COMPLEMENTARY METHOD FOR DERIVING CONCENTRATION OF DIFFUSING SUBSTANCE IN A MEDIUM FOR MULTI-DIMENSIONAL DIFFUSION

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ABSTRACT

Concentration of a diffusing substance in a medium was derived in various cases of uni-dimensional diffusion, including in a semi-infinite medium and a plate-shaped medium. Multi-dimensional diffusion involves boundary conditions in each coordinate direction. The algorithm dealing with uni-dimensional case becomes very complicated in multi-dimensional cases. This study proposes an algorithm, which is called the complementary method, that combines complementary functions of the normalized solution in uni-dimensional diffusion case by multiplication to solve those in various multi-dimensional diffusion cases with dramatically simplified mathematics. Besides, the complementary method is used to solve various kinds of boundary conditions for multi-dimensional diffusion.

Keywords: Complementary method, Diffusion, Multi-dimensional.

1. INTRODUCTION

Fick recognized that the diffusion of substance in a medium has an obvious analogy with the heat conduction and was the first to solve quantitatively the problem of diffusion using the governing equation for heat conduction that was derived by Fourier [1]. Fick's laws have been used to stand for the mechanism of various diffusion cases. Fick's first law [1] is,

$$F = -D\partial c / \partial x \tag{1a}$$

where F is the rate of transfer per unit area of section, c is the concentration of the diffusing substance in the medium, x is the space coordinate measured normal to the section, and D is the diffusivity, which is a positive constant. Fick's second law [1], which is commonly used, is,

$$\frac{\partial c}{\partial t} = D\nabla^2 c \tag{1b}$$

where t is the duration of diffusion. In spatial Cartesian coordinate system, Eq. (1b) is,

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{2a}$$

where x, y and z are the three coordinates and c is a function of t, x, y and z. For two-dimensional diffusion on the x-y plane, Eq. (2a) becomes

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) \tag{2b}$$

For uni-dimensional diffusion in the x direction, Eq. (2a) becomes

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \tag{2c}$$

Solutions for c in many cases have been derived [1-6]. They are functions of both diffusion duration and location. In 1959, Carslaw and Jaeger [1] applied Laplace transform to solve the concentration of a diffusing substance that penetrates the surface of a semi-infinite medium and is transported inward. By assuming the solution to be a uni-formed function, Carslaw and Jaeger formulated the solution as an as-

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ymptotic function. However, the solution indicates that the concentration in the medium infinitely approaches zero as the studied location moves inward but can never be zero regardless of diffusion duration. The solution indicates no diffusion front, and calculating the diffusion depth is impossible. However, transporting molecules of a diffusing substance in a medium is time consuming. The molecules of a diffusing substance do not appear deep inside the medium immediately after diffusion begins. The Carslaw and Jaeger's solution is inapplicable in practice even if it is mathematically correct.

In the same year, Carslaw and Jaeger [1] adopted a technique developed by Neumann for calculating heat conduction to solve the concentration of a diffusing substance with an advancing boundary in a semi-infinite medium. The solution obtained by their model, which they called the advancing model, is a bi-formed and continuous but not a smooth function. The solution covers a contaminated zone and an uncontaminated zone. The solution is in a series form in the contaminated zone, which is close to the surface, while the solution is zero in the uncontaminated zone, which is deeper than the contaminated zone. The bi-formed solution clearly indicates a diffusion front and makes calculating the diffusion depth possible.

Since 1959 Carslaw and Jaeger solved the diffusion case of a semi-infinite medium with the advancing model, no researchers have published any solution based on the advancing model for other cases in the past half century. Besides, no researchers have deriving the concentration of a diffusing substance in a medium for multi-dimensional diffusion case. Consequently, this study proposes the complementary method to solve various kinds of boundary conditions for multi-dimensional diffusion with the advancing model.

2. SOLUTIONS IN UNI-DIMENSIONAL DIFFUSION CASES

The simplest case of uni-dimensional diffusion is that in a semi-infinite medium. In such a case, the medium occupies the region $0 \le x$ (Fig. 1), and the diffusing substance penetrates the surface of the medium and travels toward the deep of the medium.

The initial condition is that the initial concentration of the diffusing substance in the medium is c_i. The final condition is that the diffusing substance saturates the medium and the saturated concentration is c_∞, which depends on the environmental conditions of the space surrounding the medium and the properties of both the diffusing substance and the medium. The assumed boundary condition is that when the diffusing substance comes into contact with the medium at the boundary, the boundary becomes instantly saturated and remains so throughout the process of diffusion. All cases discussed herein have the same initial, final and boundary conditions.

The problem of diffusion in a semi-infinite medium is a typical uni-dimensional diffusion problem. The normalized solution for the concentration of a diffusing substance in a medium is in the same form as that derived by Carslaw and Jaeger for heat conduction [1,2].

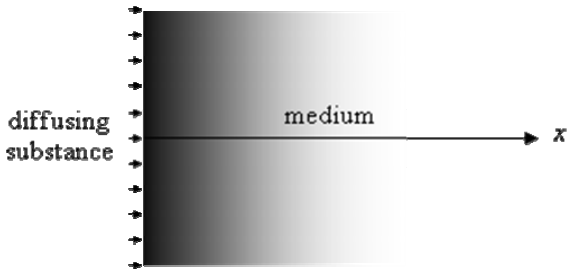


Fig. 1 Diffusion in a medium occupying the region $0 \le x$

$$C(t; x) = \frac{c(t; x) - c_i}{c_\infty - c_i} = 1 - \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2\sqrt{Dt}} \right)^{2m+1}}{m!(2m+1)} \quad (3)$$

where $C(t; x)$ is the normalized $c(t; x)$, and $c(t; x)$ depends on t and x. The first part of $C(t; x)$ is one which is the particular solution of Eq. (2c), and the second part is the complementary function of $C(t; x)$, which is

$$C_{c-s}(t; x) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2\sqrt{Dt}} \right)^{2m+1}}{m!(2m+1)} \quad (4a)$$

Interestingly, the particular solution, $C(t; x)$, the complementary function of $C(t; x)$ and $c(t; x)$ all satisfy Eqs. (2c), (2b) and (2a) simultaneously.

$C(t; x)$ and $c(t; x)$ in Eq. (3) respectively satisfy initial conditions

$$C(0; x) = 0 \quad \text{and} \quad c(0; x) = c_i \,, \quad (5)$$

final conditions

$$C(\infty; x) = 1 \ \text{and} \ c(\infty; x) = c_\infty \,, \quad (6)$$

boundary conditions

$$C(t; 0) = 1 \ \text{and} \ c(t; 0) = c_\infty \quad (7)$$

and far field conditions

$$C(t; \infty) = 0 \ \text{and} \ c(t; \infty) = c_i \quad (8)$$

As a result,

$$C_{c-s}(0; x) = 1 \quad (9)$$

$$C_{c-s}(\infty; x) = 0 \quad (10)$$

$$C_{c-s}(t; 0) = 0 \quad (11)$$

$$C_{c-s}(t; \infty) = 1 \quad (12)$$

By the same process, if the semi-infinite medium occupies the region $0 \le y$, then $C(t; x)$ in Eq. (3) becomes $C(t; y)$, and Eq. (4a) becomes

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$$C_{c-s}(t;\, y) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\dfrac{y}{2\sqrt{Dt}} \right)^{2m+1}}{m!(2m+1)} \tag{4b}$$

If the semi-infinite medium occupies the region $0 \le z$, then $C(t;\, x)$ in Eq. (3) becomes $C(t;\, z)$, and Eq. (4a) becomes

$$C_{c-s}(t;\, z) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\dfrac{z}{2\sqrt{Dt}} \right)^{2m+1}}{m!(2m+1)} \tag{4c}$$

Carslaw and Jaeger adopted Neumann's technique and derived $C(t;\, x)$ by taking into account the advancing boundary condition for the semi-infinite medium that occupies the region $0 \le x$ [1-5].

$$C(t;\, x) = \frac{c(t;\, x) - c_i}{c_\infty - c_i} = 1 - \frac{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \Phi_{ms}(t;\, x) \nu^{2m+1}}{m!(2m+1)}}{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \nu^{2m+1}}{m!(2m+1)}}, \tag{13}$$

where ν is Neumann's constant, and the complementary function of $C(t;\, x)$ is

$$C_{ca-s}(t;\, x) = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \Phi_{ms}(t;\, x) \nu^{2m+1}}{m!(2m+1)}}{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \nu^{2m+1}}{m!(2m+1)}},$$

$$\Phi_{ms}(t;\, x) = \min\left(\frac{x}{2\nu\sqrt{Dt}}, 1 \right)^{2m+1} \tag{14a}$$

Like $C(t;\, x)$ and $c(t;\, x)$ in Eq. (3), these functions in Eq. (13) satisfy Eqs. (5) ~ (8). Like $C_{c\text{-}s}(t;\, x)$ in Eq. (4a), $C_{ca\text{-}s}(t;\, x)$ in Eq. (14a) satisfies Eqs. (9) ~ (12).

By the same process, if the semi-infinite medium occupies the region $0 \le y$, then $C(t;\, x)$ in Eq. (13) becomes $C(t;\, y)$, and Eq. (14a) becomes

$$C_{ca-s}(t;\, y) = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \Phi_{ms}(t;\, y) \nu^{2m+1}}{m!(2m+1)}}{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \nu^{2m+1}}{m!(2m+1)}},$$

$$\Phi_{ms}(t;\, y) = \min\left(\frac{y}{2\nu\sqrt{Dt}}, 1 \right)^{2m+1} \tag{14b}$$

If the semi-infinite medium occupies the region $0 \le z$, then $C(t;\, x)$ in Eq. (13) becomes $C(t;\, z)$, and Eq. (14a) becomes

$$C_{ca-s}(t;\, z) = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \Phi_{ms}(t;\, z) \nu^{2m+1}}{m!(2m+1)}}{\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \nu^{2m+1}}{m!(2m+1)}},$$

$$\Phi_{ms}(t;\, z) = \min\left(\frac{z}{2\nu\sqrt{Dt}}, 1 \right)^{2m+1} \tag{14c}$$

For the case of a plate-shaped medium, the medium occupies the region $-h_x / 2 \le x \le h_x / 2$ (Fig. 2), and the diffusing substance penetrates both boundaries and travels toward the center, where h_x is the thickness of the plate-shaped medium. $C(t;\, x)$ is in the same form as that derived by Carslaw and Jaeger for the heat conduction problem [1,2], which is

$$C(t;\, x) = \frac{c(t;\, x) - c_i}{c_\infty - c_i} = 1 - 4 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\pi} (-1)^{\frac{n-1}{2}} e^{-\left(\frac{n\pi}{h_x}\right)^2 Dt} \cos\left(\frac{n\pi}{h_x} x \right) \tag{15}$$

The complementary function becomes

$$C_{c-p}(t;\, x) = 4 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\pi} (-1)^{\frac{n-1}{2}} e^{-\left(\frac{n\pi}{h_x}\right)^2 Dt} \cos\left(\frac{n\pi}{h_x} x \right) \tag{16a}$$

$C(t;\, x)$ and $c(t;\, x)$ in Eq. (15) satisfy Eqs. (5) and (6) and the boundary conditions

$$C(t; \pm h_x / 2) = 1 \quad \text{and} \quad c(t; \pm h_x / 2) = c_\infty \,, \tag{17}$$

meaning that $C_{c\text{-}p}(t;\, x)$ satisfies Eqs. (9) and (10) and

$$C_{c-p}(t; \pm h_x / 2) = 0 \tag{18}$$

By the same process, if the medium occupies the region $-h_y / 2 \le y \le h_y / 2$, $C(t;\, x)$ in Eq. (15) becomes $C(t;\, y)$, and then Eq. (16a) becomes

$$C_{c-p}(t;\, y) = 4 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\pi} (-1)^{\frac{n-1}{2}} e^{-\left(\frac{n\pi}{h_y}\right)^2 Dt} \cos\left(\frac{n\pi}{h_y} y \right) \tag{16b}$$

If the medium occupies the region $-h_z / 2 \le z \le h_z / 2$, $C(t;\, x)$ in Eq. (15) becomes $C(t;\, z)$, and then Eq. (16a) becomes

$$C_{c-p}(t;\, z) = 4 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\pi} (-1)^{\frac{n-1}{2}} e^{-\left(\frac{n\pi}{h_z}\right)^2 Dt} \cos\left(\frac{n\pi}{h_z} z \right) \tag{16c}$$

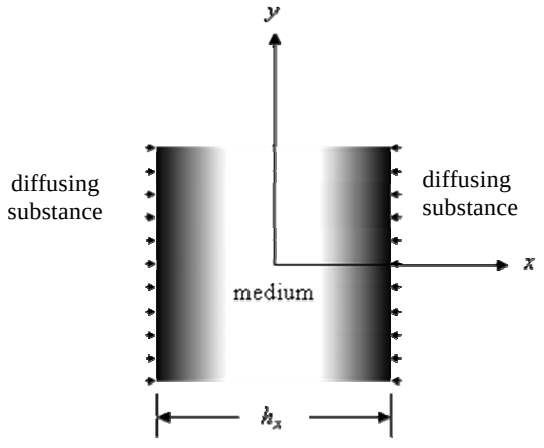


Fig. 2 Diffusion in a medium occupying the region $-h_x / 2 \le x \le h_x / 2$

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Chang *et al*. [3], Wang [4], and Lin *et al*. [5] took into account the advancing boundary condition, modified Eq. (15) by applying TSAI technique and obtained

$$C(t;x)=\frac{c(t;x)-c_i}{c_\infty-c_i}=1-\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;x)\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}} \tag{19}$$

The complementary function becomes

$$C_{ca\text{-}p}(t;x)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;x)\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}} \tag{20a}$$

where

$$\Phi_{mp}(t;x)=\begin{cases}\min\left(\dfrac{\frac{h_x}{2}+x}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_x}{2}-x}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_x^2}{16\nu^2D}\\ (-2)\sum\limits_{n=1,3,5,\ldots}^{\infty}a_{mn}e^{-\left(\frac{n\pi}{h_x}\right)^2D\left(t-\frac{h_x^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_x}x\right), & t\ge\dfrac{h_x^2}{16\nu^2D}\end{cases} \tag{21a}$$

$$a_{mn}=(2m+1)!\sum_{r=0}^{m}\left[\frac{(-1)^{r+1}}{(2m-2r)!(n\pi/2)^{2r+2}}\right] \tag{22}$$

Like $C(t;x)$ and $c(t;x)$ in Eq. (15), these functions in Eq. (19) satisfy Eqs.(5), (6) and (17). Like $C_{c\text{-}p}(t;x)$ in Eq. (16a), $C_{ca\text{-}p}(t;x)$ in Eq. (20a) satisfies Eqs. (9), (10) and (18).

By the same process, if the medium occupies the region $-h_y/2\le y\le h_y/2$, $C(t;x)$ in Eq. (19) becomes $C(t;y)$, and then Eq. (20a) becomes

$$C_{ca\text{-}p}(t;y)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;y)\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}} \tag{20b}$$

where

$$\Phi_{mp}(t;y)=\begin{cases}\min\left(\dfrac{\frac{h_y}{2}+y}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_y}{2}-y}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_y^2}{16\nu^2D}\\ (-2)\sum\limits_{n=1,3,5,\ldots}^{\infty}a_{mn}e^{-\left(\frac{n\pi}{h_y}\right)^2D\left(t-\frac{h_y^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_y}y\right), & t\ge\dfrac{h_y^2}{16\nu^2D}\end{cases} \tag{21b}$$

If the medium occupies the region $-h_z/2\le z\le h_z/2$, $C(t;x)$ in Eq. (19) becomes $C(t;z)$, and then Eq. (20a) becomes

$$C_{ca\text{-}p}(t;z)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;z)\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}} \tag{20c}$$

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where

$$\Phi_{mp}(t;z)=\begin{cases}\min\left(\dfrac{\frac{h_z}{2}+z}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_z}{2}-z}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_z^2}{16\nu^2D}\\ (-2)\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}a_{mn}e^{-\left(\frac{n\pi}{h_z}\right)^2 D\left(t-\frac{h_z^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_z}z\right), & t\ge\dfrac{h_z^2}{16\nu^2D}\end{cases} \tag{21c}$$

Carslaw and Jaeger claimed that the solution to the multi-dimensional heat conduction problem can be expressed as a product of solutions to the uni-dimensional problem [2]. In spite of the fact that the heat conduction problem and diffusion problem have the same governing equation, this claim is ambiguous and is incorrect in diffusion problem. It is hard to tract when and where Carslaw and Jaeger published their algorithm. This study presents a novel process of deriving solutions in cases of multi-dimensional diffusion by multiplication from those in cases of uni-dimensional diffusion.

3. THEORY OF COMPLEMENTARY METHOD

Equations (4a) ~ (4c), (14a) ~ (14c), (16a) ~ (16c), (20a) ~ (20c) are complementary functions of normalized solutions for the concentration of the diffusing substance in a medium in 12 cases of uni-dimensional diffusion. When the diffusion problem is multi-dimensional, the method of separation of variable can be adopted to derive the solution, but the process, like that carried out by Lin *et al*. [5] is highly complicated.

Tsai and Lin presented a much simpler algorithm, called the complementary method, for deriving the concentration of a diffusing substance in a plane medium that occupies the region $0 \le x$ and $0 \le y$ (Fig. 3) [6]. The only rule of the complementary method is that the complementary function of the normalized solution to a multi-directional diffusion problem is the product of the complementary functions of the normalized solution to the uni-dimensional diffusion problem, according to the boundary conditions in each of the coordinate directions. Consequently, the complementary method is used to solve various kinds of boundary conditions for multi-dimensional diffusion.

For example, consider the case of plane diffusion in which the medium occupies the quadrant of a Cartesian coordinate system where $0 \le x$ and $0 \le y$. The diffusing substance penetrates its two boundaries and travels toward the deep of the medium. For a two-dimensional problem, Eq. (2a) becomes Eq. (2b), and the solution for the concentration is function of x and y. The boundary condition and far field condition in the x-direction are Eq. (7 and 8), and the complementary function of the solution is Eq. (4a). Similarly, the complementary function in y-direction is Eq. (4b).

Based on complementary method and the boundaries, the complementary function of the normalized solution of concentration of the diffusing substance in the medium is the product of $C_{c\text{-}s}(t; x)$ and $C_{c\text{-}s}(t; y)$, as indicated by Eqs. (4a) and (4b). Consequently,

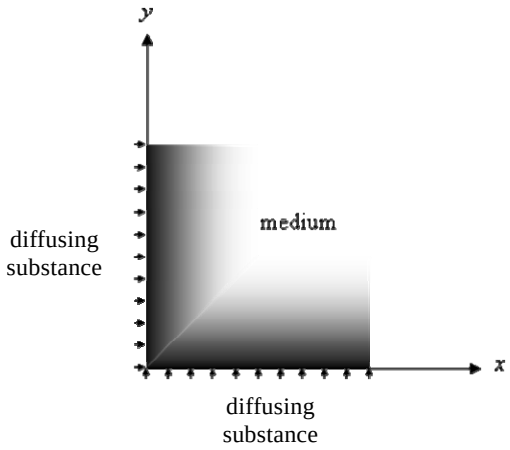


Fig. 3 Diffusion in a medium occupying the region $0 \le x$, and $0 \le y$

$$C(t; x, y)=\frac{c(t; x, y)-c_i}{c_\infty-c_i}=1-C_{c-s}(t; x)C_{c-s}(t; y) \tag{23}$$

where $C(t; x, y)$ is the normalized concentration of the diffusing substance in the medium as a function of t, x and y, and $c(t; x, y)$ is c as a function of t, x and y.

Since $C_{c\text{-}s}(t; x)$ vanishes at $x = 0$, and $C_{c\text{-}s}(t; y)$ vanishes at $y = 0$, their product must vanish on the boundaries to make $C(t; x, y) = 1$ and $c(t; x, y) = c_\infty$. Since $C_{c\text{-}s}(t; x)$ becomes one when x approaches ∞, and $C_{c\text{-}s}(t; y)$ becomes one when y approaches ∞, their product must be one when x approaches ∞ and y approaches ∞ to make $C(t; x, y) = 0$ and $c(t; x, y) = c_i$. Since $C_{c\text{-}s}(t; x)$ and $C_{c\text{-}s}(t; y)$ become one at $t = 0$, their product must be one initially to make $C(t; x, y) = 0$ and $c(t; x, y) = c_i$. Since $C_{c\text{-}s}(t; x)$ and $C_{c\text{-}s}(t; y)$ vanish when t approaches ∞, their product must also vanish finally to make $C(t; x, y) = 1$ and $c(t; x, y) = c_\infty$.

Equation (23) satisfies the governing equation and all of the conditions, so Eq. (23) is the normalized conventional solution to Eq. (2b). Because of the advancing model, $C_{c\text{-}s}(t; x)$ and $C_{c\text{-}s}(t; y)$ should be replaced by $C_{ca\text{-}s}(t; x)$ and $C_{ca\text{-}s}(t; y)$, and Eq. (23) becomes

$$C(t; x, y)=\frac{c(t; x, y)-c_i}{c_\infty-c_i}=1-C_{ca-s}(t; x)C_{ca-s}(t; y) \tag{24}$$

In another case of plane diffusion, the medium exists in the area $-h_x / 2 \le x \le h_x / 2$ and $0 \le y$ (Fig. 4), and the diffusing substance penetrates its three boundaries and travels inward. By complementary method, Eq. (23) becomes

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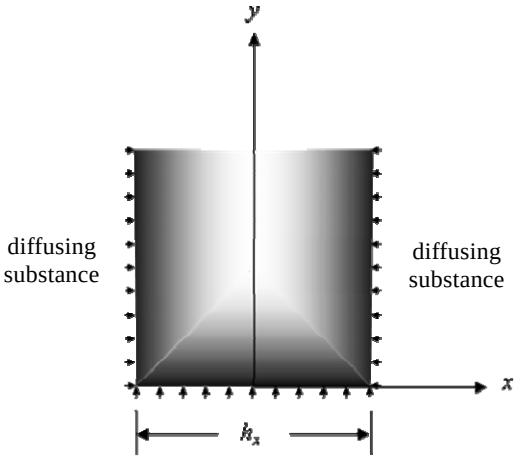


Fig. 4 Diffusion in a medium occupying the region $-h_x / 2 \le x \le h_x / 2$, and $0 \le y$

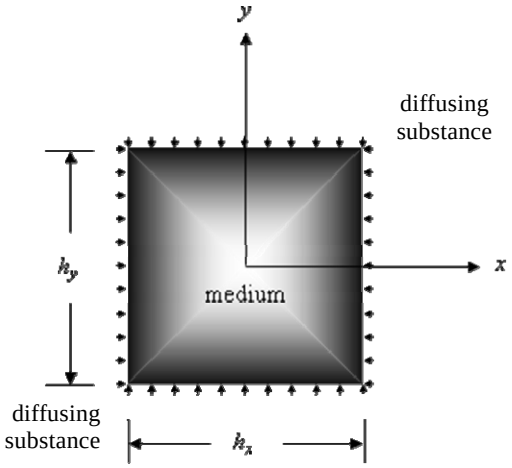


Fig. 5 Diffusion in a medium occupying the region $-h_x / 2 \le x \le h_x / 2$, and $-h_y / 2 \le y \le h_y / 2$

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-s}(t; y) \quad (25a)$$

The advancing model causes Eq. (25a) to become

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-s}(t; y) \quad (26a)$$

If the medium exists in the area $0 \le x$ and $-h_y / 2 \le y \le h_y / 2$, then the diffusing substance penetrates its three boundaries and travels inward, and Eq. (23) becomes

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{c-s}(t; x) C_{c-p}(t; y) \quad (25b)$$

The advancing model causes Eq. (25b) to become

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{ca-s}(t; x) C_{ca-p}(t; y) \quad (26b)$$

If the medium exists in the area $-h_x / 2 \le x \le h_x / 2$ and $-h_y / 2 \le y \le h_y / 2$ (Fig. 5), then the diffusing substance penetrates its four boundaries and travels toward its center, and Eq. (23) becomes [1]

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-p}(t; y) \quad (27)$$

The advancing model causes Eq. (27) to become

$$C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-p}(t; y) \quad (28)$$

For spatial diffusion in a medium occupying an octant of the Cartesian coordinate system where $0 \le x$, $0 \le y$ and $0 \le z$, the diffusing substance penetrates three boundaries and travels inward (Fig. 6), Eq. (23) becomes

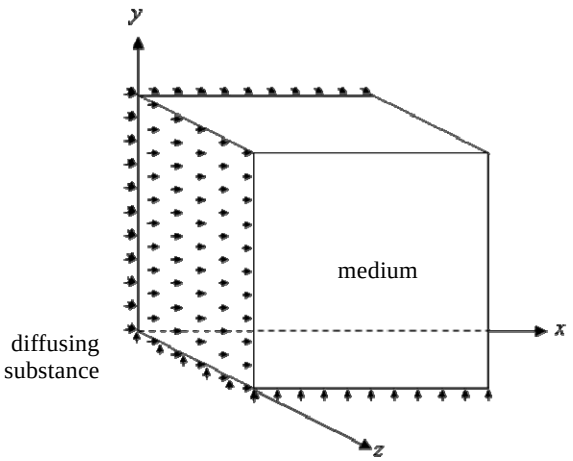


Fig. 6 Diffusion in a Medium Occupying the Region $0 \le x$, $0 \le y$, and $0 \le z$

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-s}(t; x) C_{c-s}(t; y) C_{c-s}(t; z) \quad (29)$$

where $C(t; x, y, z)$ is the normalized concentration of the diffusing substance in the medium as a function of t, x, y and z. $C(t; x, y, z)$, complementary function of $C(t; x, y, z)$ and $c(t; x, y, z)$ all satisfy the governing equation, Eq. (2a), and all of the conditions. Thus Eq. (29) is the normalized conventional solution to Eq. (2a). The advancing model causes Eq. (29) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-s}(t; x) C_{ca-s}(t; y) C_{ca-s}(t; z) \quad (30)$$

If the medium exists in the region $-h_x / 2 \le x \le h_x / 2$, $0 \le y$, $0 \le z$ (Fig. 7), then the diffusing substance penetrates the four boundaries and travels inward, and Eq. (29) becomes

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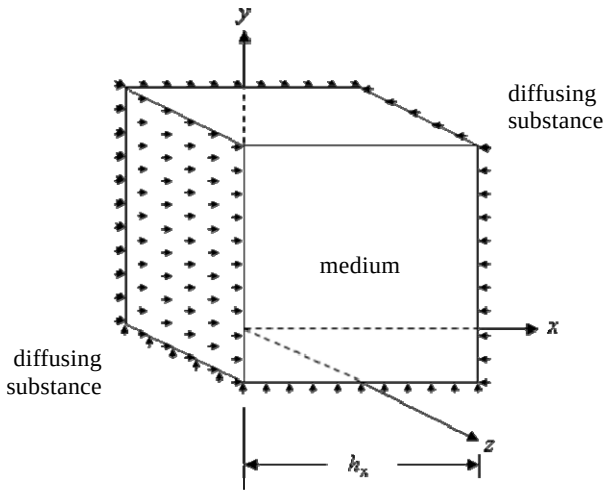


Fig. 7 Diffusion in a medium occupying the region $-h_x / 2 \le x \le h_x / 2$, $0 \le y$, and $0 \le z$

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-s}(t; y) C_{c-s}(t; z) \quad (31a)$$

The advancing model causes Eq. (31a) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-s}(t; y) C_{ca-s}(t; z) \quad (32a)$$

If the medium exists in the region $0 \le x$, $-h_y / 2 \le y \le h_y / 2$, $0 \le z$, then the diffusing substance penetrates its four boundaries and travels inward, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-s}(t; x) C_{c-p}(t; y) C_{c-s}(t; z) \quad (31b)$$

The advancing model causes Eq. (31b) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-s}(t; x) C_{ca-p}(t; y) C_{ca-s}(t; z) \quad (32b)$$

If the medium exists in the region $0 \le x$, $0 \le y$, $-h_z / 2 \le z \le h_z / 2$, then the diffusing substance penetrates its four boundaries and travels inward, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-s}(t; x) C_{c-s}(t; y) C_{c-p}(t; z) \quad (31c)$$

The advancing model causes Eq. (31c) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-s}(t; x) C_{ca-s}(t; y) C_{ca-p}(t; z) \quad (32c)$$

If the medium exists in the region $-h_x / 2 \le x \le h_x / 2$, $-h_y / 2 \le y \le h_y / 2$, $0 \le z$ (Fig. 8), then the diffusing substance penetrates its five boundaries and travels inward, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-p}(t; y) C_{c-s}(t; z) \quad (33a)$$

The advancing model causes Eq. (33a) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-p}(t; y) C_{ca-s}(t; z) \quad (34a)$$

If the medium exists in the region $-h_x / 2 \le x \le h_x / 2$, $0 \le y$, $-h_z / 2 \le z \le h_z / 2$, then the diffusing substance penetrates its five boundaries and travels inward, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-s}(t; y) C_{c-p}(t; z) \quad (33b)$$

The advancing model causes Eq. (33b) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-s}(t; y) C_{ca-p}(t; z) \quad (34b)$$

If the medium exists in the region $0 \le x$, $-h_y / 2 \le y \le h_y / 2$, $-h_z / 2 \le z \le h_z / 2$, then the diffusing substance penetrates its five boundaries and travels inward, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-s}(t; x) C_{c-p}(t; y) C_{c-p}(t; z) \quad (33c)$$

The advancing model causes Eq. (33c) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-s}(t; x) C_{ca-p}(t; y) C_{ca-p}(t; z) \quad (34c)$$

If the medium exists in the region $-h_x / 2 \le x \le h_x / 2$, $-h_y / 2 \le y \le h_y / 2$, $-h_z / 2 \le z \le h_z / 2$ (Fig. 9), then the diffusing substance penetrates its six boundaries and travels toward its center, and Eq. (29) becomes

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{c-p}(t; x) C_{c-p}(t; y) C_{c-p}(t; z) \quad (35)$$

The advancing model causes Eq. (35) to become

$$C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i} = 1 - C_{ca-p}(t; x) C_{ca-p}(t; y) C_{ca-p}(t; z) \quad (36)$$

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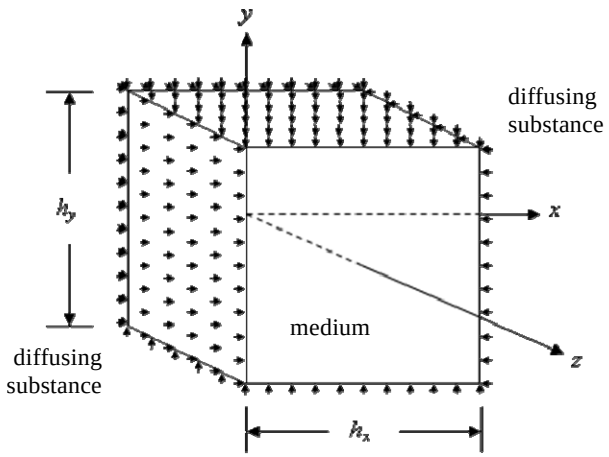


Fig. 8 Diffusion in a medium occupying the region $-h_x/2 \le x \le h_x/2$, $-h_y/2 \le \le h_y/2$, and $0 \le z$

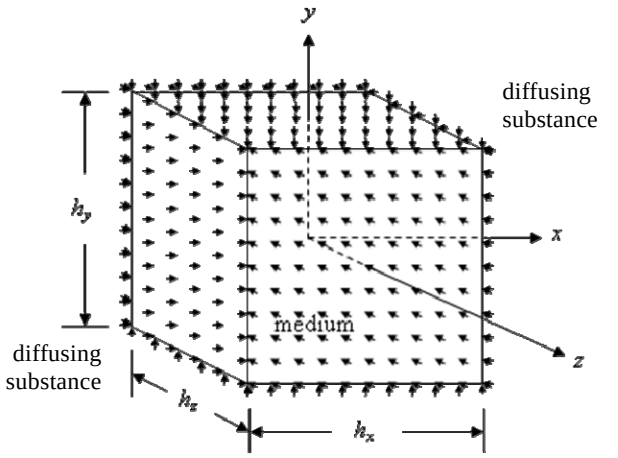


Fig. 9 Diffusion in a medium occupying the region $-h_x/2 \le x \le h_x/2$, $-h_y/2 \le y \le h_y/2$, and $-h_z/2 \le z \le h_z/2$

4. DISCUSSION AND CONCLUSIONS

In reality, a problem such as chloride diffusion in a reinforced concrete structure demands more than one parameter to quantify the rate of corrosion of the rebar and predict the service life of the reinforced concrete structure. Preliminary experiment on concrete specimens needs an exact solution to analyze data of concentration of chloride to characterize those parameters. To accelerate the preliminary experiment, multi-dimensional diffusion is commonly used and its exact solution is required.

Complementary function of normalized concentration of the diffusing substance in a medium in multi-dimensional problems can be expressed as the product of the complementary functions to uni-dimensional normalized solutions, consistent with the boundary conditions in each coordinate direction, whether or not the advancing boundary condition is taken into account. Solutions for the concentration of diffusing material in various boundary conditions for multi-dimensional diffusion have already been derived easily by the complementary method in this study. The major contribution of complementary method is the mathematical convenience for solving the concentration of the diffusion substance, while satisfying the governing equation and all of the conditions.

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NOTATIONS

a_0, a_n, b_n : constants of Fourier series

a_{mn} : $a_{mn} = \sum_{r=0}^{m} \left[\frac{(-1)^{r+1}(2m+1)!}{(2m-2r)!} \frac{1}{(n\pi/2)^{2r+2}} \right]$

b_{mn} : $b_{mn} = \sum_{r=0}^{m} \left[\frac{(-1)^{r+1}(2m+1)!}{(2m+1-2r)!} \frac{1}{(n\pi/2)^{2r+1}} \right]$

$C(t; x)$: normalized $c(t; x)$, $C(t; x) = \frac{c(t; x) - c_i}{c_\infty - c_i}$

$C(t; x, y)$: normalized $c(t; x, y)$, $C(t; x, y) = \frac{c(t; x, y) - c_i}{c_\infty - c_i}$

$C(t; x, y, z)$: normalized $c(t; x, y, z)$, $C(t; x, y, z) = \frac{c(t; x, y, z) - c_i}{c_\infty - c_i}$

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$$C_{ca\text{-}p}(t;x) \quad : \quad C_{ca\text{-}p}(t;x)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;x)\,\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}}$$

$$C_{ca\text{-}p}(t;y) \quad : \quad C_{ca\text{-}p}(t;y)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{mp}(t;y)\,\nu^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}}$$

$$C_{ca\text{-}s}(t;z) \quad : \quad C_{ca\text{-}s}(t;z)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{ms}(t;z)^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}}$$

$$C_{c\text{-}p}(t;x) \quad : \quad C_{c-p}(t;x)=4\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n\pi}(-1)^{\frac{n-1}{2}}e^{-\left(\frac{n\pi}{h_x}\right)^2 Dt}\cos\left(\frac{n\pi}{h_x}x\right)$$

$$C_{c\text{-}p}(t;y) \quad : \quad C_{c-p}(t;y)=4\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n\pi}(-1)^{\frac{n-1}{2}}e^{-\left(\frac{n\pi}{h_y}\right)^2 Dt}\cos\left(\frac{n\pi}{h_y}y\right)$$

$$C_{c\text{-}p}(t;z) \quad : \quad C_{c-p}(t;z)=4\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n\pi}(-1)^{\frac{n-1}{2}}e^{-\left(\frac{n\pi}{h_z}\right)^2 Dt}\cos\left(\frac{n\pi}{h_z}z\right)$$

$$C_{ca\text{-}s}(t;x) \quad : \quad C_{ca-s}(t;x)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{ms}(t;x)^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}}$$

$$C_{ca\text{-}s}(t;y) \quad : \quad C_{ca-s}(t;y)=\frac{\sum_{m=0}^{\infty}\frac{(-1)^m\Phi_{ms}(t;y)^{2m+1}}{m!(2m+1)}}{\sum_{m=0}^{\infty}\frac{(-1)^m\nu^{2m+1}}{m!(2m+1)}}$$

$$C_{ca\text{-}s}(t;z) \quad : \quad \Phi_{ms}(t;z)=\min\left(\frac{z}{2\nu\sqrt{Dt}},1\right)^{2m+1}$$

$$C_{c\text{-}s}(t;x) \quad : \quad C_{c-s}(t;x)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2\sqrt{Dt}}\right)^{2m+1}}{m!(2m+1)}$$

$$C_{c\text{-}s}(t;y) \quad : \quad C_{c-s}(t;y)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{y}{2\sqrt{Dt}}\right)^{2m+1}}{m!(2m+1)}$$

$$C_{c\text{-}s}(t;z) \quad : \quad C_{c-s}(t;z)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2\sqrt{Dt}}\right)^{2m+1}}{m!(2m+1)}$$

c : content of diffusion substance in a medium

c_i : initial c

c_∞ : saturated c

$c(t;x)$: c as a function of t and x

$c(t;x,y)$: c as a function of t, x and y

$c(t;x,y,z)$: c as a function of t, x, y and z

D : diffusivity

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F : the rate of transfer per unit area of section

h_x, h_y, h_z : dimensions of a medium in x, y and z directions of Cartesian Coordinate System

i, n, m, r : indexes for series

t : diffusion duration

x, y and z : the three coordinates of space Cartesian coordinate system

$$\Phi_{mp}(t;x) : \Phi_{mp}(t;x)=\begin{cases}\min\left(\dfrac{\frac{h_x}{2}+x}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_x}{2}-x}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_x^2}{16\nu^2D}\\ (-2)\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}a_{mn}\,e^{-\left(\frac{n\pi}{h_x}\right)^2 D\left(t-\frac{h_x^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_x}x\right), & t\ge\dfrac{h_x^2}{16\nu^2D}\end{cases}$$

$$\Phi_{mp}(t;y) : \Phi_{mp}(t;y)=\begin{cases}\min\left(\dfrac{\frac{h_y}{2}+y}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_y}{2}-y}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_y^2}{16\nu^2D}\\ (-2)\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}a_{mn}e^{-\left(\frac{n\pi}{h_y}\right)^2 D\left(t-\frac{h_y^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_y}y\right), & t\ge\dfrac{h_y^2}{16\nu^2D}\end{cases}$$

$$\Phi_{mp}(t;z) : \Phi_{mp}(t;z)=\begin{cases}\min\left(\dfrac{\frac{h_z}{2}+z}{2\nu\sqrt{Dt}},1\right)^{2m+1}+\min\left(\dfrac{\frac{h_z}{2}-z}{2\nu\sqrt{Dt}},1\right)^{2m+1}-1, & t\le\dfrac{h_z^2}{16\nu^2D}\\ (-2)\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}a_{mn}\,e^{-\left(\frac{n\pi}{h_z}\right)^2 D\left(t-\frac{h_z^2}{16\nu^2D}\right)}\cos\left(\dfrac{n\pi}{h_z}z\right), & t\ge\dfrac{h_z^2}{16\nu^2D}\end{cases}$$

$$\Phi_{ms}(t;x) : \Phi_{ms}(t;x)=\min\left(\frac{x}{2\nu\sqrt{Dt}},1\right)^{2m+1}$$

$$\Phi_{ms}(t;y) : \Phi_{ms}(t;y)=\min\left(\frac{y}{2\nu\sqrt{Dt}},1\right)^{2m+1}$$

$$\Phi_{ms}(t;z) : \Phi_{ms}(t;z)=\min\left(\frac{z}{2\nu\sqrt{Dt}},1\right)^{2m+1}$$

ν : Neumann's constant

REFERENCES

1. Crank, J., *The Mathematics of Diffusion*, 2nd Edition, Oxford University Press, Oxford, NY (1975).
2. Carslaw, H. S. and Jarger, J. C., *Conduction of Heat in Solids*, 2nd Edition, Oxford University Press, Oxford, NY (1986).
3. Chang, T.-W., Wang, C.-H. and Tsai, C.-L., "Advancing Boundary Model for Moisture Expansion in a Composite Laminate," *Journal of Composite Materials*, **42**, pp. 957–974 (2008).
4. Wang, C.-H., "Quantifying Diffusion Substance through Medium under Varying Environmental Conditions," Ph.D Thesis, Graduate School of Engineering Science and Technology, National Yunlin University of Science and Technology, Taiwan (2010).
5. Lin, C.-C., Tsai, C.-L., Wu, P.-K. and Lee, H.-J., "Advancing Diffusion Model for Diffusion in a Cube of Medium," *Journal of Mechanics*, **28**, pp. 345–354 (2011).
6. Tsai, C.-L. and Lin, C.-C., "Diffusion in Quadri-Infinite Medium," *The 35th National Conference on Theoretical and Applied Mechanics*, November 18–19, Tainan, Taiwan (2011).

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