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On rationally ergodic and rationally weakly mixing rank-one transformations

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Abstract. We study the notions of weak rational ergodicity and rational weak mixing as defined by J. Aaronson [Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.* **27**(2) (1977), 93–123; Rational weak mixing in infinite measure spaces. *Ergod. Th. & Dynam. Sys.* 2012, to appear. http://arxiv.org/abs/1105.3541]. We prove that various families of infinite measure-preserving rank-one transformations possess or do not possess these properties, and consider their relation to other notions of mixing in infinite measure.

1. *Definitions and preliminaries*

Let $(X, \mathcal{B}, \mu)$ be a standard Borel measure space with a σ-finite non-atomic measure μ. In most cases, we assume that μ is infinite. A transformation $T : X \to X$ is called *measurable* if $T^{-1}A \in \mathcal{B}$ for all $A \in \mathcal{B}$. A measurable transformation T is *measure-preserving* if $\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathcal{B}$. We say that T is *ergodic* if every T-invariant set (i.e., $T^{-1}A = A \bmod \mu$) is null ($\mu(A) = 0$) or full ($\mu(X \setminus A) = 0$). We say that T is *conservative* if for every measurable set A of positive measure, there exists a positive integer n such that $\mu(A \cap T^{-n}A) > 0$. It follows that T is conservative and ergodic if and only if for every set A of positive measure, $\bigcup_{n=0}^{\infty} T^{-n}A = X \bmod \mu$. An *invertible measure-preserving transformation* is a measure-preserving transformation whose inverse is also measurable. Throughout this paper, we assume that T is an invertible, conservative ergodic, measure-preserving transformation on $(X, \mathcal{B}, \mu)$, and typically use the forward images $T^n A$ instead of $T^{-n}A$. On a non-atomic space, an invertible ergodic transformation is conservative.

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When T is a measure-preserving transformation on a probability space X, the mean ergodic theorem states that ergodicity is equivalent to having the convergence

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu(A\cap T^kB)\to\mu(A)\mu(B) \tag{1}$$

as $n\to\infty$ for all measurable $A, B\subset X$. This gives a quantitative estimate for the average number of visits of one set to another. When X has infinite measure, however, the ergodic theorem implies that the Cesàro averages of (1) converge to 0 for all pairs A, B of finite measure. Moreover, in [**1, 3**] Aaronson proved that there does not exist a sequence of normalizing constants for which the averages of (1) converge to $\mu(A)\mu(B)$, and he proposed in turn the definitions of rational ergodicity and weak rational ergodicity.

For any measurable set $F\subset X$ of finite positive measure, define the *intrinsic weight sequence* of F to be

$$u_n(F)=\frac{\mu(F\cap T^nF)}{\mu(F)^2}$$

and write

$$a_n(F)=\sum_{k=0}^{n-1}u_k(F).$$

A conservative ergodic transformation T is said to be *weakly rationally ergodic* (see [**1**]) if there exists a measurable set $F\subset X$ of positive finite measure such that for all measurable $A, B\subset F$,

$$\frac{1}{a_n(F)}\sum_{k=0}^{n-1}\mu(A\cap T^kB)\to\mu(A)\mu(B) \tag{2}$$

as $n\to\infty$. If this convergence happens only along a subsequence $\{n_i\}$ of $\mathbb{N}$, we say that T is *subsequence weakly rationally ergodic*. To emphasize the set F, we will sometimes say that T is *weakly rationally ergodic on* F. Note that any measure-preserving ergodic transformation on a probability space is trivially weakly rationally ergodic by taking F to be the whole space itself. Then $a_n(F)=n$, so (2) reduces to the Cesàro sum definition of ergodicity.

A transformation T is said to be *(spectrally) weakly mixing* if, whenever $f\in L^\infty(X,\mu)$ and $f\circ T=zf$ for some $z\in\mathbb{C}$, f is constant almost everywhere. When X is a probability space, this is equivalent to ergodicity of the Cartesian square and also to the strong Cesàro convergence

$$\frac{1}{n}\sum_{k=0}^{n-1}|\mu(A\cap T^kB)-\mu(A)\mu(B)|\to 0$$

as $n\to\infty$ for all measurable $A, B\subset X$. In [**6**] it was shown that for infinite measure-preserving transformations, (spectral) weak mixing is strictly weaker than ergodicity of the Cartesian square.

Another property that we consider which is equivalent to weak mixing in the finite measure-preserving case is double ergodicity. This property was introduced by Furstenberg for probability-preserving transformations in [**11**, Theorem 4.31], and was shown to be equivalent to weak mixing but was not given a specific name. A transformation T is said

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to be *doubly ergodic* if for every pair of sets A and B with positive measure, there exists a positive integer n for which $\mu(A \cap T^n A)$ and $\mu(B \cap T^n A)$ are simultaneously non-zero. In the infinite measure-preserving case, double ergodicity is strictly stronger than spectral weak mixing and is properly implied by the ergodic Cartesian square [**8**].

More recently, Aaronson has introduced a notion of weak mixing in infinite measure relying on a 'strong Cesàro generalization' of (2). An ergodic transformation T is said be *rationally weakly mixing* (see [**5**]) if there exists a measurable set $F \subset X$ of positive finite measure such that for all measurable $A, B \subset F$,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |\mu(A \cap T^k B) - \mu(A)\mu(B)u_k(F)| \to 0 \tag{3}$$

as $n \to \infty$. Note that the convergence of (3) implies the convergence of (2). Again, it is clear that rational weak mixing reduces to the usual definition of weak mixing in the finite measure-preserving case.

We now describe our main results. In §2 we establish weak rational ergodicity for a large class of rank-one transformations and discuss the notions of rational ergodicity and bounded rational ergodicity in this context. In §3 we construct a class of rank-one transformations that are not rationally weakly mixing; in particular, we obtain a transformation that is rationally ergodic and spectrally weakly mixing but not rationally weakly mixing. This negatively answers a question of Aaronson's. (After our work was completed, we learned that Aaronson had also independently answered this question in [**4**].) Section 4 shows that rational weak mixing implies double ergodicity and constructs a transformation that is not rationally weakly mixing and which we conjecture to be doubly ergodic. Section 5 proves that the notion of zero-type for infinite measure-preserving transformations (whose spectral definition is similar to the mixing condition in the probability-preserving case) is independent of rational weak mixing. Finally, in §6 we present a class of rank-one positive-type transformations that are rationally weakly mixing. As remarked therein, all the examples of rationally weakly mixing transformations constructed in [**5**] are of the type $T \times S$, where T is an infinite measure-preserving K-automorphism and S is a mildly mixing probability-preserving transformation. These examples have countable Lebesgue spectrum and are of a different nature than our rank-one constructions.

1.1. *Rank-one transformations (basics).* We briefly review rank-one cutting-and-stacking transformations (see, for example, [**15**]). A *column* is an ordered collection of pairwise disjoint intervals (called *levels*) in $\mathbb{R}$, each of the same measure. We think of the levels in a column as being stacked on top of each other, so that the $(j+1)$th level is directly above the jth level. Every column $C = \{J_j\}$ is associated with a natural column map T_C sending each point in J_j to the point directly above it in J_{j+1}. Note that T_C is undefined on the top level of C. A (*rank-one*) *cutting-and-stacking* construction for T consists of a sequence of columns C_n such that:

(a) the first column C_0 is the unit interval;

(b) each column C_{n+1} is obtained from C_n by cutting C_n into $r_n \geq 2$ subcolumns of equal width, adding any number of new levels (called *spacers*) above each

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subcolumn, and stacking every subcolumn under the subcolumn to its right—in this way, C_{n+1} consists of r_n copies of C_n, possibly separated by spacers;

(c) the collection of levels $\bigcup_n C_n$ forms a generating semiring for $\mathcal{B}$.

Observing that $T_{C_{n+1}}$ agrees with T_{C_n} everywhere where T_{C_n} is defined, we then take T to be the pointwise limit of T_{C_n} as $n \to \infty$. It is easily seen that these rank-one transformations are invertible, measure-preserving, and conservative ergodic.

1.2. *Rank-one transformations (notation).* Let T be a rank-one transformation, and fix any column C_n of T. We denote the number of levels in C_n by h_n and write w_n for the width of each level. We denote the height of any level J in C_n by $h(J)$, with the convention that $0 \le h(J) < h_n$. For each $0 \le k < r_n$, let $s_{n,k}$ be the number of spacers added above the kth subcolumn of C_n, and denote the number of levels in the kth subcolumn (after adding spacers) by $h_{n,k} = h_n + s_{n,k}$.

Define T to be *normal* if $s_{n,r_n-1} > 0$ for infinitely many values of n. This means that at least one spacer is added above the rightmost subcolumn infinitely many times. In addition, we say that T has a *bounded number of cuts* if $\sup\{r_n\} < \infty$; this implies that T is partially rigid and of infinite conservative index [7].

Given any level J from C_n and any column C_m of T with $m \ge n$, we define the *descendants* of J in C_m to be the collection of levels in C_m whose disjoint union is J. We denote this set by $D(J, m)$. Occasionally, we will also use $D(J, m)$ to refer to the heights of the descendants of J in C_m. In the case when J is the unit interval I, for each $m \in \mathbb{N}$ it will be convenient to define $M_m = \max(D(I, m))$. That is, M_m is the height of the uppermost descendant of I in C_m.

We say that T *grows exponentially* if $2s_{n,r_n-1} \ge h_{n+1}$ for every n. Intuitively, this means that the upper half of every column C_n consists of spacers added during the $(n-1)$th stage of construction. In particular, the descendants of any level J from an earlier column must lie in the lower half of C_n. Note that any T which grows exponentially is clearly normal.

2. *Rational ergodicity*

In this section we establish some introductory ideas and prove that a large class of rank-one transformations are rationally ergodic.

We begin with a computational lemma. Suppose that T is a normal rank-one transformation. We claim that the partial sums $a_n(J)$ for any level J can be computed from the descendant heights $D(J, N)$ for N sufficiently large. More precisely, we have the following lemma.

LEMMA 2.1. *Let T be a normal rank-one transformation. Fix any level J and $n \in \mathbb{N}$. Then for every N sufficiently large,*

$$\mu(J \cap T^k J) = w_N \cdot |D(J, N) \cap (k + D(J, N))|$$

for all $0 \le k < n$. Consequently,

$$\sum_{k=0}^{n-1} \mu(J \cap T^k J) = w_N \cdot \sum_{k=0}^{n-1} |D(J, N) \cap (k + D(J, N))|.$$

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Proof. Fix any level J, and let $n \in \mathbb{N}$ be arbitrary. Since T is normal, we can find some column C_N in which all the heights $D(J, N)$ are bounded above by $h_N - n$. For any $0 \le k < n$ and level $J_i \in D(J, N)$, the image $T^k(J_i)$ is then the level in C_N of height $h(J_i) + k$. The conclusion follows immediately. □

We will sometimes need to compute $\mu(J \cap T^k J)$ for $k < 0$. For this, simply observe that

$$\mu(J \cap T^k J) = \mu(T^{-k} J \cap J)$$

and

$$|D(J, N) \cap (k + D(J, N))| = |(-k + D(J, N)) \cap D(J, N)|,$$

so in fact Lemma 2.1 holds for all $-n < k < n$.

We now calculate $D(J, N)$. Suppose that J is a level in C_j of height $h(J)$. Then J splits into r_j levels in C_{j+1} of heights

$$\{h(J)\} \cup \left\{ h(J) + \sum_{k=0}^{i} h_{j,k} : 0 \le i < r_j - 1 \right\}.$$

Letting

$$H_j = \{0\} \cup \left\{ \sum_{k=0}^{i} h_{j,k} : 0 \le i < r_j - 1 \right\},$$

it follows inductively that

$$D(J, N) = h(J) + H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1}.$$

We now show that every normal rank-one transformation satisfies condition (2) for A, B finite unions of levels and F the unit interval. In this context, we note that Aaronson has shown every set F of finite measure contains a dense algebra of sets satisfying (2) [**1**, Theorem 6.1], but at the same time it is never true that (2) is satisfied for all measurable subsets in every set F of finite positive measure [**1**, Theorem 6.2].

THEOREM 2.2. *Let T be a normal rank-one transformation. Then T satisfies condition (2) for A, B finite unions of levels and F the unit interval.*

Proof. Let $F = I$ denote the unit interval. We begin by proving (2) for $A = B = J$, where J is the bottom level of any column C_j. We need to show that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(J \cap T^k J) \to \mu(J)^2$$

as $n \to \infty$. For N sufficiently large (as a function of n),

$$\sum_{k=0}^{n-1} \mu(J \cap T^k J) = w_N \left(\sum_{k=0}^{n-1} |D(J, N) \cap (k + D(J, N))| \right)$$

by Lemma 2.1. Now, writing

$$D(I, N) = H_0 \oplus H_1 \oplus \cdots \oplus H_{N-1}$$

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and

$$D(J, N) = H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1},$$

we have $D(I, N) = U \oplus V$ with $D(J, N) = V$ and $U = H_0 \oplus H_1 \oplus \cdots \oplus H_{j-1}$. Noting that $\mu(J) = 1/|D(I, j)| = 1/|U|$, we thus wish to show that

$$\frac{w_N}{a_n(I)}\left(|U|^2 \sum_{k=0}^{n-1} |V \cap (k + V)|\right) \to 1.$$

We give the term inside the parentheses a combinatorial interpretation. Let $P(n)$ denote the number of ordered quadruplets (u, u', v, v') with $u, u' \in U$ and $v, v' \in V$ for which $0 \le v - v' < n$. Then the above quotient is precisely $w_N P(n)/a_n(I)$, since $|V \cap (k + V)|$ counts the number of pairs $v, v' \in V$ with $k = v - v'$.

Now let M be the maximum value of $U - U$. We claim that the following inequality holds:

$$\sum_{k=M}^{n-1-M} |(U \oplus V) \cap (k + U \oplus V)| \le P(n) \le \sum_{k=-M}^{n-1+M} |(U \oplus V) \cap (k + U \oplus V)|.$$

Indeed, the sum on the left counts the number of quadruplets (u, u', v, v') with $M \le u - u' + v - v' < n - M$; the sum on the right counts the number of quadruplets with $-M \le u - u' + v - v' < n + M$. Clearly, any quadruplet with $M \le u - u' + v - v' < n - M$ has $0 \le v - v' < n$. Similarly, any quadruplet with $0 \le v - v' < n$ has $-M \le u - u' + v - v' < n + M$. Recalling that $U \oplus V = D(I, N)$, it thus follows that

$$\frac{1}{a_n(I)} \sum_{k=M}^{n-1-M} \mu(I \cap T^k I) \le \frac{w_N}{a_n(I)} P(n) \le \frac{1}{a_n(I)} \sum_{k=-M}^{n-1+M} \mu(I \cap T^k I).$$

Now, M is a fixed constant, independent of n. Furthermore, the sequence $\mu(I \cap T^k I)$ is bounded above by 1 but has divergent sum. Hence both sides of the above inequality tend to 1 as $n \to \infty$, showing that $w_N P(n)/a_n(I) \to 1$, as desired. This proves (2) for $A = B = J$, where J is the bottom level of any column.

We now prove (2) for J and J' any two levels in the same column. By applying T^{-1} and using the fact that T is measure-preserving, we may assume that one of the two levels (say, J) is actually the bottom level of the column. Letting $J' = T^d(J)$ for some d, we wish to show that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(J \cap T^{k+d} J) \to \mu(J)^2.$$

Now, we have from before that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(J \cap T^k J) \to \mu(J)^2.$$

Since $\mu(J \cap T^k J)$ is bounded and $a_n(I) \to \infty$, the conclusion follows immediately.

Finally, we extend to finite unions of levels. Without loss of generality, we may assume that J and J' are both disjoint unions of images of the same level K. The desired statement then follows from summing together the limits (2) for each pair of images. □

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We now show that under certain conditions, we can extend the results of Theorem 2.2 to all subsets A and B of F, thus proving weak rational ergodicity.

THEOREM 2.3. *Let T be an exponentially growing rank-one transformation with a bounded number of cuts. Then T is weakly rationally ergodic.*

Proof. We show that for T satisfying the above hypotheses, it suffices to prove (2) for finite unions of levels (as in Theorem 2.2). Given arbitrary measurable sets $A, B \subset I$, choose $D \subset I$ a finite union of levels for which $\mu(D \triangle B) < \varepsilon$. We claim that there is some constant c such that

$$\left| \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(A \cap T^k B) - \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(A \cap T^k D) \right| \leq c\varepsilon \tag{4}$$

for every n. Let $B_0 = B \cap D$, and write B and D as the disjoint unions $B = B_0 \sqcup B_1$ and $D = B_0 \sqcup D_1$. Then the above difference reduces to

$$\left| \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(A \cap T^k B_1) - \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(A \cap T^k D_1) \right|.$$

Now, we claim that we can bound

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(A \cap T^k B_1) \leq c\mu(B_1) \tag{5}$$

for some c independent of n, A, and B_1. Applying this bound in conjunction with a similar inequality for D_1 will bound (4) by $c(\mu(B_1) + \mu(D_1)) \leq 2c\varepsilon$, as desired.

Recall that for $m \in \mathbb{N}$ we have defined $M_m = \max(D(I, m))$. Clearly, $\{M_m\}$ is an increasing sequence. For any fixed n, if we choose m such that $M_{m-1} \leq n - 1 < M_m$, then

$$\sum_{k=0}^{n-1} \mu(A \cap T^k B_1) \leq \sum_{k=0}^{n-1} \mu(I \cap T^k B_1) \leq \sum_{k=0}^{M_m} \mu(I \cap T^k B_1)$$

and

$$\sum_{k=0}^{M_{m-1}} \mu(I \cap T^k I) \leq \sum_{k=0}^{n-1} \mu(I \cap T^k I) = a_n(I).$$

To prove (5), it thus suffices to find some c such that

$$\sum_{k=0}^{M_m} \mu(I \cap T^k B_1) \leq c\mu(B_1) \sum_{k=0}^{M_{m-1}} \mu(I \cap T^k I) \tag{6}$$

for every m.

Now observe that the sets $T^k I$ with $-M_m \leq k \leq M_m$ cover each point of I exactly $|D(I, m)|$ times. Indeed, consider the column C_m and fix any $x \in I$. Let x be contained in J, where J is some level from $D(I, m)$. For any level J' in $D(I, m)$, we claim that there is exactly one value of k between $-M_m$ and M_m for which $T^k J' \cap J \neq \emptyset$. Suppose that $0 \leq k \leq M_m$ and $T^k J' \cap J \neq \emptyset$. Any forward image $T^k J'$ with $0 \leq k \leq M_m$ is just a translation upwards by k levels, since $h_m \geq 2M_m$. (This is implied by our hypothesis

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that T is exponentially growing.) Hence in this case k must equal $h(J) - h(J')$. On the other hand, suppose that $-M_m \le k < 0$ and $T^k J' \cap J \neq \emptyset$. Then $J' \cap T^{-k} J \neq \emptyset$, and exactly the same argument shows that $-k = h(J') - h(J)$ (i.e., $k = h(J) - h(J')$). The claim is then immediate.

We thus have

$$\sum_{k=-M_m}^{M_m} \mu(I \cap T^k B_1) = \sum_{k=-M_m}^{M_m} \mu(T^k I \cap B_1) = |D(I, m)|\mu(B_1)$$

and

$$\sum_{k=-M_{m-1}}^{M_{m-1}} \mu(I \cap T^k I) = |D(I, m-1)|.$$

Hence

$$\sum_{k=-M_m}^{M_m} \mu(I \cap T^k B_1) = \left(\frac{|D(I, m)|}{|D(I, m-1)|}\right)\mu(B_1)\left(\sum_{k=-M_{m-1}}^{M_{m-1}} \mu(I \cap T^k I)\right)$$

and so

$$\sum_{k=0}^{M_m} \mu(I \cap T^k B_1) \le \left(\frac{|D(I, m)|}{|D(I, m-1)|}\right)\mu(B_1)\left(2\left(\sum_{k=0}^{M_{m-1}} \mu(I \cap T^k I)\right) - 1\right).$$

But $|D(I, m)|/|D(I, m-1)| = r_{m-1}$, and T has a bounded number of cuts. We thus easily obtain (6). Hence (4) holds, and we can approximate B with D a finite union of levels. Applying a similar argument to A shows that it suffices to prove (2) for all A, B finite unions of levels, which is the content of Theorem 2.2. □

We now consider some alternate notions of rational ergodicity, also due to Aaronson [**1**]. For any measurable function f, write

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k.$$

We say that T is *rationally ergodic* if it is conservative ergodic and there exists a set F of positive finite measure that satisfies a *Rényi inequality*; i.e., there is some constant M such that

$$\int_F (S_n(1_F))^2 \, dm \le M\left(\int_F S_n(1_F) \, dm\right)^2 \tag{7}$$

for every $n \in \mathbb{N}$. If this inequality holds only on a subset $\{n_i\} \subset \mathbb{N}$, we say that T is *subsequence rationally ergodic*. Some authors adopt this as the definition of rational ergodicity instead (see, for example, [**10**]). It was shown in [**1**] that rational ergodicity implies weak rational ergodicity. It is not currently known whether these notions are equivalent.

We say that T is *boundedly rationally ergodic* (see [**2**]) if it is conservative ergodic and there exists a set F of positive finite measure such that

$$\sup_{n \ge 1} \left\| \frac{1}{a_n(F)} S_n(1_F) \right\|_\infty < \infty.$$

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In [**2**] it was shown that bounded rational ergodicity is a strictly stronger property than rational ergodicity. It is not difficult to see that the proof of Theorem 2.3 (in particular, the establishment of (5) for all B_1) yields bounded rational ergodicity for the transformations in question. Indeed, set $A = I$ in (5). Then there is a constant c such that for any n and $B_1 \subset I$,

$$\int_{B_1} \frac{1}{a_n(I)} S_n(1_I) dm = \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(I \cap T^k B_1) \le c\mu(B_1).$$

This means that the average value of $S_n(1_I)/a_n(I)$ on B_1 is bounded above by c. Since this holds for every B_1, the essential supremum of $S_n(1_I)/a_n(I)$ must also be bounded above by c. Hence T is boundedly rationally ergodic.

Aaronson proved in [**2**] that every dyadic tower over the adding machine is boundedly rationally ergodic; Theorem 2.3 extends this result to a larger class of transformations and uses a different approach. Some interesting examples of exponentially growing rank-one transformations with a bounded number of cuts are as follows.

(a) Hajian–Kakutani skyscraper-type constructions [**14**]:

$$r_n = 2, \quad \{s_{n,0} = 0, s_{n,1} \ge 2h_n\}.$$

(When $s_{n,1} = 2h_n + 1$ the transformation is spectral weakly mixing, see [**7**].)

(b) Chacón-like constructions:

$$r_n = 3, \quad \{s_{n,0} = 0, s_{n,1} = 1, s_{n,2} \ge 3h_n + 1\}.$$

(When $s_{n,2} = 3h_n + 1$ the transformation has infinite ergodic index, see [**7**], but is not power weakly mixing, see [**12**].)

We now prove a slightly different version of Theorem 2.3 without the hypothesis of a bounded number of cuts but only obtain rational ergodicity along a subsequence.

THEOREM 2.4. *Let T be an exponentially growing rank-one transformation. Then T is subsequence rationally ergodic on $F = I = (0, 1)$ along the sequence $\{n_m = M_m + 1\}$.*

Proof. We verify the Rényi inequality (7) with $n = M_m + 1$ and $M = 2$. Let $D(I, m) = \{I_j\}$ be the descendants of I in column C_m, and set $N = |D(I, m)|$. Order $\{I_j\}$ by height of appearance in C_m so that I_1 is the lowermost level of $\{I_j\}$ in C_m and I_N is the uppermost. Denote the heights of $\{I_j\}$ in C_m by $\{h(I_j)\}$.

Now $S_n(1_I)(x)$ is equal to the number of k with $0 \le k \le n - 1$ such that $T^k(x) \in I$. Since T is exponentially growing, the descendants of I in C_m all lie in the lower half of C_m (i.e., $M_m \le C_m/2$). As $n = M_m + 1$, this implies that $S_n(1_I)$ is constant on each I_j and that the value of $S_n(1_I)$ on any fixed I_l is the cardinality of the intersection $(h(I_l) + \{0, 1, \ldots, M_m\}) \cap \{h(I_j)\}$. (The relevant forward images $T^k I_l$ are simply upward translations.) On the other hand, it is obvious that $h(I_j) \in (h(I_l) + \{0, 1, \ldots, M_m\})$ exactly when $j \ge l$. Hence $S_n(1_I)$ takes the value $N + 1 - l$ on I_l. Restricting the domain of $S_n(1_I)$ to I, we thus have

$$S_n(1_I) = \sum_{l=1}^{N} (N + 1 - l) 1_{I_l}.$$

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Proving (7) is thus equivalent to showing that

$$\sum_{l=1}^{N}(N+1-l)^2 w_m \le 2\left(\sum_{l=1}^{N}(N+1-l)w_m\right)^2.$$

Now $w_m = 1/|D(I,m)| = 1/N$. Multiplying through by N^2 and reindexing yields the equivalent inequality

$$N\left(\sum_{l=1}^{N} l^2\right) \le 2\left(\sum_{l=1}^{N} l\right)^2.$$

The result then follows from the formulas for power sums. □

3. *Rational weak mixing*

In this section we present a large class of transformations that are not rationally weakly mixing. We obtain as a corollary the existence of transformations that are rationally ergodic and spectrally weakly mixing but not rationally weakly mixing.

We begin with an example of a rank-one transformation that is subsequence rationally weakly mixing.

First, consider the Chacón rank-one transformation T constructed by starting with the unit interval, cutting each column in half, and adding a single spacer on top of the right subcolumn at every step [**9**]. This transformation is finite measure-preserving and weakly mixing; thus, it is rationally weakly mixing by taking F to be the entire domain. We claim that in fact T is rationally weakly mixing on the unit interval $I = (0, 1)$.

It follows from the definition of weak rational ergodicity that if T is weakly rationally ergodic on F, then T is weakly rationally ergodic on any subset of F. To see this, let $\tilde{F}$ be any subset of F and take $A = B = \tilde{F}$ in (2). Dividing both sides by $\mu(\tilde{F})^2$ shows that $a_n(\tilde{F})/a_n(F) \to 1$ as $n \to \infty$; hence in general we may pass from the convergence of (2) with $a_n(F)$ to the convergence of (2) with $a_n(\tilde{F})$. Moreover, it was shown in [**5**] that for T rationally weakly mixing, the class of sets F satisfying (2) is the same as the class of sets F satisfying (3). This establishes the claim.

Now let

$$\phi_n(A, B) = \frac{1}{a_n(I)} \sum_{k=0}^{n-1} |\mu(A \cap T^k B) - \mu(A)\mu(B)u_k(I)|$$

be the quotient from (3), and let D_m denote the collection of dyadic intervals of the form $(i/2^m, (i+1)/2^m)$ for $0 \le i < 2^m$. Since D_1 is a finite collection and T is rationally weakly mixing, there exists some natural number m_1 such that for all $A, B \in D_1$ we have $\phi_{m_1}(A, B) < 1/2$. We claim that in fact this inequality is true for every rank-one transformation $\tilde{T}$ which shares its first m_1 stages of construction with T (i.e., $\tilde{C}_n = C_n$ for all $n < m_1$). Indeed, for $A, B \subset I$, the value of $\phi_{m_1}(A, B)$ depends only on the first m_1 stages of the construction of T, since the heights $D(I, m_1)$ are all less than $h_{m_1} - m_1$.

We now define our desired transformation. We begin by following the construction of the transformation T as described above, until we reach C_{m_1}. Then, at the m_1th iteration, we add $2h_{m_1}$ spacers above the right subcolumn. Now, adding one spacer at each subsequent iteration gives another finite measure-preserving transformation, which

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is also weakly mixing. Hence, there is some $m_2 > m_1$ such that $\phi_{m_2}(A, B) < 1/4$ for all $A, B \in D_1 \cup D_2$.

We thus continue adding a single spacer at each step until we reach C_{m_2}, at which point we add $2h_{m_2}$ spacers. Proceeding inductively in this manner, we obtain a cutting-and-stacking transformation T and a sequence $\{m_i\}$ such that for each i, $\phi_{m_i}(A, B) < 1/2^i$ for all $A, B \in D_1 \cup D_2 \cup \cdots \cup D_i$. The result is an invertible, infinite measure-preserving transformation that is rationally weakly mixing along $\{m_i\}$ for dyadic intervals.

In order to extend to all subsets of I, we use the following result due to Aaronson [**5**].

LEMMA 3.1. *Let T be an invertible measure-preserving transformation on a Polish space X, and assume that T is rationally ergodic on some open set F. Suppose that there is a countable base $\mathcal{C}$ for the topology of F such that for every finite subcollection $\{C_i\} \subset \mathcal{C}$, there exists a finite subcollection $\{D_i\} \subset \mathcal{C}$ which is disjoint and has the same union. Then to establish rational weak mixing, it suffices to prove condition (3) for elements of $\mathcal{C}$.*

Lemma 3.1 also holds for establishing subsequence rational weak mixing, so long as rational ergodicity is known along the same subsequence. Since the transformation T above may expressed as a dyadic tower over the adding machine, T is rationally ergodic [**2**]. It follows from Lemma 3.1 that T is subsequence rationally weakly mixing.

We now present a large class of examples that are not rationally weakly mixing. It will be convenient to write

$$u_k(A, B) = \frac{\mu(A \cap T^k B)}{\mu(A)\mu(B)}$$

so that for A, B of positive measure, we can divide (3) by $\mu(A)\mu(B)$ to obtain

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(A, B) - u_k(F)| \to 0.$$

It is not difficult to see that in this case we must have $a_n(F)/a_n(A, B) \to 1$ (see [**5**]). This yields the following theorem.

THEOREM 3.2. *Let T be a rank-one transformation constructed by cutting C_n in half and adding at least $c_n \geq 2h_n$ spacers on top of the right subcolumn at every step. Then T is not rationally weakly mixing.*

Proof. We prove by contradiction. Suppose that T were rationally weakly mixing on some set F. Choose a level J that is at least $(3/4)$-full of F, and let J_1 and J_2 be the left and right halves of J, respectively. By applying T^{-1} to F, we may assume that J is the bottom level of some column C_j. Now, both J_1 and J_2 intersect F in positive measure, so by the discussion preceding the theorem we have

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F) - u_k(F)| \to 0$$

by taking $A = B = J_1 \cap F$, and

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F, J_2 \cap F) - u_k(F)| \to 0$$

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by taking $A = J_1 \cap F$ and $B = J_2 \cap F$. Moreover, $a_n(F)/a_n(J_1 \cap F) \to 1$. Multiplying through by this limit and using the triangle inequality, we obtain

$$\frac{1}{a_n(J_1 \cap F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F, J_2 \cap F) - u_k(J_1 \cap F)| \to 0. \tag{8}$$

Now, fix k and suppose that $u_k(J_1 \cap F) > 0$. Then $\mu(J_1 \cap T^k J_1) > 0$, so for sufficiently large N we have $k \in D(J_1, N) - D(J_1, N)$. Similarly, if $u_k(J_1 \cap F, J_2 \cap F) > 0$, then $\mu(J_1 \cap T^{k+h_j} J_1) = \mu(J_1 \cap T^k J_2) > 0$, which implies that $k + h_j \in D(J_1, N) - D(J_1, N)$. Hence we cannot have both $u_k(J_1 \cap F)$ and $u_k(J_1 \cap F, J_2 \cap F)$ non-zero, since then we would have $h_j \in (D(J_1, N) - D(J_1, N)) - (D(J_1, N) - D(J_1, N))$. As $D(J_1, N) = \{0, h_{j+1}\} \oplus \{0, h_{j+2}\} \oplus \cdots \oplus \{0, h_{N-1}\}$, this is easily seen to be impossible (given the fact that $c_n \geq 2h_n$ for all n).

It is then immediate that

$$|u_k(J_1 \cap F, J_2 \cap F) - u_k(J_1 \cap F)| \geq u_k(J_1 \cap F)$$

for every k. Indeed, if $u_k(J_1 \cap F) = 0$ then we are done; otherwise, $u_k(J_1 \cap F, J_2 \cap F)$ is 0. It follows that the quotient (8) is bounded below by 1, which is a contradiction. This shows that T is not rationally weakly mixing. □

In particular, we obtain the following corollary.

COROLLARY 3.3. *Let T be the transformation constructed by cutting each column C_n in half and adding $2h_n + 1$ spacers on top of the right subcolumn. Then T is rationally ergodic and spectrally weakly mixing but not rationally weakly mixing.*

Proof. Rational ergodicity follows from Theorem 2.3 and the subsequent discussion (see also [**2**, Theorem 1]), and spectral weak mixing follows from [**7**, Proposition 1.1]. By Theorem 3.2, however, T is not rationally weakly mixing. □

This negatively answers a question of Aaronson's. (As noted previously, this result was obtained independently by Aaronson in [**4**, §1.1].)

We now extend Theorem 3.2 to other rank-one transformations. Define

$$H = \bigcup_{j=0}^{\infty} H_j \backslash \{0\}$$

and observe that the elements of H are increasing when listed in the obvious order. (Begin with successive elements of $H_0\backslash\{0\}$, followed by successive elements of $H_1\backslash\{0\}$, and so on.) We say that a rank-one transformation is *steep* if $t_{i+1} \geq 4t_i$ for every pair of successive $t_i, t_{i+1} \in H$. Clearly, the transformations of Theorem 3.2 are steep. In general, such transformations can be constructed by adding an exponentially increasing number of spacers above successive subcolumns.

Steep transformations satisfy several nice properties, chief among which is a linear independence condition that allows us to extend Theorem 3.2. Suppose that we have a linear combination

$$\sum_{t \in H} c_t t = 0 \tag{9}$$

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with the coefficients $c_t \in \{-2, -1, 0, 1, 2\}$. Then it is easily seen that all the c_t must be 0. Similarly, we also obtain a uniqueness condition that will be useful in §5: every integer k has at most one representation

$$k = \sum_{t \in H} c_t t$$

with the $c_t \in \{-1, 0, 1\}$. Altering the definition of steepness slightly yields stronger forms of these properties; for example, requiring $t_{i+1} \geq 5t_i$ results in uniqueness of representation with $c_t \in \{-2, -1, 0, 1, 2\}$.

THEOREM 3.4. *Let T be a normal, steep rank-one transformation. Then T is not rationally weakly mixing.*

Proof. We sketch the proof and leave the details to the reader. As before, we proceed by contradiction. Suppose that T is rationally weakly mixing on F, and let J be a level $(3/4)$-full of F. Without loss of generality, we may assume that J is the bottom level of some column C_j. Now, there must exist at least two descendants J_1 and J_2 of J in C_{j+1} that have positive intersection with F. For these levels, we have $J_2 = T^d J_1$ for some $d \in H_j - H_j$. It then suffices to show that d cannot be contained in $(D(J_1, N) - D(J_1, N)) - (D(J_1, N) - D(J_1, N))$, which follows from the linear independence (9). □

4. *Relation to double ergodicity*

In this section we show that rational weak mixing implies double ergodicity and present an example suggesting the converse implication is false.

We begin by proving that rational weak mixing on F implies double ergodicity for subsets of F.

THEOREM 4.1. *Suppose that T is rationally weakly mixing on F. Then T is doubly ergodic for all $A, B \subset F$.*

Proof. Let $A, B \subset F$ be of positive measure, and fix $\delta > 0$ such that

$$\delta < \tfrac{1}{2} \min(\mu(A)^2, \mu(A)\mu(B)).$$

Since T is rationally weakly mixing on F,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |\mu(A \cap T^k A) - \mu(A)^2 u_k(F)| \to 0$$

and

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |\mu(A \cap T^k B) - \mu(A)\mu(B) u_k(F)| \to 0.$$

Summing these together, we obtain (by contradiction) that there exists a positive integer k for which $u_k(F) > 0$ and

$$|\mu(A \cap T^k A) - \mu(A)^2 u_k(F)| + |\mu(A \cap T^k B) - \mu(A)\mu(B) u_k(F)| < \delta u_k(F).$$

We thus have

$$|\mu(A \cap T^k A) - \mu(A)^2 u_k(F)| < \delta u_k(F)$$

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and so

$$\mu(A \cap T^k A) > u_k(F)(\mu(A)^2 - \delta) > 0$$

for this k. Similarly,

$$\mu(A \cap T^k B) > u_k(F)(\mu(A)\mu(B) - \delta) > 0.$$

By construction of δ, this shows that T is doubly ergodic on F. □

We now extend this result to all of X. It was shown in [**1**] that if T is weakly rationally ergodic on F, it is weakly rationally ergodic on any finite union $F_N = F \cup T(F) \cup \cdots \cup T^{N-1}(F)$. It follows that the analogous statement holds for rational weak mixing, giving the following theorem.

THEOREM 4.2. *Suppose that T is rationally weakly mixing. Then T is doubly ergodic.*

Proof. Let T be rationally weakly mixing on F, and suppose that T is not doubly ergodic. Fix $A, B \subset X$ for which the double ergodicity condition fails; i.e., choose A and B such that for every n, either $\mu(A \cap T^n A) = 0$ or $\mu(A \cap T^n B) = 0$. Since F sweeps out X, there is some N for which F_N intersects both A and B in positive measure. Then $\tilde{A} = F_N \cap A$ and $\tilde{B} = F_N \cap B$ are sets of positive measure that fail the double ergodicity condition. But T is doubly ergodic on F_N, a contradiction. □

It is worth noting that (in general) the class of sets on which T is doubly ergodic is not a hereditary ring. For example, let T be any doubly ergodic transformation on X, and define S on $X \times \{0, 1\}$ by $S(x, 0) = (T(x), 1)$ and $S(x, 1) = (x, 0)$. Then S is doubly ergodic on both $X \times \{0\}$ and $X \times \{1\}$, but not doubly ergodic on all of $X \times \{0, 1\}$. (Let $A = X \times \{0\}$ and $B = X \times \{1\}$.)

We now investigate whether rational weak mixing is strictly stronger than double ergodicity. It will be useful for us consider transformations that are 'almost' steep. Recall that T is steep if for any pair of successive elements t_i, t_{i+1} in $H = (H_0 \cup H_1 \cup \cdots)\backslash\{0\}$, we have $t_{i+1} \geq 4t_i$. Now, suppose that T is constructed so that:

(a) each column C_n is cut into at least three subcolumns ($r_n \geq 3$);
(b) we add zero spacers above the first subcolumn and one spacer above the second ($s_{n,0} = 0$ and $s_{n,1} = 1$);
(c) we add a sufficient number of spacers above each subsequent subcolumn so that

$$\sum_{k=0}^{i} h_{n,k} \geq 4\left(\sum_{k=0}^{i-1} h_{n,k}\right)$$

for every $2 \leq i \leq r_n - 1$.

Then T is 'almost' steep, in the sense that $t_{i+1} < 4t_i$ only when t_i and t_{i+1} are the first two non-zero elements of some H_n. For such T, we can still extract a (slightly technical) algebraic uniqueness condition in the spirit of (9). Indeed, let

$$B_n = \{h_{n,0}, h_{n,0} + h_{n,1}\} \times \{h_{n,0}, h_{n,0} + h_{n,1}\}$$

and define

$$A_n = (H_n \times H_n)\backslash(\Delta H_n \cup B_n).$$

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Here, $\Delta H_n = \{(x, x) : x \in H_n\}$. Then for any $(a, b), (a', b') \in A_n$ and $-M_n \le k, k' \le M_n$, the equality

$$k + a - b = k' + a' - b' \tag{10}$$

implies

$$a = a', \quad b = b', \quad k = k'.$$

The proof of this is not difficult and is left to the reader. Before we proceed, it will be useful to establish following lemma.

LEMMA 4.3. *Let J be any level, and fix N sufficiently large. Suppose that $(a, b) \in A_N$ and $-M_N \le k \le M_N$. Then*

$$\mu(J \cap T^{k+a-b} J) = \frac{1}{r_N} \mu(J \cap T^k J).$$

Proof. By Lemma 2.1,

$$\mu(J \cap T^k J) = w_N \cdot |D(J, N) \cap (k + D(J, N))|$$

and

$$\mu(J \cap T^{k+a-b} J) = w_{N+1} |D(J, N+1) \cap (k + a - b + D(J, N+1))|.$$

By uniqueness of (10), every representation of $k + a - b$ as an element of $D(J, N+1) - D(J, N+1)$ corresponds to exactly one representation of k as an element of $D(J, N) - D(J, N)$, and vice versa. Hence

$$\mu(J \cap T^{k+a-b} J) = \frac{w_{N+1}}{w_N} \mu(J \cap T^k J) = \frac{1}{r_N} \mu(J \cap T^k J),$$

as desired. □

We now show that if T is almost steep and $\{r_n\}$ is sufficiently large, T cannot be rationally weakly mixing.

THEOREM 4.4. *Let T be a rank-one transformation. Suppose that T is almost steep (as described above), and that*

$$\sum_{n=0}^{\infty} \frac{1}{r_n} < \infty.$$

Then T is not rationally weakly mixing.

Proof. We begin by proving that T is not rationally weakly mixing on levels. Let J be the bottom level of any column C_j, and let J_1 and J_2 be any two descendants of J in C_{j+1}. Then $J_2 = T^d J_1$ for some $d \in H_j - H_j$. As in Theorem 3.2, it suffices to disprove the convergence

$$\frac{1}{a_n(J_1)} \sum_{k=0}^{n-1} |u_k(J_1) - u_k(J_1, J_2)| \to 0. \tag{11}$$

To do this, define

$$P_m = \sum_{k=-M_m}^{M_m} |\mu(J_1 \cap T^k J_1) - \mu(J_1 \cap T^{k+d} J_1)|$$

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and

$$Q_m = \sum_{k=-M_m}^{M_m} \mu(J_1 \cap T^k J_1).$$

For m sufficiently large, $R_m = P_m/Q_m$ approximates the quotient (11), so it is enough to show that R_m is bounded below by some positive constant. Let

$$\widetilde{P}_m = \sum_{k=-M_m}^{M_m-d} |\mu(J_1 \cap T^k J_1) - \mu(J_1 \cap T^{k+d} J_1)|.$$

Any choice of $(a, b) \in A_m$ and $-M_m \le k \le M_m - d$ yields a unique number $k + a - b$ between $-M_{m+1}$ and $M_{m+1} - d$. Hence

$$\begin{aligned}
\widetilde{P}_{m+1} &= \sum_{k=-M_{m+1}}^{M_{m+1}-d} |\mu(J_1 \cap T^k J_1) - \mu(J_1 \cap T^{k+d} J_1)| \\
&\ge \sum_{(a,b)\in A_m} \sum_{k=-M_m}^{M_m-d} |\mu(J_1 \cap T^{k+a-b} J_1) - \mu(J_1 \cap T^{k+a-b+d} J_1)| \\
&= \frac{1}{r_m}\Bigg(\sum_{(a,b)\in A_m} \sum_{k=-M_m}^{M_m-d} |\mu(J_1 \cap T^k J_1) - \mu(J_1 \cap T^{k+d} J_1)| \Bigg) \\
&= \frac{|A_m|}{r_m} \widetilde{P}_m,
\end{aligned}$$

where in the third line we have used Lemma 4.3. Moreover, the same argument as in Theorem 2.3 shows that

$$Q_m = \sum_{k=-M_m}^{M_m} \mu(J_1 \cap T^k J_1) = |D(J_1, m)|\mu(J_1),$$

from which it follows that

$$Q_{m+1} = r_m Q_m.$$

Since $\widetilde{P}_m/P_m \to 1$ as $m \to \infty$, we obtain

$$R_{m+1} \ge \frac{|A_m|}{r_m^2} R_m$$

in the limit of large m. Now, $|A_m| = r_m^2 - r_m - 2$, so R_m is asymptotically bounded below by a constant multiple of

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{r_k} - \frac{2}{r_k^2}\right).$$

This is strictly positive by the hypotheses of the theorem. Hence (10) is bounded from below along the sequence $\{M_m + 1\}$. This shows that T is not rationally weakly mixing on levels.

To pass from levels to arbitrary sets, we apply the discussion at the beginning of §3. By Theorem 2.4, T is subsequence rationally ergodic on $F = I$ along $\{M_m + 1\}$. Thus, if T were rationally weakly mixing, then in particular it would be rationally weakly mixing along $\{M_m + 1\}$ on the same set F. But the fact that this is not the case is precisely the content of the above remarks. □

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We now show that T is doubly ergodic for levels, suggesting that rational weak mixing is strictly stronger than double ergodicity.

LEMMA 4.5. *The transformation T above is doubly ergodic for levels.*

Proof. We check that for any pair of levels A and B in the same column, there exists an integer n such that both $\mu(A \cap T^n A) > 0$ and $\mu(B \cap T^n A) > 0$. Without loss of generality, let A and B be levels in C_j with $A = T^d B$. It then suffices to prove that there exists an n such that both n and $n + d$ are in $D(A, N) - D(A, N)$ for N sufficiently large. This is easy; choose

$$n = h_{j+1,0} + \cdots + h_{j+d,0}$$

and let $h(A)$ be the height of A in C_j. Clearly,

$$\begin{aligned} x &= h(A), \\ y &= h(A) + h_{j+1,0} + \cdots + h_{j+d,0}, \quad \text{and} \\ z &= h(A) + (2h_{j+1,0} + 1) + \cdots + (2h_{j+d,0} + 1) \end{aligned}$$

are all elements of $D(A, N) = h(A) \oplus H_{j+1} \oplus \cdots \oplus H_{j+d}$. Writing $n = y - x$ and $n + d = z - y$ establishes that T is doubly ergodic for two levels in the same column; the passage to arbitrary pairs of levels is immediate. □

5. *Independence from zero-type*

We now show that (subsequence) rational weak mixing and zero-type are independent (i.e., do not imply each other). We say that T is *zero-type* if $\mu(A \cap T^n A) \to 0$ for all sets A of finite measure [**13**]. It is well known that in order to show that a conservative ergodic transformation is zero-type, it suffices to check this convergence for a single set A of positive finite measure [**13**]. We show that every steep transformation with an increasing number of cuts is zero-type.

THEOREM 5.1. *Let T be a normal, steep rank-one transformation, and suppose that $\{r_n\}$ is nondecreasing with $\sup\{r_n\} = \infty$. Then T is zero-type.*

Proof. Consider $I = (0, 1)$. For N sufficiently large,

$$\mu(I \cap T^k I) = \frac{|D(I, N) \cap (k + D(I, N))|}{|D(I, N)|}.$$

Now, $|D(I, N) \cap (k + D(I, N))|$ counts the number of representations

$$k = \sum_{i=0}^{N-1} (d_i - d_i') \tag{12}$$

with $d_i, d_i' \in H_i$. (Recall that $D(I, N) = H_0 \oplus H_1 \oplus \cdots \oplus H_{N-1}$.) If $k \notin D(I, N) - D(I, N)$, then $\mu(I \cap T^k I) = 0$, so suppose that $k \in D(I, N) - D(I, N)$. Then there is at least one representation

$$k = \sum_{i=0}^{N-1} (x_i - x_i') \tag{13}$$

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with $x_i, x_i' \in H_i$. Now fix n and suppose that $x_n - x_n' \neq 0$. By uniqueness of representation, any other representation (12) of k must have $d_n = x_n$ and $d_n' = x_n'$. In particular, the only indices i at which (12) can differ from (13) are those for which $x_i - x_i' = 0$. In these cases we must have $d_i = d_i'$, but otherwise there are no restrictions (i.e., $d_i = d_i'$ can be any element of H_i). Hence

$$|D(I, N) \cap (k + D(I, N))| = \prod_{x_i - x_i' = 0} |H_i|$$

with the product being taken over all i for which $x_i - x_i' = 0$. Since

$$|D(I, N)| = \prod_{i=0}^{N-1} |H_i|$$

it follows that

$$\mu(I \cap T^k I) = \left(\prod_{x_i - x_i' \neq 0} |H_i| \right)^{-1}.$$

Now, if $k > M_n$, then the representation (13) of k must have $x_m - x_m' \neq 0$ for some $m \geq n$. This implies that

$$\mu(I \cap T^k I) \leq \frac{1}{|H_m|} = \frac{1}{r_m} \leq \frac{1}{r_n},$$

which shows $\mu(I \cap T^k I) \to 0$ as $k \to \infty$. Hence T is zero-type, as desired. □

We thus have the following theorem.

THEOREM 5.2. *There exist rank-one transformations that are zero-type but not rationally weakly mixing.*

In [**4**] Aaronson recently constructed a zero-type transformation of the form $T \times S$, where S is a Markov shift, such that $T \times S$ is not subsequence rationally weakly mixing. Our examples, however, are rank-one, so of a different nature, and were constructed independently.

We note that it follows from Aaronson [**5**, Theorem F] that there exist subsequence rationally weakly mixing transformations of positive-type; a rank-one example is given by the subsequence rationally weakly mixing transformation of §3. (Indeed, this is partially rigid since $\mu(I \cap T^{h_i} I) \geq 1/2$ for every i.) Aaronson [**4**] also constructed positive-type, rank-one, transformations that are not subsequence rationally ergodic.

6. *Examples of rational weak mixing*

We end with a construction of a positive-type, rank-one transformation that is rationally weakly mixing. Consider any Chacón-like transformation ($r_n = 3$, $\{s_{n,0} = 0, s_{n,1} = 1, s_{n,2} \geq 3h_n + 1\}$) with enough spacers added above every third subcolumn so as to have $h_{n+1} = 3^c h_n$ for some fixed $c \geq 2$. Then $h_n = 3^{cn}$ and $D(I, n) = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1}$, where $H_i = \{0, h_i, 2h_i + 1\}$.

THEOREM 6.1. *Any transformation T from the above class is rationally weakly mixing.*

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Proof. We prove rational weak mixing for levels. Fix $j \in \mathbb{N}_0$ and let $J = J_1$ be the bottom level of the jth column C_j. Let J_2 be the dth level of C_j, so that $J_2 = T^d J_1$. As in the proof of Theorem 4.4, it suffices to show the convergence (11) for arbitrary j and d. Now, for any n, we may choose m such that $M_{m-1} \le n - 1 < M_m$. Then the quotient (11) is asymptotically bounded above by $P_m/Q_{m-1} = 3P_m/Q_m$, so it suffices to prove that $P_m/Q_m \to 0$ as $m \to \infty$.

By the triangle inequality,

$$\begin{aligned} P_m &= \sum_{k=-M_m}^{M_m} |\mu(J \cap T^k J) - m(J \cap T^{k+d} J)| \\ &\le \sum_{\ell=0}^{d-1} \sum_{k=-M_m}^{M_m} |\mu(J \cap T^{k+\ell} J) - m(J \cap T^{k+1+\ell} J)|. \end{aligned}$$

Since each of the d outer sums on the right differs from the $\ell = 0$ sum by a finite number of terms, it suffices to prove the convergence with only the $\ell = 0$ sum. That is, we show that

$$\frac{\sum_{k=-M_m}^{M_m} |\mu(J \cap T^k J) - \mu(J \cap T^{k+1} J)|}{\sum_{k=-M_m}^{M_m} \mu(J \cap T^k J)} \to 0.$$

To this end, we introduce the following combinatorial auxiliary functions.

Vector expression function. Fix $m \ge j$. We define a function

$$g : D(J, m) \times D(J, m) \to \{-2, -1, 0, 1, 2\}^{m-j}$$

as follows. For any $d \in D(J, m)$, write

$$d = \sum_{i=j}^{m-1} d_i$$

with each $d_i \in H_i = \{0, h_i, 2h_i + 1\}$. It is easily seen that this representation is unique. Given two representations of d, subtracting one from the other shows that it suffices to prove $h_n > \sum_{k=j}^{n-1} (2h_k + 1)$ for all $n > j$. This follows immediately by induction and the fact that $h_n = 3^c h_{n-1}$ for all $n > j$.

We may thus view d as a vector $v = (v_i)_{i=j}^{m-1} \in \{0, 1, 2\}^{m-j}$ by setting $v_i = 0$ if $d_i = 0$, $v_i = 1$ if $d_i = h_i$, and $v_i = 2$ if $d_i = 2h_i + 1$. Let v be the vector associated with d and v' be the vector associated with d'. We then define $g(d, d')$ to be the vector $\varepsilon = v - v' \in \{-2, -1, 0, 1, 2\}^{m-j}$. Explicitly, given $(d, d') \in D(J, m) \times D(J, m)$, we have

$$d - d' = \left(\sum_{i=j}^{m-1} d_i \right) - \left(\sum_{i=j}^{m-1} d'_i \right). \tag{14}$$

Replacing each instance of $2h_i + 1$ with $2h_i$ in the above expression yields a sum of the form

$$\sum_{i=j}^{m-1} \varepsilon_i 3^{ci}, \tag{15}$$

with each $\varepsilon_i \in \{-2, -1, 0, 1, 2\}$. The components of ε are then the above ε_i.

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Multiplicity function. Fix $m \geq j$ and let $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$. Observe that in general $g^{-1}(\varepsilon)$ will have multiple elements. For example, if $\varepsilon_i = 1$, then we can have either $d_i = h_i$ and $d_i' = 0$ or $d_i = 2h_i + 1$ and $d_i' = h_i$. In the first case $d_i - d_i' = h_i$, while in the second $d_i - d_i' = h_i + 1$. Hence different preimages (d, d') of ε need not have the same value of $d - d'$. It will be useful to subdivide $g^{-1}(\varepsilon)$ based on the values of $d - d'$; to this end, we define the multiplicity function

$$\tilde{\varepsilon} : D(J, m) - D(J, m) \to \mathbb{N}_0$$

by

$$\tilde{\varepsilon}(k) = |g^{-1}(\varepsilon) \cap \{(d, d') | d, d' \in D(J, m) \text{ and } d - d' = k\}|.$$

That is, $\tilde{\varepsilon}(k)$ counts the number of pairs (d, d') in $g^{-1}(\varepsilon)$ with $d - d' = k$.

Note that both the above functions are implicitly functions of m, but we have suppressed this dependence for brevity.

LEMMA 6.2. *The following properties hold.*

(a) *Fix $k \in D(J, m) - D(J, m)$. Then*

$$\sum_{\varepsilon} \tilde{\varepsilon}(k) = |D(J, m) \cap (k + D(J, m))|,$$

where the sum on the left is taken over all $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$.

(b) *Fix $\varepsilon = (\varepsilon_i)_{i=j}^{m-1} \in \{-2, -1, 0, 1, 2\}^{m-j}$. For each $p \in \{-2, -1, 0, 1, 2\}$, let a_p be the number of ε_i equal to p. Then*

$$\sum_{k} \tilde{\varepsilon}(k) = 3^{a_0} 2^{a_1 + a_{-1}},$$

where the sum on the left is taken over all $k \in D(J, m) - D(J, m)$.

Proof. For (a), simply observe that both the left- and right-hand expressions count the number of pairs $(d, d') \in D(J, m) \times D(J, m)$ for which $d - d' = k$. For (b), observe that the sum on the left counts the number of pairs (d, d') whose associated vector is ε. Now, if $\varepsilon_i = 0$ in (15), then we must have $d_i = d_i'$ in (14), and there are three ways that this can happen. Similarly, if $\varepsilon_i = 1$, then either $d_i = h_i$ and $d_i' = 0$ or $d_i = 2h_i + 1$ and $d_i' = h_i$. Proceeding in this manner, a counting argument yields the desired equality. □

(Proof of Theorem 6.1, continued.) Applying Lemmas 6.2(a) and 2.1, we thus need to show that

$$\frac{\sum_{k=-M_m}^{M_m} |\sum_{\varepsilon} \tilde{\varepsilon}(k) - \sum_{\varepsilon} \tilde{\varepsilon}(k+1)|}{\sum_{k=-M_m}^{M_m} \sum_{\varepsilon} \tilde{\varepsilon}(k)} \to 0 \tag{16}$$

as $m \to \infty$. (Here, the inner summations are again taken over all $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$.) By the triangle inequality, it suffices to prove the convergence after exchanging the order of summation in both the numerator and denominator. To this end, we exhibit a non-increasing function c which converges to 0 such that

$$R(m, \varepsilon) := \frac{\sum_{k=-M_m}^{M_m} |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)|}{\sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)} \leq c(a_1 + a_{-1}) \tag{17}$$

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for each m and $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$. Once we have such a function, we obtain the following bound for large enough N:

$$\begin{aligned}\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)| &\le \sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k) c(a_1 + a_{-1}) \\ &\le c(N) \sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} \ge N}} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k) \\ &\quad + c(0) \sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < N}} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k).\end{aligned}$$

Dividing this by the denominator of (16) yields

$$\frac{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)|}{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)} \le c(N) + c(0) d(N, m),$$

where

$$d(N, m) = \frac{\sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < N}} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)}{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)}.$$

We claim that $d(N, m) \to 0$ as $m \to \infty$. Combined with the fact (still to be proven) that $c(N) \to 0$ as $N \to \infty$, this will imply the convergence of (16) to zero.

By Lemma 6.2(b),

$$\begin{aligned} d(N, m) &= \frac{\sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < N}} 3^{a_0} 2^{a_1 + a_{-1}}}{\sum_{\varepsilon} 3^{a_0} 2^{a_1 + a_{-1}}} \\ &\le \frac{2^N \cdot \sum_{\varepsilon} 3^{a_0}}{\sum_{\varepsilon} 3^{a_0} 2^{a_1 + a_{-1}}}\end{aligned}$$

with the sums taken over all $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$. Now, $\sum_{p=-2}^{2} a_p = m - j$ for each such ε, so we can view the sums over ε as sums over 5-tuples $(a_0, a_{-1}, a_1, a_{-2}, a_2)$ of non-negative integers summing to $m - j$. That is, the above expression is equal to

$$2^N \left(\sum \binom{m-j}{a_0, a_{-1}, a_1, a_{-2}, a_2} 3^{a_0} \right) \Big/ \left(\sum \binom{m-j}{a_0, a_{-1}, a_1, a_{-2}, a_2} 3^{a_0} 2^{a_1} 2^{a_{-1}} \right)$$

with the summation as described above and $\binom{m-j}{a_0, a_{-1}, a_1, a_{-2}, a_2}$ the multinomial coefficient '$m - j$ choose $a_0, \ldots, a_2$'. By the identity

$$(x_1 + x_2 + x_3 + x_4 + x_5)^n = \sum_{e_1 + e_2 + e_3 + e_4 + e_5 = n} \binom{n}{e_1, e_2, e_3, e_4, e_5} x_1^{e_1} \cdots x_5^{e_5},$$

this is equal to

$$\frac{2^N (3 + 1 + 1 + 1 + 1)^{m-j}}{(3 + 2 + 2 + 1 + 1)^{m-j}}.$$

Hence $D(N, m) \le 2^N (7/9)^{m-j}$, which clearly goes to zero as $m \to \infty$.

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We are thus left with the construction of the function c. Let

$$s(n) := \sup_{\substack{m,\varepsilon \text{ with} \\ a_1+a_{-1}=n}} R(\varepsilon, m),$$

where the supremum is taken over all $m \geq j$ and $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$ with $a_1 + a_{-1} = n$. We show that $s(n)$ converges to 0 as $n \to \infty$ and is in fact finite for every n; setting

$$c(t) := \sup_{n \geq t} s(n)$$

then produces a non-increasing function c with the desired properties and completes the proof.

Let $m \geq j$ and $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$ be arbitrary with $a_1 + a_{-1} = n$. We bound $R(m, \varepsilon)$ from above by a function of n and show that this function converges to zero as $n \to \infty$. To do this, we begin by obtaining a combinatorial expression for $\tilde{\varepsilon}(k)$. (Recall that $\tilde{\varepsilon}(k)$ counts the number of pairs (d, d') such that $g(d, d') = \varepsilon$ and $d - d' = k$.)

Let a be the least element of $D(J, m) - D(J, m)$ for which $\tilde{\varepsilon}(a) > 0$. Fix i and consider ε_i. If $\varepsilon_i = -2$, then we must have $(d_i, d_i') = (0, 2h_i + 1)$; similarly, if $\varepsilon_i = 2$ then $(d_i, d_i') = (2h_i + 1, 0)$. If $\varepsilon_i = -1$, then in general we can have either $(d_i, d_i') = (0, h_i)$ or $(d_i, d_i') = (h_i, 2h_i + 1)$, but minimality of $d - d' = a$ implies that the latter must the case. Similarly, if $\varepsilon_i = 1$, then in general there are two possibilities for (d_i, d_i'), but minimality of a rules out one of these. Finally, if $\varepsilon_i = 0$, then $d_i = d_i'$ can be any element of $\{0, h_i, 2h_i + 1\}$. We thus see that $\tilde{\varepsilon}(a) = 3^{a_0}$, since each i for which $\varepsilon_i = \pm 1$ or ± 2 uniquely determines (d_i, d_i'), while each $\varepsilon_i = 0$ allows for three possible pairs (d_i, d_i').

Now let us consider $\tilde{\varepsilon}(a + 1)$. As before, each i for which $\varepsilon_i = \pm 2$ uniquely determines (d_i, d_i'), and each $\varepsilon_i = 0$ allows for three pairs (d_i, d_i'). Previously, the minimality of $d - d' = a$ forced $(d_i, d_i') = (h_i, 2h_i + 1)$ whenever $\varepsilon_i = -1$ and $(d_i, d_i') = (h_i, 0)$ whenever $\varepsilon_i = 1$. However, since we now have $d - d' = a + 1$, we must 'raise' precisely one of these indices—that is, we select an index i such that $\varepsilon_i = \pm 1$ and set $(d_i, d_i') = (0, h_i)$ (if $\varepsilon = -1$) or $(d_i, d_i') = (2h_i + 1, h_i)$ (if $\varepsilon = 1$) so that $d_i - d_i'$ is one greater than before. Since there are $a_1 + a_{-1} = n$ such indices to choose from, we see that $\tilde{\varepsilon}(a + 1) = 3^{a_0} n$. In general, when computing $\tilde{\varepsilon}(a + k)$, we raise exactly k out of these n indices, so that

$$\tilde{\varepsilon}(a + k) = 3^{a_0} \binom{a_1 + a_{-1}}{k} = 3^{a_0} \binom{n}{k}$$

for all $0 \leq k \leq n$, and is 0 otherwise.

Substituting this into the definition of $R(m, \varepsilon)$ (see (17)), we have

$$\begin{aligned} R(m, \varepsilon) &= 3^{a_0} \left(1 + \sum_{k=0}^{n-1} \left| \binom{n}{k} - \binom{n}{k+1} \right| + 1 \right) \Big/ 3^{a_0} \left(\sum_{k=0}^{n} \binom{n}{k} \right) \\ &= \frac{1}{2^n} \left(\sum_{k=0}^{n-1} \left| \binom{n}{k} - \binom{n}{k+1} \right| + 2 \right). \end{aligned}$$

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We now obtain a simple bound on this by defining the random variable X to be the number of 'heads' in $n+1$ tosses of a fair coin. Then observe that

$$\begin{aligned} E\left[\left|\frac{X}{n+1}-\frac{1}{2}\right|\right] &= \sum_{k=0}^{n+1}\left|\frac{k}{n+1}-\frac{1}{2}\right|\frac{1}{2^{n+1}}\binom{n+1}{k} \\ &= \frac{1}{2^{n+1}}\left(\sum_{k=1}^{n}\left|\binom{n}{k-1}-\frac{1}{2}\binom{n+1}{k}\right|+1\right) \\ &= \frac{1}{2^{n+1}}\left(\sum_{k=1}^{n}\left|\frac{1}{2}\binom{n}{k-1}-\frac{1}{2}\binom{n}{k}\right|+1\right) \\ &= \frac{1}{2^{n+2}}\left(\sum_{k=0}^{n-1}\left|\binom{n}{k}-\binom{n}{k+1}\right|+2\right). \end{aligned}$$

Thus $R(m,\varepsilon)=4E[|X/(n+1)-1/2|]$. By Jensen's inequality, we have

$$\begin{aligned} R(m,\varepsilon) &= 4E[|X/(n+1)-1/2|] \\ &\le 4\sqrt{E[|X/(n+1)-1/2|^2]} \\ &= 2/\sqrt{n} \end{aligned}$$

which goes to 0 as a function of n. This establishes the desired properties of c and proves that T is rationally weakly mixing on levels. By Theorem 2.3 and Lemma 3.1, it follows that T is rationally weakly mixing. □

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Note added in proof. Recently, Bozgan, Sanchez, Silva, Stevens and Wang removed the assumption of exponential growth in Theorems 2.3 and 2.4.

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