H_P \times G_P, A) \cong H^n(H \times G, A)$, we will employ the LHS cohomology spectral sequences associated with the extensions $1 \to H \to H \times G \to G \to 1$ and $1 \to H_P \to H_P \times G_P \to G_P \to 1$. Since H is in $\mathfrak{U}_P$, the map $H^q(H_P, A) \to H^q(H, A)$ induced by $l_P: H \to H_P$ is an isomorphism for every $q \geqslant 0$. Moreover, since G acts P-locally on A, the induced action of G on $H^q(H, A)$ is also P-local by Lemma 2·3. In addition, that H is of type FP_∞ guarantees that $H^q(H, A)$ is finite for all $q \geqslant 0$. Consequently, since G is in $\mathfrak{U}_P$, the map $H^p(G_P, H^q(H_P, A)) \to H^p(G, H^q(H, A))$ induced by $l_P: H \times G \to H_P \times G_P$ is an isomorphism for all p, $q \geqslant 0$. Hence all the maps induced by $l_P: H \times G \to H_P \times G_P$ between corresponding terms of the spectral sequences are isomorphisms. Therefore, the map $H^n(H_P \times G_P, A) \to H^n(H \times G, A)$ is an isomorphism for all $n \geqslant 0$.

Remark. The argument above may also be employed to show that the direct product of any group in $\mathfrak{U}_P$ with a group that is either free or nilpotent is in $\mathfrak{U}_P$.

We conclude with one more proposition which may be used to construct groups in $\mathfrak{U}_P$. The proof of this one, however, must be postponed until the next section (Proposition 4·9), as it requires Theorem 4·8.

PROPOSITION 3·6. *Let* $1 \to N \to G \to Q \to 1$ *be a group extension with coupling* $\chi: Q \to \mathrm{Out}(N)$. *Then* G *is in* $\mathfrak{U}_P$ *if each of the following conditions is satisfied*:

(i) Q *is in* $\mathfrak{U}_P$;

(ii) N *is a finite group in* $\mathfrak{U}_P$;

(iii) Im χ *is a* P*-group.*

4. *Group extensions with a finite kernel*

In this section we discuss the effect of P-localization on a group extension with finite kernel. Our principal result, Theorem 4·8, provides a complete description of the P-localization of such an extension when the quotient is in $\mathfrak{U}_P$.

We begin with two elementary lemmas on finite subgroups.

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LEMMA 4·1. *Let H be a finite subgroup of a P-local group G. If $g \in G$ such that $g^n \in N_G(H)$ for some P'-number n, then $g \in N_G(H)$.*

Proof. Since H is finite, there exists $m \in \mathbb{Z}$ such that $g^m h g^{-m} = h$ for all $h \in H$. Hence $(hgh^{-1})^m = g^m$ for all $h \in G$. This means that, for each $h \in H$, $(hgh^{-1})^k = g^k$, where k is the largest P-number dividing m. It follows that $g^k \in C_G(H) \subseteq N_G(H)$. Therefore, since n and k are relatively prime, $g \in N_G(H)$.

LEMMA 4·2. *Let N be a finite normal subgroup of G. Then the image of N under $l_P: G \to G_P$ is normal in G_P.*

Proof. As described in [**17**], we can view G_P as the direct limit of

$$G = G_1 \to G_2 \to G_3 \to \cdots,$$

where G_i is obtained from G_{i-1} by adjoining any missing nth roots for all P'-numbers n and dividing out all the elements of the form gh^{-1} such that $g^n = h^n$ in G_{i-1} for some P'-number n. Hence every element of G_i can be expressed as a product of elements which have P'-powers in the image of G_{i-1}. Therefore, by Lemma 4·1, if the image of N in G_P is normalized by the image of G_{i-1} in G_P, then the image of N must also be normalized by the image of G_i. It follows, then, by induction, that the image of N in G_P is normalized by the image of G_i for all i. Thus the image of N in G_P is normal.

Now we are prepared to consider the P-localization of a group extension with finite kernel.

THEOREM 4·3. *Assume $1 \to N \xrightarrow{\iota} G \xrightarrow{\epsilon} Q \to 1$ is a group extension in which N is finite. Then the sequence*

$$N_P \xrightarrow{\iota_P} G_P \xrightarrow{\epsilon_P} Q_P \longrightarrow 1$$

is exact. Moreover, $\Gamma^G_{P'} N_P \subseteq \operatorname{Ker} \iota_P$.

Proof. By Lemma 2·10, $\Gamma^G_{P'} N_P \subseteq \operatorname{Ker} \iota_P$. Let $\hat{N}$ be the image of $\iota_P: N_P \to G_P$. The kernel of the epimorphism $N \to \hat{N}$ induced by $l_P: G \to G_P$ is normal in G, which means that the action of G on N by conjugation induces an action of G on $\hat{N}$. Moreover, since $\hat{N}$ is a G-homomorphic image of the G-group $N/T^G_{P'}N$ with P-local action, G must also act P-locally on $\hat{N}$ by Corollary 2·7. Let ω be the homomorphism $G \to \operatorname{Aut}(\hat{N})$ arising from the action of G on $\hat{N}$. By Lemma 4·2, we have $\hat{N} \trianglelefteq G_P$, making $\hat{N}$ a G_P-group. Let ω^P be the homomorphism $G_P \to \operatorname{Aut}(\hat{N})$ that arises from the action of G_P on $\hat{N}$. The two maps ω and ω^P make the diagram

$$\begin{array}{ccc} G & \xrightarrow{\omega} & \operatorname{Aut}(\hat{N}) \\ {\scriptstyle l_P}\downarrow & \nearrow_{\omega^P} & \\ G_P & & \end{array}$$

commute. It follows from Lemma 2·8 that $\operatorname{Im} \omega = \operatorname{Im} \omega^P$. Therefore, the action of G_P on $\hat{N}$ is also P-local. Hence, by Proposition 2·1, $R = G_P/\hat{N}$ is a P-local group. Let $\phi: Q \to R$ be the map induced by $l_P: G \to G_P$. We claim that ϕ P-localizes Q, which will yield the desired result. To show this, let $\psi: Q \to S$ be a homomorphism such that S is P-local. The universal property of $l_P: G \to G_P$ provides a unique map $\theta: G_P \to S$

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such that $\theta l_P = \psi\epsilon$. The map θ, then, induces a unique homomorphism $\eta: R \to S$ such that $\eta\phi = \psi$. Thus ϕ P-localizes Q.

In the following proposition, we observe that the above result fails to hold if N is only assumed to be torsion rather than finite.

PROPOSITION 4·4. *Assume p is an odd prime. Let G be the wreath product of $\mathbb{Z}/p$ with $\mathbb{Z}$. If A is the normal factor in G, then the image of A under $l_{\{p\}}: G \to G_{\{p\}}$ is not normal in $G_{\{p\}}$.*

Proof. First we observe that A injects into $G_{\{p\}}$. To see this, for each $n \in \mathbb{N}$, let $G(n)$ be the wreath product of $\mathbb{Z}/p$ with $\mathbb{Z}/p^n$. Then $G(n)$ is a $\{p\}$-local homomorphic image of G. Moreover, for every nontrivial element of A, there is an n such the image of that element in $G(n)$ is nontrivial. Therefore, A injects into $G_{\{p\}}$.

Now we observe that the image of A in $G_{\{p\}}$ is not normal. If this image were normal, then there would have to be an element α of $\mathrm{Aut}(A)$ such that α^2 is the automorphism that shifts each element of A one coordinate to the right. However, by Lemma 4·5 below, such an automorphism α does not exist.

LEMMA 4·5. *Assume K is a field and V is a vector space over K with basis $\{u_i: i \in \mathbb{Z}\}$. Let σ be the automorphism of V that maps u_i to u_{i+1} for each integer i. Then there is no automorphism of V whose square is σ.*

Proof. Suppose $\alpha: V \to V$ is a linear transformation such that $\alpha^2 = \sigma$. Set $\alpha(u_0) = \sum_i a_i u_i$. Since α commutes with σ, we have that $\alpha(u_j) = \sum_i a_i u_{i+j}$ for each $j \in \mathbb{Z}$. Thus

$$u_1 = \alpha\left(\sum_i a_i u_i\right) = \sum_{i,j} a_i a_j u_{i+j}.$$

Hence $\sum_{i+j=k} a_i a_j = 0$ for every integer $k \neq 1$. Let n be the smallest integer such that $a_n \neq 0$. Then $a_n^2 = \sum_{i+j=2n} a_i a_j = 0$, which yields a contradiction.

We now show that, under the hypotheses of Theorem 4·3, $\Gamma_{P'}^G N_P = \mathrm{Ker}\, \iota_P$ if Q is in the class $\mathfrak{U}_P$. To prove this, we require the following properties of a group extension.

LEMMA 4·6. *Assume $1 \to N \to G \to Q \to 1$ is a group extension such that N is finite and the action of G on N is P-local. Let $\chi: Q \to \mathrm{Out}(N)$ be the coupling of this extension. Then the following statements hold.*

(i) *There exists a unique homomorphism $\chi^P: Q_P \to \mathrm{Out}(N)$ such that the diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\chi} & \mathrm{Out}(N) \\ {\scriptstyle l_P}\downarrow & \nearrow_{\chi^P} & \\ Q_P & & \end{array}$$

commutes. Moreover, $\mathrm{Im}\, \chi^P = \mathrm{Im}\, \chi$.

(ii) *If Q is in the class $\mathfrak{U}_P$, then there exists an extension of N by Q_P with coupling $\chi^P: Q_P \to \mathrm{Out(N)}$.*

(iii) *If Q is in the class $\mathfrak{U}_P$, then $l_P: Q \to Q_P$ induces a bijection $\mathcal{E}(Q_P, N, \chi^P) \to \mathcal{E}(Q, N, \chi)$.*

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Proof.

(i) This can be deduced similarly to Lemma 2·8.

(ii) Following S. Mac Lane's [**14**] theory of group extensions, we can associate an element ζ_P of $H^3(Q_P, Z(N))$ to the map $\chi^P: Q_P \to \mathrm{Out}(N)$ such that the existence of an extension of N by Q with coupling χ^P is equivalent to ζ_P being trivial. Since Q acts P-locally on $Z(N)$, the map $H^3(Q_P, Z(N)) \to H^3(Q, Z(N))$ induced by $l_P: Q \to Q_P$ is an isomorphism. Let ζ be the image of ζ_P under this map. Then $\zeta = 0$ because we have an extension of N by Q with coupling χ. Thus $\zeta_P = 0$, and the conclusion follows.

(iii) According to [**14**], the cohomology group $H^2(Q, Z(N))$ acts freely and transitively on $\mathcal{E}(Q, N, \chi)$, and $H^2(Q_P, Z(N))$ does the same on $\mathcal{E}(Q_P, N, \chi^P)$. Moreover, if the equivalence class $E_P \in \mathcal{E}(Q_P, N, \chi^P)$ is mapped to $E \in \mathcal{E}(Q, N, \chi)$ by the function $\mathcal{E}(Q_P, N, \chi^P) \to \mathcal{E}(Q, N, \chi)$ induced by $l_P: Q \to Q_P$, then, for any $\xi_P \in H^2(Q_P, Z(N))$, $\xi_P \cdot E_P$ is mapped to $\xi \cdot E$, where ξ is the image of ξ_P in $H^2(Q, Z(N))$ under the map induced by $l_P: Q \to Q_P$. However, the homomorphism $H^2(Q_P, Z(N)) \to H^2(Q, Z(N))$ is an isomorphism. Therefore, the function $\mathcal{E}(Q_P, N, \chi^P) \to \mathcal{E}(Q, N, \chi)$ must be a bijection.

The following consequence of Lemma 4·6(i) is a useful tool in calculating G_P for G as in Theorem 4·3, assuming N_P, Q_P and $\mathrm{Ker}\ \iota_P$ are known.

LEMMA 4·7. *Let G be an extension of the finite group N by the group Q. Assume $\hat{N}$ is the image of N under $l_P: G \to G_P$, and let ξ be the epimorphism $N \to \hat{N}$ induced by $l_P: G \to G_P$. If H is an extension of $\hat{N}$ by Q_P such that there is a homomorphism $\phi: G \to H$ that makes the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle l_P} & & \\ 1 & \longrightarrow & \hat{N} & \longrightarrow & H & \longrightarrow & Q_P & \longrightarrow & 1 \end{array}$$

commute, then ϕ P-localizes G.

Proof. The action of G on N induces an action of G on $\hat{N}$. Moreover, since $\Gamma^G_{P'}N \subseteq \mathrm{Ker}\ \xi$, the action of G on $\hat{N}$ must be P-local in view of Corollary 2·7. This action induces a homomorphism $\hat{\chi}: Q \to \mathrm{Out}(\hat{N})$. Let $\chi^P: Q_P \to \mathrm{Out}(\hat{N})$ be the coupling of the extension $1 \to \hat{N} \to H \to Q_P \to 1$. By Lemma 4·6(i), $\mathrm{Im}\ \hat{\chi} = \mathrm{Im}\ \chi^P$. Hence the action of H on $\hat{N}$ by conjugation is P-local, so that H is a P-local group by Proposition 2·1. Applying the universal property of P-localization to the map $\phi: G \to H$, we obtain a map $\psi: G_P \to H$ such that $\psi l_P = \phi$. Then ψ induces the identity map on both $\hat{N}$ and Q_P. Thus ψ is an isomorphism, which means that ϕ P-localizes G.

Armed with the preceding two lemmas, we are ready to prove our main theorem.

THEOREM 4·8. *Assume $1 \to N \xrightarrow{\iota} G \xrightarrow{\epsilon} Q \to 1$ is a group extension in which N is finite and Q is in the class $\mathfrak{U}_P$. Then the sequence*

$$1 \longrightarrow \Gamma^G_{P'}N_P \xrightarrow{\subseteq} N_P \xrightarrow{\iota_P} G_P \xrightarrow{\epsilon_P} Q_P \longrightarrow 1 \tag{4·1}$$

is exact.

Proof. Set $\bar{N} = N/T^G_{P'}N \cong N_P/\Gamma^G_{P'}N_P$ and $\bar{G} = G/T^G_{P'}N$. Let $\hat{N}$ be the image of N under $l_P: G \to G_P$. The couplings of the extensions $1 \to \bar{N} \to \bar{G} \to Q \to 1$ and

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$1 \to \hat{N} \to G_P \to Q_P \to 1$ will be denoted by $\bar{\chi}$ and $\hat{\chi}^P$, respectively. Let $\hat{\chi}\colon Q \to \mathrm{Out}(\hat{N})$ be the composition of $l_P\colon Q \to Q_P$ with $\hat{\chi}^P$. The map $l_P\colon Q \to Q_P$ induces a function $\mathcal{E}(Q_P, \hat{N}, \hat{\chi}^P) \to \mathcal{E}(Q, \hat{N}, \hat{\chi})$. The kernel of the map $\xi\colon \bar{N} \to \hat{N}$ induced by $l_P\colon G \to G_P$ is a normal subgroup of every extension of $\bar{N}$ by Q with coupling $\bar{\chi}$. Hence we have a function $\mathcal{E}(Q, \bar{N}, \bar{\chi}) \to \mathcal{E}(Q, \hat{N}, \hat{\chi})$ that maps the class of $1 \to \bar{N} \to \bar{G} \to Q \to 1$ to the image of the class of $1 \to \hat{N} \to G_P \to Q_P \to 1$ in $\mathcal{E}(Q, \hat{N}, \hat{\chi})$ (see [**13**, proposition 3·3]). By Lemma 4·6(i), there is a homomorphism $\bar{\chi}^P\colon Q_P \to \mathrm{Out}(\bar{N})$ such that $\bar{\chi}^P l_P = \bar{\chi}$. Furthermore, since $\mathrm{Im}\ \bar{\chi}^P = \mathrm{Im}\ \bar{\chi}$, the kernel of $\xi\colon \bar{N} \to \hat{N}$ is a normal subgroup of every extension of $\bar{N}$ by Q_P with coupling $\bar{\chi}^P$. This means that every extension of $\bar{N}$ by Q_P with coupling $\bar{\chi}^P$ yields an extension of $\hat{N}$ by Q_P; moreover, by the "uniqueness part" of Lemma 4·6(i), the coupling of this extension of $\hat{N}$ by Q_P must be $\hat{\chi}^P$. Hence there is a function $\mathcal{E}(Q_P, \bar{N}, \bar{\chi}^P) \to \mathcal{E}(Q_P, \hat{N}, \hat{\chi}^P)$.

The above discussion establishes that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(Q_P, \bar{N}, \bar{\chi}^P) & \longrightarrow & \mathcal{E}(Q_P, \hat{N}, \hat{\chi}^P) \\ \downarrow & & \downarrow \\ \mathcal{E}(Q, \bar{N}, \bar{\chi}) & \longrightarrow & \mathcal{E}(Q, \hat{N}, \hat{\chi}). \end{array} \tag{4·2}$$

By Lemma 4·6(iii), the vertical functions in (4·2) are bijections. Let $1 \to \bar{N} \to G_* \to Q_P \to 1$ be a representative of the inverse image of the class of $1 \to \bar{N} \to \bar{G} \to Q \to 1$ with respect to the first vertical function in (4·2). Then the image of the class of $1 \to \bar{N} \to G_* \to Q_P \to 1$ in $\mathcal{E}(Q_P, \hat{N}, \hat{\chi}^P)$ and the class of $1 \to \hat{N} \to G_P \to Q_P \to 1$ both have the same image in $\mathcal{E}(Q, \hat{N}, \hat{\chi})$. Therefore, the image of the class of $1 \to \bar{N} \to G_* \to Q_P \to 1$ and the class of $1 \to \hat{N} \to G_P \to Q_P \to 1$ must be identical. This yields the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 1 & \longrightarrow & \bar{N} & \longrightarrow & G_* & \longrightarrow & Q_P & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 1 & \longrightarrow & \hat{N} & \longrightarrow & G_P & \longrightarrow & Q_P & \longrightarrow & 1. \end{array}$$

By Lemma 4·7, the homomorphism $\psi\phi\colon G \to G_P$ P-localizes G. Moreover, since G_* acts P-locally on $\bar{N}$, G_* is a P-local group by Proposition 2·1. Hence, applying the universal property of P-localization, we obtain a map $\theta\colon G_P \to G_*$ such that $\theta\psi\phi = \phi$. Then $\psi\theta$ is the identity map $G_P \to G_P$, and θ maps $\hat{N}$ isomorphically onto $\bar{N}$. It follows that ψ is an isomorphism, which yields the desired result.

Theorem 4·8 allows us to determine some additional groups that belong to the class $\mathfrak{U}_P$.

Proposition 4·9. *Let* $1 \to N \to G \to Q \to 1$ *be a group extension with coupling* $\chi\colon Q \to \mathrm{Out}(N)$. *Then* G *is in* $\mathfrak{U}_P$ *if all of the following conditions are satisfied*:

(i) Q *is in* $\mathfrak{U}_P$;

(ii) N *is a finite group in* $\mathfrak{U}_P$;

(iii) Im χ *is a* P*-group.*

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Proof. By Theorem 4·8, we have the exact sequence $1 \to N_P \to G_P \to Q_P \to 1$. Assume A is a finite G-module such that the action of G on A is P-local. Consider the LHS cohomology spectral sequences associated with the extensions $1 \to N \to G \to Q \to 1$ and $1 \to N_P \to G_P \to Q_P \to 1$. Since N is in $\mathfrak{U}_P$, the map $H^q(N_P, A) \to H^q(N, A)$ induced by $l_P: G \to G_P$ is an isomorphism for all $q \geqslant 0$. Moreover, for each $g \in G$, there exists a P-number n such that $g^n \in C_G(N_P)$ and $g^n \cdot a = a$ for all $a \in A$. This means that the action of Q on $H^q(N_P, A)$ is P-local for each $q \geqslant 0$. Therefore, since Q is in $\mathfrak{U}_P$, the map $H^p(Q_P, H^q(N_P, A)) \to H^p(Q, H^q(N, A))$ induced by $l_P: G \to G_P$ is an isomorphism for all p, $q \geqslant 0$. Hence all the maps induced by $l_P: G \to G_P$ between corresponding terms of the spectral sequences are isomorphisms. Consequently, the map $H^n(G_P, A) \to H^n(G, A)$ is an isomorphism for all $n \geqslant 0$.

If we dispense with the hypothesis concerning Q in Theorem 4·8, then the sequence (4·1) may not be exact; that is, $\Gamma^G_{P'} N_P$ may not equal Ker ι_P. This is illustrated by the following example.

Example. It is shown in [**19**, p. 332] that $H_2(A_4) = \mathbb{Z}/2$. Hence, by the universal coefficient theorem, $H^2(A_4, \mathbb{Z}/2) \neq 0$, where A_4 is assumed to act on $\mathbb{Z}/2$ trivially. This means there is a nonsplit central extension G of $\mathbb{Z}/2$ by A_4. If we let $P = \{2\}$, then $\Gamma^G_{P'}(\mathbb{Z}/2) = 1$ in this extension. Since A_4 localized at P is trivial, we have, by Theorem 4·3, the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & G & \longrightarrow & A_4 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \mathbb{Z}/2 & \longrightarrow & G_P & \longrightarrow & 1 & & \end{array} \tag{4·3}$$

with exact rows. Since the first row in (4·3) does not split, the map $\mathbb{Z}/2 \to G_P$ in the second row cannot be injective. Therefore, for the extension $1 \to \mathbb{Z}/2 \to G \to A_4 \to 1$, the sequence (4·1) is not exact.

Remark. Another such example is provided by [**4**, example 2·7].

Our result in Theorem 4·8 may be employed– especially in conjunction with Lemma 4·7– to calculate the P-localization of a group that fits into an extension like that in the theorem. Below we provide a simple example of such a calculation.

Example. Assume p and q are positive primes such that q is a divisor of $p-1$. Let k be an integer relatively prime to p whose order modulo p^n is q. Assume, further, that l and m are also integers relatively prime to p. Let G be the group with the presentation

$$\begin{aligned} G = \langle a, b, x, y: a^{p^n} = b^q = 1; bab^{-1} = a^k; xax^{-1} = a^l; \\ yay^{-1} = a^m; xbx^{-1} = yby^{-1} = b; [x, y] = a \rangle. \end{aligned}$$

Let N be the subgroup of G generated by by a and b. Then N is a semidirect product of $\mathbb{Z}/p^n$ and $\mathbb{Z}/q$, and G is an extension of N by $\mathbb{Z} \oplus \mathbb{Z}$. We will localize G at P. If $q \notin P$, then $T^G_{P'} N = N$, which means that $G_P = \mathbb{Z}_P \oplus \mathbb{Z}_P$. If $q \in P$ and either $p \notin P$, the order of l modulo p is not a P-number, or the order of m modulo p is not a P-number, then $T^G_{P'} N = \langle a \rangle$, implying that $G_P = \mathbb{Z}/q \oplus \mathbb{Z}_P \oplus \mathbb{Z}_P$. Finally, if p and q are both in P and the orders of both l and m modulo p are P-numbers, then $T^G_{P'} N = 1$, which means that G_P fits into an extension $1 \to N \to G_P \to \mathbb{Z}_P \oplus \mathbb{Z}_P \to 1$.

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5. *Finite-by-nilpotent groups*

The case of Theorem 4·8 for Q nilpotent is of special interest. First of all, it implies that the class of finite-by-nilpotent groups is closed under P-localization. Also, we may use this special case of Theorem 4·8 to generalize some of the familiar properties of P-localization for nilpotent groups to this larger class. This is accomplished in the following two propositions: the first describes the kernel of the P-localizing map of a finite-by-nilpotent group, and the second extends Hilton's [**9**] characterization of the maps of nilpotent groups that P-localize (see Section 1) to finite-by-nilpotent groups.

PROPOSITION 5·1. *Let $1 \to N \to G \to Q \to 1$ be a group extension in which N is finite and Q is nilpotent. Then the kernel of $l_P: G \to G_P$ is given by*

$$\operatorname{Ker} l_P = \{g \in G : g^n \in T^G_{P'}N \quad \text{for some } P'\text{-number } n\}.$$

Proof. This follows from Theorem 4·8 together with the fact that the kernel of $l_P: Q \to Q_P$ is the set of all elements of Q whose order is a P'-number.

PROPOSITION 5·2. *Let $1 \to N \to G \to Q \to 1$ be a group extension in which N is finite and Q is nilpotent. A homomorphism $\phi: G \to H$ P-localizes G if and only if the following three statements hold*:

(i) *H is a P-local group*;

(ii) *for every $g \in \operatorname{Ker} \phi$, there exists a P'-number n such that g^n is an element of $T^G_{P'}N$*;

(iii) *for every $h \in H$, there exists a P'-number n such that $h^n \in \operatorname{Im} \phi$.*

Proof.

($\Longrightarrow$) This direction follows immediately from Theorem 4·8 and Hilton's characterization of the nilpotent group homomorphisms that P-localize.

($\Longleftarrow$) Let $\bar{N} = \phi(N) \cong N/T^G_{P'}N$. First we observe that $\bar{N}$ is normal in H– this follows from (iii) and Lemma 4·1. Now, by Hilton's result, the map $Q \to H/\bar{N}$ induced by ϕ P-localizes. Therefore, ϕ P-localizes by Theorem 4·8 and Lemma 4·7.

By a result of P. Hall [**8**], the class of finite-by-nilpotent groups is contained in the class of nilpotent-by-finite groups. Moreover, P-localization in the latter class of groups is investigated in [**4**, **6**]. In particular, in [**6**, theorem 4·3] it is shown that, if G is nilpotent-by-finite, the kernel of $l_P: G \to G_P$ is the minimal normal subgroup $U_{P'}G$ of G such that the nth power function $x \mapsto x^n$ from $G/U_{P'}G$ to $G/U_{P'}G$ is injective for every P'-number n. We conclude the article by observing that this result may be used to prove the special case of Theorem 4·8 where Q is a nilpotent group. In contrast to the proof given above, this new proof does not make any use of cohomology.

Proof. Set $\bar{N} = N/T^G_{P'}N$ and $\bar{G} = G/T^G_{P'}N$. In view of Theorem 4·3, the conclusion will follow if we show that the kernel, $U_{P'}\bar{G}$, of $l_P: \bar{G} \to \bar{G}_P$ intersects $\bar{N}$ trivially. Let

$$\bar{K} = \{x \in \bar{G} : x^n = 1 \quad \text{for some } P'\text{-number } n\}.$$

Then $\bar{K} \subseteq U_{P'}\bar{G}$. Also, since $\bar{G}$ acts on $\bar{N}$ P-locally, $\bar{K} \subseteq C_{\bar{G}}(\bar{N})$ and $\bar{K} \cap \bar{N} = 1$. We will establish that $\bar{K} = U_{P'}\bar{G}$, which will prove the result. First we verify that K is a subgroup of $\bar{G}$. To show this, let x and y be elements of $\bar{K}$. Then $x^n = y^n = 1$ for some P'-number n. By [**12**, lemma 3·4], $(xy)^{n^i} \equiv 1 \bmod \Gamma^i\langle x, y\rangle$ for each $i \in \mathbb{N}$,

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where $\Gamma^i\langle x,y\rangle$ is the ith term of the lower central series of $\langle x,y\rangle$ (in which Γ^1 is the commutator subgroup). Moreover, if c is the nilpotency class of Q, then $\Gamma^{c+1}\langle x,y\rangle = 1$. Thus $(xy)^{n^{c+1}} = 1$. Therefore, $\bar{K}$ is indeed a subgroup of $\bar{G}$.

Next we show that $\bar{K} = U_{P'}\bar{G}$. First we observe that $\bar{G}/\bar{K}\bar{N}$ does not contain any nontrivial P'-torsion. To see this, assume $g \in \bar{G}$ such that $g^n \in \bar{N}$ for some P'-number n. Then $g^{nm} = 1$, where m is a P-number. Let λ and μ be integers such that $\lambda m + \mu n = 1$. We have $g = g^{m\lambda}g^{n\mu}$, which means that $g \in \bar{K}\bar{N}$. Thus $\bar{G}/\bar{K}\bar{N}$ has no nontrivial P'-torsion. Now consider the extension

$$1 \longrightarrow \bar{N} \longrightarrow \bar{G}/\bar{K} \longrightarrow \bar{G}/\bar{K}\bar{N} \longrightarrow 1.$$

Since $\bar{G}/\bar{K}\bar{N}$ is nilpotent and contains no nontrivial P'-torsion, the nth power function $\bar{G}/\bar{K}\bar{N} \to \bar{G}/\bar{K}\bar{N}$ is injective for any P'-number n. Hence, by a modified version of Proposition 2·1, the nth power function from $\bar{G}/\bar{K}$ to $\bar{G}/\bar{K}$ is injective for every P'-number n. Therefore, $\bar{K} = U_{P'}\bar{G}$, completing the proof.

Remark. The case of Theorem 4·8 for Q free may also be proved without cohomology by exploiting the fact that, in that case, the extension splits. However, no proofs that avoid cohomology are known for the groups in $\mathfrak{U}_P$ described in Propositions 3·4 and 3·5.

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