Math. Proc. Camb. Phil. Soc. (1985), **97**, 63
Printed in Great Britain

a-convexity

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(*Received* 14 *February* 1984; *revised* 6 *July* 1984)

For $a \in \mathbf{R}$, we define a subset V of $\mathbf{R}^n$ to be *a-convex* if $\mathbf{x}, \mathbf{y} \in V$ implies

$$a\mathbf{x} + (1-a)\,\mathbf{y} \in V.$$

Clearly V is $(1-a)$-convex iff it is a-convex: V is convex iff it is a-convex for all $a \in [0, 1]$, and any set is 1-convex. We define the *a-convex hull* of V to be the intersection of all a-convex subsets of $\mathbf{R}^n$ containing V. Put $D = \{0, 1\}$ and $D(a)$ the a-convex hull of D in $\mathbf{R}^1$. If V is a-convex and $\mathbf{x}, \mathbf{y} \in V$, then V must contain the points of the line through $\mathbf{x}, \mathbf{y}$ given by $\{t\mathbf{x} + (1-t)\,\mathbf{y} : t \in D(a)\}$ and so in this paper we concentrate on the study of $D(a)$.

Let $S \subseteq \mathbf{R}$ and let X be an indeterminate. We extend the definition to call a subset U of $\mathbf{R}[X]$ *$f(X)$-convex*, for a polynomial $f(X)$, if $g, h \in U$ implies $fg + (1-f)\,h \in U$. Define sets of polynomials $S_j = S_j(X)$ by $S_0(X) = S$,

$$S_{n+1}(X) = \{Xg + (1-X)\,h : g, h \in S_n(X)\}.$$

Putting $g = h$ in this definition, we see that $S_n \subseteq S_{n+1}$. Put $S(X) = \cup\, S_n(X)$. For a polynomial $f \in S(X)$, define the *level*

$$L(f) = \min\{n : f \in S_n(X)\}.$$

Clearly $S(X)$ is X-convex. For $a \in \mathbf{R}$, put

$$S(a) = \{f(a) : f \in S(X)\} \quad \text{and} \quad S_n(a) = \{f(a) : f \in S_n(X)\}.$$

PROPOSITION 1. *$S(a)$ is the a-convex hull of S in* $\mathbf{R}$.

Proof. Clearly $S(a)$ is a-convex. Now suppose that V is an a-convex subset of $\mathbf{R}$ containing S. Then $V \supseteq S = S_0(a)$ and if $V \supseteq S_n(a)$ then $V \supseteq S_{n+1}(a)$. So $V \supseteq S(a)$, which is therefore the a-convex hull of S. |

PROPOSITION 2. *If S is closed under multiplication, then so is $S(X)$.*

Proof. Let $f, g \in S(X)$ with $L(f) = m$ and $L(g) = n$. We proceed by induction on $m+n$. If $m+n = 0$ then $f, g \in S$ and by hypothesis $fg \in S$. If $m+n > 0$ then suppose, without loss of generality, that $m > 0$. So $f = Xh + (1-X)\,k$ with $L(h), L(k) < m$. By induction, $hg, kg \in S(X)$. Then $fg = Xhg + (1-X)\,kg \in S(X)$. So $S(X)$ is closed under multiplication. |

Clearly if S is closed under multiplication, then so is $S(a)$ for any $a \in \mathbf{R}$.

PROPOSITION 3. *Suppose that S is s-convex for every $s \in S$. Then*

(i) *$S(X)$ is f-convex for every $f \in S(X)$,*

(ii) *$S(X)$ is closed under composition of polynomials.*

Proof. (i) Consider $fg + (1-f)\,h$ with $f, g, h \in S(X)$. We proceed by induction on $L(f)$.

https://doi.org/10.1017/S0305004100062587 Published online by Cambridge University Press

If $L(f) = 0$, then $f \in S$. We show that $fg + (1-f)h \in S(X)$ by induction on $L(g) + L(h)$. If $L(g) + L(h) = 0$, then $g, h \in S$ and $fg + (1-f)h \in S$ since S is f-convex by hypothesis. If $L(g) > 0$, say, then $g = Xp + (1-X)r$ with $L(p), L(r) < L(g)$. Now

$$\begin{aligned} fg + (1-f)h &= f(Xp + (1-X)r) + (1-f)(Xh + (1-X)h) \\ &= X(fp + (1-f)h) + (1-X)(fr + (1-f)h), \end{aligned}$$

and since $L(p) + L(h), L(r) + L(h) < L(g) + L(h)$, by induction

$$fp + (1-f)h, \quad fr + (1-f)h \in S(X) \quad \text{and so} \quad fg + (1-f)h \in S(X).$$

Hence $S(X)$ is f-convex for every $f \in S$.

Now suppose $L(f) > 0$, so that $f = Xp + (1-X)r$ with $L(p), L(r) < L(f)$. Then

$$\begin{aligned} fg + (1-f)h &= (Xp + (1-X)r)g + (1 - Xp - (1-X)r)h \\ &= X(pg + (1-p)h) + (1-X)(rg + (1-r)h). \end{aligned}$$

By induction, $S(X)$ is p-convex and r-convex, so $pg + (1-p)h, rg + (1-r)h \in S(X)$ and so $fg + (1-f)h \in S(X)$. Hence $S(X)$ is f-convex for every $f \in S(X)$.

(ii) Let $g, h \in S(X)$ and consider $g(f(X))$. We proceed by induction on $L(g)$. If $L(g) = 0$, then $g \in S$ is a constant, so $g(f) = g \in S$. If $L(g) > 0$, then

$$g = Xh(X) + (1-X)k(X)$$

with $L(h), L(k) < L(g)$. So $g(f(X)) = f(X)h(f(X)) + (1 - f(X))(k(f(X))$. By induction, $h(f(X)), k(f(X)) \in S(X)$ and by (i) $S(X)$ is f-convex, so $f(g(X)) \in S(X)$. Hence $S(X)$ is closed under composition of polynomials. |

If S is s-convex for every $s \in S$, then $S(a)$ is b-convex for every $b \in S(a)$. Note that $D = \{0, 1\}$ satisfies the conditions of both Propositions 2 and 3, so $D(X)$ is closed under multiplication and composition of polynomials.

We next derive an alternative description of $S(X)$. For $S, T \subseteq \mathbf{R}$, let

$$S + T = \{s + t : s \in S, t \in T\}$$

and $S.n$ denote $S + S + \ldots + S$ to n terms. Clearly $S.(m+n) = S.m + S.n$.

PROPOSITION 4. *$t \in S_n(X)$ if and only if $f = \sum_{r=0}^{n} a_r X^r (1-x)^{n-r}$ with $a_r \in S.{}_nC_r$.*

Proof. We proceed by induction on n. If $n = 0$ then $S.{}_0C_0 = S.1 = S$ and the assertion is true.

If the assertion is true for $n = k-1$, then let $f, g \in S_{k-1}(X)$, say

$$f = \sum_{r=0}^{k-1} a_r X^r (1-X)^{k-1-r} \quad \text{and} \quad g = \sum_{r=0}^{k-1} b_r X^r (1-X)^{k-1-r}$$

with $a_r, b_r \in S.{}_{k-1}C_r$. Define $b_k = 0$. Then $Xf + (1-X)g = \sum_{r=0}^{k} (a_{r+1} + b_r) X^r (1-X)^{k-r}$, and $a_{r+1} + b_r \in S.{}_{k-1}C_{r-1} + S.{}_{k-1}C_r = S.{}_kC_r$. Conversely, if $c_r \in S.{}_kC_r$ then we can express $c_r = a_{r-1} + b_r$ with $a_{r-1} \in S.{}_{k-1}C_{r-1}$ and $b_r \in S.{}_{k-1}C_r$, and the result follows. |

COROLLARY 4·1. $D_n(X) = \{\sum_0^n a_r X^r (1-X)^{n-r} : 0 \leqslant a_r \leqslant {}_nC_r\}$. |

COROLLARY 4·2. $|S_n(X)| = \prod_{r=0}^{n} |S.{}_nC_r|$. *In particular* $|D_n(X)| = \prod_0^n (1 + {}_nC_r)$. |

We fix the notation $B = X/(1-X)$, $b = a/(1-a)$, $q = -b$. So

$$S_n(X) = \{(1-X)^n \textstyle\sum_{r=0}^{n} a_r B^r : a_r \in S.{}_nC_r\}.$$

We next consider the topological properties of $D(a)$.

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PROPOSITION 5. *Suppose* $a \notin [0, 1]$ *and let* V *be an a-convex subset of* $\mathbf{R}$. *If* V *has any limit points, then* 0 *is a limit point of* V.

Proof. Replacing a by $1-a$ if necessary, we may assume that $a > 1$. Suppose, without loss of generality, that V has a positive limit point. Let

$$x = \inf\{y \geqslant 0 : y \text{ is a limit point of } V\}.$$

Then x is a limit point of V. Suppose, if possible, that $x \neq 0$. Let $x_n \to x$ with $x_n \in V$ and all $x_n \neq x$. Choose N large enough that $|x - x_N| < x/a$. Define

$$y = ax_N + (1-a)x = a(x_N - x) + x \quad \text{if } x_N < x:$$
$$y = ax + (1-a)x_N = a(x - x_N) + x \quad \text{if } x_N > x.$$

Then $y \in V$ and

$$y = \lim ax_N + (1-a)x_n \quad \text{if } x_N < x:$$
$$y = \lim ax_n + (1-a)x_N \quad \text{if } x_N > x,$$

so that in either case y is a limit point of V with $0 < y < x$, a contradiction. So 0 is a limit point of V. ∎

PROPOSITION 6. *If* $0 < a < 1$, *then* $D(a)$ *is dense in* $[0, 1]$.

Proof. $D(a)$ is contained in $[0, 1]$, the convex hull of $\{0, 1\}$. Put $\lambda = \max\{a, 1-a\} < 1$. By induction on n it is clear that the gap between consecutive elements of $D_n(a)$ is less than λ^n. It follows that $D(a)$ is dense in $[0, 1]$. ∎

PROPOSITION 7. *Suppose* $a \notin (0, 1)$. $D(a)$ *either is dense in* $\mathbf{R}$ *or has no limit points.*

Proof. We may assume $a \geqslant 1$, since $D(a) = D(1-a)$. If $D(a)$ has any limit point, then by Proposition 5, 0 is a limit point. If $x_n \to 0$, then $x_n^2 \to 0$ and $x_n^2 \in D(a)$, so 0 is a limit point of positive elements of $D(a)$. In particular, there is $y \in D(a) \cap (0, 1)$. By Proposition 3, $D(a) \supseteq D(y)$ and by Proposition 6, $D(y)$ is dense in $[0, 1]$. So every point of $[0, 1]$ is also a limit point of $D(a)$. Since $D(a)$ is closed under multiplication, every point of $[0, a^m]$ and $[1-a^m, 0]$ is a limit point of $D(a)$ and the union of these intervals is $\mathbf{R}$. So if $D(a)$ has any limit point, it is dense in $\mathbf{R}$. ∎

We define a to be *sparse* if $D(a)$ has no limit points. If $a \in \mathbf{Z}$, then $D(a) \subseteq \mathbf{Z}$ and so every rational integer is sparse.

We recall some of the terminology of algebraic number theory: see, for example, Samuel [2], chapter 2. For $x \in \mathbf{C}$, we say that x is *algebraic* if x satisfies a monic polynomial $f(X) \in \mathbf{Q}[X]$ and *integral*, or an *algebraic integer*, if $f(X) \in \mathbf{Z}[X]$. The monic polynomial of minimal degree satisfied by an algebraic x is its *minimal polynomial* and the *degree* of x is the degree of its minimal polynomial. An algebraic of degree 2 is a *quadratic*. The *conjugates*, $x^{(i)}$, $i = 1, \ldots, n$ of an algebraic x of degree n are the roots of its minimal polynomial: x is *totally real* if all the $x^{(i)} \in \mathbf{R}$. The *norm* of an element y of the field $\mathbf{Q}(x)$ generated by x over the rationals is the determinant Ny of the linear map of $\mathbf{Q}(x)$ to itself as a $\mathbf{Q}$-vector space given by multiplication by y. Ny is a power of the product of the conjugates of y, and is equal to this product if the degree of y is equal to that of x (see Samuel [2], proposition 2·6·1). Since the product is the constant term in the minimal polynomial for y, the norm of an integral element y of $\mathbf{Q}(x)$ is a rational integer, $Ny \in \mathbf{Z}$. If y is integral, then so is $g(y)$ for any $g \in \mathbf{Z}[Y]$.

THEOREM 8. *If* a *is sparse, then* a *is an algebraic integer.*

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Proof. Assume that $a > 1$ is sparse: recall that $q = a/(a-1)$, so $q > 1$. Fix k with $1 < k < q$. We define an equivalence relation on $D(a)$ by defining $x < y \in D(a)$ to be *k-connected* if there exist $x_i \in D(a)$ with

$$x = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = y \quad \text{and} \quad x_i/x_{i-1} < k,\ 1 \leqslant i \leqslant n.$$

We call an equivalence class for this relationship a *k-block.*

We now show that there are only finitely many k-blocks in $D(a) \cap [1, \infty)$. Suppose that $[x, y] \cap D(a)$ is a k-block. Then ax, ay are k-connected. Let $r \in \mathbf{N}$ and put $x' = ax + (1-a)a^r$: then $x' \in D(a)$ and $x' < ax$. So

$$ax/x' = ax/(ax + (1-a)a^r) = 1/(1-a^r/qx).$$

Suppose $a^r < xq(k-1)/k \leqslant a^{r+1}$: this will be possible for $r \in \mathbf{N}$ if x is large enough, $x > k/q(k-1)$. If so, then $ax/x' < k$, so that x' and ay are k-connected. Now $ay/x' \geqslant uy/x$ where $u = ak/(ak-k+1) > 1$. We put

$$x^{(0)} = x, \quad y^{(0)} = y \quad \text{and} \quad x^{(n+1)} = ax^{(n)} + (1-a)a^{r(n)}, \quad y^{(n+1)} = ay^{(n)}$$

where $r(n)$ is defined by $a^{r(n)} < x^{(n)}q(k-1)/k \leqslant a^{r(n)+1}$. So $x^{(n)}$, $y^{(n)}$ are k-connected and $y^{(n)}/x^{(n)} \geqslant u^n(y/x)$. Since $u > 1$, $y^{(n)}/x^{(n)} > a$ for all n sufficiently large. But then $x^{(n)} < x^{(n+1)} < y^{(n)} < y^{(n+1)}$ so that $x^{(n)}$, $y^{(n+1)}$ are k-connected for all large enough n. But then $D(a) \cap [x^{(n)}, \infty)$ is k-connected and there are only finitely many k-blocks in $D(a) \cap [1, \infty)$.

Now let $1 < k < \sqrt{q}$ and choose M as above so that $K = D(a) \cap [M, \infty)$ is k-connected. Put $A = D(a) \cap [1, M]$.

We show that if $x \in K$, then x is a $\mathbf{Z}$-linear combination of elements of A. We proceed by induction: this is possible since K has no limit points. Let $x \in K$. If $x = M$ then $x \in A$ and the result holds. If $x > M$, let $u < x < v$ with $u, v \in K$ and $x/u,\ v/x < k$. Put $r = au + (1-a)x$, $s = au + (1-a)v$, $t = ax + (1-a)v$. Since $k^2 < q$, we have $r, s, t > 0$ and $s < r < u < x, t < x$. But $x = r - s + t$: by induction r, s, t are $\mathbf{Z}$-linear combinations of elements of A and so the same is true of x.

We now prove the Theorem. A is finite and its elements can be represented by polynomials in a of degree $\leqslant P$, say. Choose N so that $N > P$ and $a^N > M$. Then a^N is a $\mathbf{Z}$-linear combination of integral polynomials in a of degree $< N$, so a satisfies a monic integral polynomial and a is an algebraic integer. ∎

This result was announced by Calvert [1]: unfortunately his proof assumes that $q^n \in D(a)$ for sufficiently large n. If $a = 3 + \sqrt{7}$, then a is sparse (by Proposition 12 below), but $q = (1+\sqrt{7})/3$, and hence no power q^n is integral, and q^n cannot be of the form $f(a)$ if f has integer coefficients

Write $\mathscr{S}$ for the set of sparse numbers.

PROPOSITION 9. *$\mathscr{S}$ is closed.*

Proof. Suppose $a_n \to a$ with $a_n \in \mathscr{S}$. If $a \notin \mathscr{S}$, then by Proposition 7 there is $b \in D(a)$ with $b \in (0, 1)$. Now $b = f(a)$ for some polynomial $f \in D(X)$, and $f(a_n) \to f(a) = b$, so for n large enough, $f(a_n) \in (0, 1)$. But $f(a_n) \in D(a_n)$ so a_n cannot be sparse. ∎

PROPOSITION 10. *Suppose for $a \in \mathbf{R}$, there are $f(X), g(X) \in D(X)$ with*

$$0 < |f(a) - g(a)| < 1.$$

Then a is not sparse.

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Proof. We show that under this condition, $D(a)$ has a limit point. Without loss of generality, assume that $1 < f(a) < g(a) < f(a)+1$. Define polynomials $f_n(X)$, $g_n(X)$ by $f_0 = f$, $g_0 = g$ and

$$f_{n+1} = g_n f_n + (1-g_n) g_n = g_n + (f_n - g_n) g_n,$$
$$g_{n+1} = f_n f_n + (1-f_n) g_n = g_n - (f_n - g_n) f_n.$$

$f_n(X), g_n(X) \in D(X)$ by Proposition 3 (i). Now $g_{n+1}(a) - f_{n+1}(a) = (g_n(a) - f_n(a))^2$ and so $g_n(a) - f_n(a) \to 0$: in particular, $0 < g_n(a) - f_n(a) < 1$ for all n. We have

$$g_{n+1}(a) < g_n(a)$$

since $f_n(a) < g_n(a)$.

Further $f_n(a), g_n(a) > 0$: this is true for $n = 0$ by hypothesis. Now

$$f_{n+1}(a) = g_n(a)(1 + f_n(a) - g_n(a)) \quad \text{and} \quad 1 + f_n(a) - g_n(a) > 0.$$

$g_n(a) > 0$ by induction and the assertion follows.

So $0 < f_n(a) < g_n(a) < g(a)$ and there are infinitely many points of $D(a)$ in the interval $[0, g(a)]$: hence $D(a)$ has a limit point and a is not sparse. ∎

We can now strengthen Proposition 9.

PROPOSITION 11. *Every point of $\mathscr{S}$ is isolated.*

Proof. Let $a \in \mathscr{S}$. By Theorem 8, a is an algebraic integer and so $b = a/(1-a)$ is algebraic, with minimal polynomial $f(B) = \sum_{i=0}^{n} f_i B^i$, say, where the $f_i \in \mathbf{Z}$. Let $L = \max\{n, |f_i|\}$. Then $h(X) = (1-X) L_f(B) \in T(X)$, where $B = X/(1-X)$ and $T = \{-1, 0, 1\}$, and $h(a) = 0$. Pick $\delta > 0$ such that if $0 < |a - a'| < \delta$ then $0 < |h(a')| < \frac{1}{2}$. But $h(x)$ can be expressed as $h(X) = g(X) - k(X)$ for $g, k \in D(X)$, so by Proposition 10 such an a' cannot be sparse. So the interval $(a-\delta, a+\delta)$ contains no other point of $\mathscr{S}$. ∎

PROPOSITION 12. *Suppose a is a totally real algebraic integer, $a \notin (0, 1)$, and that all the non-trivial conjugates $a^{(i)} \in (0, 1)$, $i > 1$. Then a is sparse.*

Proof. Suppose, if possible, that a is not sparse. Then there is $f(X) \in D(X)$ such that $f(a) \in (0, 1)$. But for the conjugates $a^{(i)}$ with $i > 1$, $D(a^{(i)}) \subseteq [0, 1]$, so $f(a^{(i)}) \in (0, 1)$. Now $f(a^{(i)}) = f(a)^{(i)}$ since $f \in \mathbf{Z}[X]$. Thus all the non-trivial conjugates of $f(a)$ lie in $(0, 1)$, so their product, and hence $Nf(a)$, also lie in $(0, 1)$. But a is an algebraic integer, $f \in \mathbf{Z}[X]$ and so $f(a)$ is also an algebraic integer. So $Nf(a) \in \mathbf{Z}$, a contradiction. Hence a is sparse. ∎

This result answers the conjecture of Calvert [1]. For if $a = (3+\sqrt{5})/2$, the conjugate $a' = (3-\sqrt{5})/2 \in (0, 1)$ and so a is sparse by this Proposition.

Proposition 12 is best possible in the sense that if all but two conjugates of a lie in $(0, 1)$, then a need not be sparse. Consider the irreducible quartic $x^4 - 9x^2 + 9x - 2 = 0$ with roots $a = a^{(1)} \doteq 2.3592$, $a^{(2)} \doteq -3.4338$, $a^{(3)} \doteq 0.7514$, $a^{(4)} \doteq 0.3295$. Then by Corollary 4·2, $D_3(X)$ has $2 \cdot 4 \cdot 4 \cdot 2 = 64$ elements and the corresponding elements of $D_3(a)$ are all distinct, for distinct cubics in a cannot be equal, and so the average difference between elements of $D_3(a)$ is $(a^3 - (1-a^3))/64 < 1$. So a is not sparse by Proposition 10. (This also follows from theorem 2 of Calvert [1].)

We can provide a partial converse to Proposition 12. First we improve Proposition 10.

PROPOSITION 13. *Suppose that a is irrational, and that there exist $f, g \in D(a)$ with $0 < f(a) = g(a) - 1$. Then a is not sparse.*

Proof. We show first that $D(a)$ is closed under translation by $f(a)$. For the translate $D(a) - f(a)$ is a-convex and contains $0 = f(a) - f(a)$ and $1 = g(a) - f(a)$. So

$$D(a) - f(a) \supseteq D(a).$$

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But the translate $D(a)+f(a)$ is again a-convex and contains $0 = (1-g(a))-f(a)$ and $1 = (1-f(a))-f(a)$. So $D(a)+f(a) \supseteq D(a)$ and so $D(a) = D(a)+f(a)$.

Now $D(a)$ can be written as $\{nf(a)+c: n \in \mathbf{Z}, c \in C\}$ where $C = D(a) \cap [0, f(a))$. Let $m, p \in \mathbf{Z}$ and put $n = m+p$. Then $(pa+m)f(a) = anf(a)+(1-a)mf(a) \in D(a)$ since $nf(a)$, $mf(a) \in D(a)$. But a is irrational, and $f(a) \neq 0$, so $\{(pa+m)f(a): p, m \in \mathbf{Z}\}$ is dense in $\mathbf{R}$ and so a cannot be sparse. ∎

PROPOSITION 14. *Suppose $a > 1$, $a \neq 2$, satisfies a non-zero polynomial $f(X) \in D(X)$. Then a is not sparse.*

Proof. If $a \in (1, 2)$ then $(1-a)^2 \in (0, 1)$ and so a is not sparse. So suppose $a > 2$, and that $0 = f(a) = g(b)(1-a)^n$ where $n = L(f)$ is the level of f and g is a polynomial of degree n in $B = X/(1-X)$. $f(X) = 0$, so $g(B) = 0$. Suppose the coefficient g_r of B^r in $g(B)$ is non-zero. Then $h(X) = (1-X)^n(g(B)-B^r) \in D(X)$ by Production 4. Now $h(a) = -a^r(1-a)^{n-r}$. But $k(a) = 1-a^r(1-a)^{n-r} \in D(a)$, and $h(a) > 1$, so $k(a) > 0$, so by Proposition 13, a is not sparse. ∎

Note that 2 is a root of $1-(1-X)^2 \in D_2(X)$, although 2 is sparse.

THEOREM 15. *Let a be a totally real algebraic integer, $a \notin \mathbf{Z}$, and suppose a is sparse. Then at least one conjugate of a lies in $(0, 1)$.*

Proof. We may assume that $a > 1$: further, since a is sparse and $a \notin \mathbf{Z}$, $a > 2$. Suppose, if possible, that all the conjugates of a lie outside $(0, 1)$: then all the conjugates of $b = a/(1-a)$ are negative, so the minimal polynomial $g(B) = \sum_{i=0}^{n} g_i B^i$ of b has positive coefficients g_i. But $f(X) = (1-X)^N g(B) \in D(X)$ provided $N > \max\{n, |g_i|\}$: $f(X) = 0$ and $f(a) = (1-a)^N g(b) = 0$. Since a is irrational and $a > 2$, by Proposition 14 a cannot be sparse: a contradiction. So at least one conjugate of a lies in $(0, 1)$. ∎

COROLLARY 15·1. *Let a be a real quadratic integer. a is sparse if and only if its conjugate $a' \in (0, 1)$.*

Thus the condition of Proposition 12 is necessary as well as sufficient in the case of real quadratic a. It is natural to conjecture that the condition is necessary for all totally real integral a.

I am grateful to Dr. B. J. Birch for his help and encouragement, and especially for pointing out Proposition 10 to me, as also to Dr W. W. Stothers for helpful discussions and to the referee. I acknowledge the financial support of the Science Research Council and Christ Church, Oxford, during part of the work described in this paper.

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