INEQUALITIES FOR FUNCTIONS REGULAR IN THE UNIT CIRCLE

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1. Let $f(z) = u(z) + iv(z)$ be a function regular for $|z| < 1$ with real and imaginary parts $u(z)$ and $v(z)$ respectively. The following theorems have been proved by Miss M. L. Cartwright (1) and (2).

Theorem A. *If* $v(0) = 0$, $0 < \alpha < \infty$ *and*

$$|u(r e^{i\theta})| < A(1-r)^{-\alpha}, \tag{1·1}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$ *then*

$$|f(r e^{i\theta})| < C(\alpha) A(1-r)^{-\alpha}, \tag{1·2}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

Theorem B. *If* $u(0) \geqslant 0$, $v(0) = 0$, $1 < \alpha < \infty$ *and*

$$u(r e^{i\theta}) < A(1-r)^{-\alpha}, \tag{1·3}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$ *then*

$$|f(r e^{i\theta})| < K(\alpha) A(1-r)^{-\alpha}, \tag{1·4}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

Theorem C. *If* $u(0) \geqslant 0$, $v(0) = 0$ *and* (1·3) *holds with* $0 \leqslant \alpha < 1$ *then*

$$|f(r e^{i\theta})| < K(\alpha) A(1-r)^{-1}, \tag{1·5}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

Although the behaviour of $C(\alpha)$ and $K(\alpha)$ has not been considered in detail an examination of the methods given in (1) and (2) for the proofs of Theorems A, B and C shows that the conclusions of these theorems are valid with certain functions $C(\alpha)$ and $K(\alpha)$ which are bounded except in neighbourhoods of $\alpha = 0$ and $\alpha = 1$ respectively and which satisfy

$$C(\alpha) = O(1/\alpha) \quad \text{as} \quad \alpha \to 0+0, \tag{1·6}$$

$$K(\alpha) = O((\alpha-1)^{-2}) \quad \text{as} \quad \alpha \to 1+0, \tag{1·7}$$

$$K(\alpha) = O((1-\alpha)^{-1/(1-\alpha)}) \quad \text{as} \quad \alpha \to 1-0. \tag{1·8}$$

In this work it will be shown that we can sharpen (1·8) by substituting for $K(\alpha)$ a function which is $O((1-\alpha)^{-2})$ as $\alpha \to 1-0$ and counter examples are given which show that this together with (1·6) and (1·7) represents a result of the best possible order of magnitude. The methods employed allow us to obtain bounds to $K(\alpha)$ which are sharper than those obtainable by the methods of (1) and (2) in both cases $\alpha > 1$ and $\alpha < 1$.

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2. The three main theorems which are proved enable us to substitute particular functions $C(\alpha)$ and $K(\alpha)$ in Theorems A, B and C. We state the results as follows.

THEOREM 1. *The conclusion* (1·2) *of Theorem A is valid if we write*

$$C(\alpha) = \frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1} + 1. \tag{2·1}$$

THEOREM 2. *The conclusion* (1·4) *of Theorem B is valid* (i) *if we write*

$$K(\alpha) = 4\left\{\frac{\alpha-2}{\alpha-1}\right\}\left\{\frac{\alpha}{\alpha-2}\right\}^{\alpha}\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\} \tag{2·2}$$

when $3 < \alpha < \infty$ *and* (ii) *if we write*

$$K(\alpha) = 2\left\{\frac{\alpha+1}{\alpha-1}\right\}\left\{\frac{2\alpha}{\alpha-1}\right\}^{\alpha}\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\} \tag{2·3}$$

when $1 < \alpha \leqslant 3$.

THEOREM 3. *The conclusion* (1·5) *of Theorem C is valid* (i) *if we write*

$$K(\alpha) = 2^{2+\alpha}\left[2^{\frac{1}{2}(3\alpha-1)}\left\{\frac{1}{\alpha}\left(\sin\frac{\pi(1-\alpha)}{4\alpha}\right)^{-\frac{1}{2}(3\alpha+1)} + \frac{2}{3\alpha-1}\left\{\left(\frac{2\alpha}{1-\alpha}\right)^{\frac{1}{2}(3\alpha-1)}-1\right\}\right\}+\frac{3}{\pi}\right]\left(\frac{16}{\pi}+1\right) \tag{2·4}$$

when $\frac{5}{12} < \alpha < 1$ *and* (ii) *if we write*

$$K(\alpha) = 2^{2+\alpha}\left[2^{\frac{3}{4}\alpha}\left\{\frac{(\sin\frac{1}{8}\pi)^{-\frac{2}{4}\alpha}}{4-5\alpha} + \frac{4-2^{\frac{9}{4}\alpha}}{4-9\alpha}\right\}+\frac{3}{\pi}\right]\left(\frac{16}{\pi}+1\right) \tag{2·5}$$

when $0 \leqslant \alpha \leqslant \frac{5}{12}$.

If $\epsilon > 0$ it is seen that the function $C(\alpha)$ defined by (2·1) is bounded in the interval (ϵ, ∞). Further

$$C(\alpha) \sim \frac{4}{\pi\alpha} \quad \text{as} \quad \alpha \to 0+0$$

and

$$C(\alpha) \sim \frac{4e}{\pi}+1 \quad \text{as} \quad \alpha \to \infty.$$

The behaviour of the function $K(\alpha)$ can similarly be deduced from the definitions (2·2), (2·3), (2·4) and (2·5). Thus, if $0 < \epsilon < 1$, $K(\alpha)$ is bounded in each of the intervals $[0, 1-\epsilon]$ and $[1+\epsilon, \infty)$, and we have

$$K(\alpha) \sim 8(\alpha-1)^{-2}\left(\frac{16}{\pi}+1\right) \quad \text{as} \quad \alpha \to 1+0,$$

$$K(\alpha) \sim 256\pi^{-2}(1-\alpha)^{-2}\left(\frac{16}{\pi}+1\right) \quad \text{as} \quad \alpha \to 1-0,$$

$$K(\alpha) \sim 4e^2\left\{\frac{4e}{\pi}+1\right\} \quad \text{as} \quad \alpha \to \infty.$$

3. The method by which we prove Theorem 1 was given by Miss Cartwright (1) in her proof of Theorem A. However, a little more detail is now necessary.

Since $f(z)$ is regular for $|z| < 1$ the Poisson formula gives

$$u(r\,e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(R\,e^{i\phi})\,(R^2-r^2)}{R^2-2Rr\cos(\phi-\theta)+r^2}\,d\phi, \tag{3·1}$$

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for $0 \leqslant r < R < 1$. It follows that

$$\frac{\partial v(r\,e^{i\theta})}{\partial r} = -\frac{1}{r}\frac{\partial u(r\,e^{i\theta})}{\partial \theta} = -\frac{R(R^2-r^2)}{\pi}\int_0^{2\pi} \frac{u(R\,e^{i\phi})\sin(\phi-\theta)}{(R^2-2Rr\cos(\phi-\theta)+r^2)^2}\,d\phi,$$

and hence

$$\begin{aligned}\left|\frac{\partial v(r\,e^{i\theta})}{\partial r}\right| &\leqslant \frac{R(R^2-r^2)}{\pi}\int_0^{2\pi} \frac{|u(R\,e^{i\phi})|\,|\sin(\phi-\theta)|}{(R^2-2Rr\cos(\phi-\theta)+r^2)^2}\,d\phi \\ &< \frac{AR(R^2-r^2)(1-R)^{-\alpha}}{\pi}\int_0^{2\pi} \frac{|\sin(\phi-\theta)|}{(R^2-2Rr\cos(\phi-\theta)+r^2)^2}\,d\phi \\ &= \frac{4AR(1-R)^{-\alpha}}{\pi(R^2-r^2)} \\ &< \frac{4A(1-R)^{-\alpha}}{\pi(R-r)}.\end{aligned}$$

Now let $R = (\alpha r+1)/(\alpha+1)$. Then

$$\left|\frac{\partial v(r\,e^{i\theta})}{\partial r}\right| < \frac{4A}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha}(\alpha+1)(1-r)^{-\alpha-1},$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. Along any radius of the circle $|z| < 1$ the angle θ is a constant. Integration with respect to r yields

$$|v(r\,e^{i\theta})| < \frac{4A}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}(1-r)^{-\alpha},$$

since $v(0) = 0$ and it follows that

$$\begin{aligned}|f(r\,e^{i\theta})| &\leqslant |u(r\,e^{i\theta})| + |v(r\,e^{i\theta})| \\ &< \left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\}A(1-r)^{-\alpha},\end{aligned}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. This completes the proof of Theorem 1.

4. The proofs of Theorems 2 and 3 are based on the following lemma which is a special case of a result proved by Hayman (3) for subharmonic functions. We employ the usual notation $U^+ = \max(U, 0)$, $U^- = U - U^+$.

LEMMA 1. *If $F(w) = U(w) + iV(w)$ is regular for $|w| \leqslant \sigma$ then*

$$|U(\rho\,e^{i\phi})| \leqslant \frac{1}{\pi}\left(\frac{\sigma+\rho}{\sigma-\rho}\right)\left\{\int_0^{2\pi} U^+(\sigma\,e^{i\psi})\,d\psi - \pi U(0)\right\}, \tag{4·1}$$

for $0 \leqslant \rho < \sigma$, $0 \leqslant \phi < 2\pi$.

We write the formula (3·1) as

$$U(\rho\,e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\sigma^2-\rho^2)\{U^+(\sigma\,e^{i\psi})+U^-(\sigma\,e^{i\psi})\}}{\sigma^2-2\sigma\rho\cos(\psi-\phi)+\rho^2}\,d\psi. \tag{4·2}$$

Since $0 \leqslant \rho < \sigma$ we have

$$0 < \frac{\sigma^2-\rho^2}{\sigma^2-2\sigma\rho\cos(\psi-\phi)+\rho^2} \leqslant \frac{\sigma+\rho}{\sigma-\rho},$$

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and hence $$|U(\rho e^{i\phi})| \leqslant \frac{1}{2\pi}\left(\frac{\sigma+\rho}{\sigma-\rho}\right)\int_0^{2\pi}\{U^+(\sigma e^{i\psi}) - U^-(\sigma e^{i\psi})\}\,d\psi. \tag{4·3}$$

When $\rho = 0$ (4·2) may be written as

$$U(0) = \frac{1}{2\pi}\int_0^{2\pi}\{U^+(\sigma e^{i\psi}) + U^-(\sigma e^{i\psi})\}\,d\psi,$$

and the inequality (4·1) follows from (4·3).

5. We shall deduce Theorem 2 as a corollary of the following theorem. The statement involves a parameter β which will be defined later as a function of α.

THEOREM 4. *Suppose that $f(z)$ is regular for $|z| < 1$, $u(0) \geqslant 0$, $v(0) = 0$, $1 < \alpha < \infty$ and*

$$u(r e^{i\theta}) < A(1-r)^{-\alpha}, \tag{5·1}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. Then, for each constant β such that $\frac{1}{2} \leqslant \beta < 1$ and $\alpha\beta > 1$, we have

$$|f(r e^{i\theta})| < \frac{2A}{\beta}\left(\frac{\alpha\beta-1}{1-\beta}\right)^{1/\beta}\left\{\frac{\alpha\beta}{(\alpha\beta-1)(1-r)}\right\}^{\alpha} \mathrm{B}\left(\frac{1}{\beta}, \alpha-\frac{1}{\beta}\right)\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\} \tag{5·2}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, where

$$\mathrm{B}(p,q) = \int_0^{\infty} x^{p-1}(1+x)^{-p-q}\,dx.$$

In proving Theorem 4 we shall need the following lemma.

LEMMA A. *The transformation*

$$1-z = (1-w)^{\beta}, \quad \tfrac{1}{2} \leqslant \beta < 1 \tag{5·3}$$

maps the circle $|w| \leqslant 1$ onto a region which, except for the point $z = 1$, is entirely contained in $|z| < 1$.

Later we shall require some of the elements of the proof (4) of Lemma A and we quote them now in sketching the proof. Let us consider that branch of $(1-w)^{\beta}$ in (5·3) which associates real positive values of $1-w$ with real positive values of $1-z$. Then under the transformation (5·3) there corresponds to each point $w = e^{i\psi}$ $(0 \leqslant \psi < 2\pi)$ a point z such that

$$|z|^2 = 1 - 2(2\sin\tfrac{1}{2}\psi)^{\beta}\cos\tfrac{1}{2}\beta(\pi-\psi) + (2\sin\tfrac{1}{2}\psi)^{2\beta}.$$

Since it is shown (4) that

$$2\cos\tfrac{1}{2}\beta(\pi-\psi) - (2\sin\tfrac{1}{2}\psi)^{\beta} > 2(1-\beta) \tag{5·4}$$

for $0 < \beta < 1$, $0 \leqslant \psi < 2\pi$ it follows that

$$|z|^2 \leqslant 1 - 2(1-\beta)(2\sin\tfrac{1}{2}\psi)^{\beta}. \tag{5·5}$$

Hence $|z| < 1$ except when $\psi = 0$, that is $z = w = 1$. This is the assertion of Lemma A.

We now consider the function

$$F(w) = f(R(1-(1-w)^{\beta})) = f(Rz), \tag{5·6}$$

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where $0 < R < 1$. Since the function $f(Rz)$ is regular for $|z| \leqslant 1$ it follows from Lemma A that the function $F(w) = U(w) + iV(w)$ is regular for $|w| \leqslant 1$ except possibly for $w = 1$, where $F(w)$ is bounded. Thus we can apply Lemma 1 to the function $F(w)$ to obtain

$$|U(\rho e^{i\phi})| \leqslant \frac{1}{\pi}\left(\frac{\sigma+\rho}{\sigma-\rho}\right)\int_0^{2\pi} U^+(\sigma e^{i\psi})\,d\psi$$

for $0 \leqslant \rho < \sigma < 1$, $0 \leqslant \phi < 2\pi$ since $U(0) = u(0) \geqslant 0$. Because of the continuity of the latter integral we may let σ tend to 1 so that we have

$$|U(\rho e^{i\phi})| \leqslant \frac{1}{\pi}\left(\frac{1+\rho}{1-\rho}\right)\int_0^{2\pi} U^+(e^{i\psi})\,d\psi \tag{5·7}$$

for $0 \leqslant \rho < 1$, $0 \leqslant \phi < 2\pi$.

The definition (5·6) implies that $U(w) = u(Rz)$. Hence, if $w = e^{i\psi}$ and z are related by (5·3), we have

$$\begin{aligned} U^+(e^{i\psi}) &< A(1-R|z|)^{-\alpha} \\ &< A\{1-\tfrac{1}{2}R(1+|z|^2)\}^{-\alpha} \\ &\leqslant A\{1-R+R(1-\beta)(2\sin\tfrac{1}{2}\psi)^\beta\}^{-\alpha} \end{aligned}$$

by (5·1) and (5·5). It follows from (5·7) that

$$\begin{aligned} |U(\rho e^{i\phi})| &< \frac{A}{\pi}\left(\frac{1+\rho}{1-\rho}\right)\int_0^{2\pi}\{1-R+R(1-\beta)(2\sin\tfrac{1}{2}\psi)^\beta\}^{-\alpha}\,d\psi \\ &< \frac{2A}{\pi}\left(\frac{1+\rho}{1-\rho}\right)\int_0^{\pi}\left\{1-R+R(1-\beta)\left(\frac{2\psi}{\pi}\right)^\beta\right\}^{-\alpha}\,d\psi \end{aligned}$$

and writing

$$x = \frac{R(1-\beta)}{1-R}\left(\frac{2\psi}{\pi}\right)^\beta, \quad X = \frac{2^\beta R(1-\beta)}{1-R},$$

we obtain

$$\begin{aligned} |U(\rho e^{i\phi})| &< \frac{A}{\beta}\left(\frac{1+\rho}{1-\rho}\right)(1-R)^{1/\beta-\alpha}\{R(1-\beta)\}^{-1/\beta}\int_0^X (1+x)^{-\alpha}x^{1/\beta-1}\,dx \\ &< \frac{2A}{\beta(1-\rho)}(1-R)^{1/\beta-\alpha}\{R(1-\beta)\}^{-1/\beta}\,\mathrm{B}\left(\frac{1}{\beta}, \alpha-\frac{1}{\beta}\right) \end{aligned}$$

for $0 \leqslant \rho < 1$, $0 \leqslant \phi < 2\pi$.

According to (5·6) we have $u(r) = U(\rho)$, where $r = R - R(1-\rho)^\beta$ and thus

$$|u(r)| < \frac{2A}{\beta}\left\{\frac{1-R}{(R-r)(1-\beta)}\right\}^{1/\beta}\mathrm{B}\left(\frac{1}{\beta}, \alpha-\frac{1}{\beta}\right)(1-R)^{-\alpha}.$$

Letting

$$R = r + \frac{1-r}{\alpha\beta}$$

we deduce

$$|u(r)| < \frac{2A}{\beta}\left(\frac{\alpha\beta-1}{1-\beta}\right)^{1/\beta}\left\{\frac{\alpha\beta}{(\alpha\beta-1)(1-r)}\right\}^{\alpha}\mathrm{B}\left(\frac{1}{\beta}, \alpha-\frac{1}{\beta}\right),$$

an inequality which readily generalizes to give

$$|u(re^{i\theta})| < \frac{2A}{\beta}\left(\frac{\alpha\beta-1}{1-\beta}\right)^{1/\beta}\left\{\frac{\alpha\beta}{(\alpha\beta-1)(1-r)}\right\}^{\alpha}\mathrm{B}\left(\frac{1}{\beta}, \alpha-\frac{1}{\beta}\right)$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. The inequality (5·2) of Theorem 4 follows by application of Theorem 1 to the function $f(z)$.

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6. Any function $f(z)$ which satisfies the hypothesis of Theorem B must satisfy the hypothesis of Theorem 4 and it follows that the inequality (5·2) is valid. By the hypothesis of Theorem 2 (i) we have $3 < \alpha < \infty$ and setting $\beta = \frac{1}{2}$ in (5·2) we have

$$|f(re^{i\theta})| < 4A(\alpha-2)^2\left\{\frac{\alpha}{(\alpha-2)(1-r)}\right\}^{\alpha} \mathrm{B}(2,\alpha-2)\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\}$$

$$= 4A\left\{\frac{\alpha-2}{\alpha-1}\right\}\left\{\frac{\alpha}{(\alpha-2)(1-r)}\right\}^{\alpha}\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, since

$$\mathrm{B}(2,\alpha-2) = \frac{1}{(\alpha-2)(\alpha-1)}.$$

We have shown that the function $K(\alpha)$ of (2·2) can be substituted in the inequality (1·4) of Theorem B when $3 < \alpha < \infty$. Thus Theorem 2 (i) has been proved.

According to the hypothesis of Theorem 2 (ii) the inequality (5·2) is valid with $1 < \alpha \leqslant 3$. We set $1/\beta = \frac{1}{2}(\alpha+1)$ so that

$$\alpha\beta - 1 = 1-\beta = \frac{\alpha-1}{\alpha+1}$$

and consequently

$$|f(re^{i\theta})| < A(\alpha+1)\left\{\frac{2\alpha}{(\alpha-1)(1-r)}\right\}^{\alpha} \mathrm{B}(\tfrac{1}{2}(\alpha+1),\tfrac{1}{2}(\alpha-1))\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\}$$

$$< 2A\left\{\frac{\alpha+1}{\alpha-1}\right\}\left\{\frac{2\alpha}{(\alpha-1)(1-r)}\right\}^{\alpha}\left\{\frac{4}{\pi}\left(1+\frac{1}{\alpha}\right)^{\alpha+1}+1\right\}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, since

$$\mathrm{B}(\tfrac{1}{2}(\alpha+1),\tfrac{1}{2}(\alpha-1)) = \int_0^{\infty} x^{\frac{1}{2}(\alpha-1)}(1+x)^{-\alpha}\,dx$$

$$< \int_0^{\infty}(1+x)^{-\frac{1}{2}(\alpha+1)}\,dx$$

$$= \frac{2}{\alpha-1}.$$

We have proved that the inequality (1·4) of Theorem B is valid when $1 < \alpha \leqslant 3$ provided that $K(\alpha)$ is defined by (2·3). This completes the proof of Theorem 2.

7. The following proof of Theorem 3 is similar to that of Theorem 2 but in place of the auxiliary Theorem 4 we have

THEOREM 5. *Suppose that $f(z)$ is regular for $|z| < 1$, $u(0) \geqslant 0$, $v(0) = 0$, $0 \leqslant \alpha < 1$ and*

$$u(re^{i\theta}) < A(1-r)^{-\alpha} \tag{7·1}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. Then if $0 < \gamma < 1$ and $\alpha(\gamma+1) < 1$ we have

$$|f(re^{i\theta})| \leqslant \frac{A\,2^{2+\alpha}}{\pi(1-r)}\{2^{1+\alpha-\alpha\gamma}\Phi(\gamma,\alpha)+3\}\left\{\frac{16}{\pi}+1\right\} \tag{7·2}$$

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for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, *where*

$$\Phi(\gamma, \alpha) = \int_0^{\frac{1}{2}\pi} \{(\cos\phi)^{\gamma+1} \cos(1-\gamma)\phi\}^{-\alpha} d\phi.$$

Corresponding to Lemma A in the proof of Theorem 4 we have

LEMMA 2. *If* $0 < \gamma < 1$ *the transformation*

$$1 - z = (1-w)\{1 - \tfrac{1}{4}(1-w)^{\gamma}\} \tag{7·3}$$

maps the circle $|w| \leqslant 1$ *onto a region which, except for the point* $z = 1$, *is entirely contained in* $|z| < 1$.

Let us consider that branch of the right-hand side of (7·3) which transforms the interval $(0, 1)$ of the real w-axis into the interval $(\frac{1}{4}, 1)$ of the real z-axis. When $w = e^{i\psi}$, (7·3) gives

$$\begin{aligned} |z|^2 &= |e^{i\psi} + \tfrac{1}{4}(1 - e^{i\psi})^{\gamma+1}|^2 \\ &= |1 - \tfrac{1}{4}(2\sin\tfrac{1}{2}\psi)^{\gamma+1} e^{\frac{1}{2}i(1-\gamma)(\pi-\psi)}|^2 \\ &= 1 - \tfrac{1}{2}(2\sin\tfrac{1}{2}\psi)^{\gamma+1} \cos\tfrac{1}{2}(1-\gamma)(\pi-\psi) + \tfrac{1}{16}(2\sin\tfrac{1}{2}\psi)^{2\gamma+2}. \end{aligned} \tag{7·4}$$

Now by (5·4) we have $\quad 2\cos\tfrac{1}{2}\gamma(\pi-\psi) > (2\sin\tfrac{1}{2}\psi)^{\gamma}$

for $0 < \gamma < 1$, $0 \leqslant \psi < 2\pi$. Hence

$$\begin{aligned} 4\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi) - (2\sin\tfrac{1}{2}\psi)^{\gamma+1} &> 4\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi) - 4\cos\tfrac{1}{2}\gamma(\pi-\psi)\sin\tfrac{1}{2}\psi \\ &= 4\cos\tfrac{1}{2}\psi \sin\tfrac{1}{2}\gamma(\pi-\psi) \\ &\geqslant 0 \end{aligned}$$

and (7·4) yields $\quad |z|^2 \leqslant 1 - \tfrac{1}{4}(2\sin\tfrac{1}{2}\psi)^{\gamma+1}\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi).$ (7·5)

Thus, when $0 \leqslant \psi < 2\pi$, we have $|z| < 1$ unless $\psi = 0$, that is $w = 1$. This completes the proof of Lemma 2.

8. In our proof of Theorem 5 we consider the function

$$F(w) = f(R(w + \tfrac{1}{4}(1-w)^{\gamma+1})) = f(Rz), \tag{8·1}$$

where $0 < R < 1$. Lemma 2 shows that $F(w)$ is regular for $|w| \leqslant 1$ except possibly for $w = 1$, where it is bounded. Writing $F(w) = U(w) + iV(w)$ we have, by application of Lemma 1,

$$|U(\rho e^{i\phi})| \leqslant \frac{2}{\pi(1-\rho)}\left\{\int_0^{2\pi} U^+(e^{i\psi})\, d\psi + \pi |U(0)|\right\} \tag{8·2}$$

for $0 \leqslant \rho < 1$, $0 \leqslant \phi < 2\pi$. We determine upper bounds to $|U(\rho e^{i\phi})|$ in a more convenient form.

Let $w = e^{i\psi}$ and z be related by (7·3). Then (8·1) and (7·1) yield

$$\begin{aligned} U(e^{i\psi}) = u(Rz) &< A(1 - R|z|)^{-\alpha} \\ &< A\{2^{\gamma-2}(\sin\tfrac{1}{2}\psi)^{\gamma+1}\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi)\}^{-\alpha} \end{aligned} \tag{8·3}$$

because $0 < R < 1$ and

$$|z| \leqslant \tfrac{1}{2}(1 + |z|^2) \leqslant 1 - 2^{\gamma-2}(\sin\tfrac{1}{2}\psi)^{\gamma+1}\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi)$$

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by (7·5). Since the right-hand side of (8·3) is positive whenever $0 \leqslant \psi < 2\pi$ it follows that

$$\begin{aligned}\int_0^{2\pi} U^+(e^{i\psi})\,d\psi &< A2^{\alpha(2-\gamma)}\int_0^{2\pi}\{(\sin\tfrac{1}{2}\psi)^{\gamma+1}\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi)\}^{-\alpha}\,d\psi\\ &= A2^{1+\alpha(2-\gamma)}\int_0^{\pi}\{(\sin\tfrac{1}{2}\psi)^{\gamma+1}\cos\tfrac{1}{2}(1-\gamma)(\pi-\psi)\}^{-\alpha}\,d\psi\\ &= A2^{2+\alpha(2-\gamma)}\int_0^{\frac{1}{2}\pi}\{(\cos\phi)^{\gamma+1}\cos(1-\gamma)\phi\}^{-\alpha}\,d\phi\\ &= A2^{2+\alpha(2-\gamma)}\Phi(\gamma,\alpha) \qquad (8\cdot4)\end{aligned}$$

according to the definition of $\Phi(\gamma,\alpha)$.

The definition (8·1) implies that $U(0) = u(\frac{1}{4}R)$ and an application of Lemma 1 with $\sigma = \frac{1}{2}$, $\rho = \frac{1}{4}R$, $u(0) \geqslant 0$ shows that

$$\begin{aligned}|U(0)| = |u(\tfrac{1}{4}R)| &= \frac{1}{\pi}\left(\frac{\frac{1}{2}+\frac{1}{4}R}{\frac{1}{2}-\frac{1}{4}R}\right)\int_0^{2\pi} u^+(\tfrac{1}{2}e^{i\psi})\,d\psi\\ &< \frac{3}{\pi}\int_0^{2\pi} A2^{\alpha}\,d\psi\\ &= 3A2^{\alpha+1},\end{aligned}$$

by (7·1). Combining this inequality with (8·2) and (8·4) we have

$$|U(\rho\,e^{i\phi})| < \frac{A2^{2+\alpha}}{\pi(1-\rho)}\{2^{1+\alpha-\alpha\gamma}\Phi(\gamma,\alpha)+3\}. \qquad (8\cdot5)$$

According to (8·1) we have $u(r) = U(\rho)$, where

$$\frac{r}{R} = \rho + \tfrac{1}{4}(1-\rho)^{\gamma+1}. \qquad (8\cdot6)$$

As ρ increases from 0 to 1 the right-hand side of (8·6) increases from $\frac{1}{4}$ to 1. Thus any value r such that $\frac{1}{4}R \leqslant r < R$ corresponds to a value ρ such that $0 \leqslant \rho < 1$ and for such a value r we have by (8·5)

$$|u(r)| = |U(\rho)| < \frac{AR2^{2+\alpha}}{\pi(R-r)}\{2^{1+\alpha-\alpha\gamma}\Phi(\gamma,\alpha)+3\}.$$

Since $0 < R < 1$ we have, by continuity as $R \to 1-0$,

$$|u(r)| \leqslant \frac{A2^{2+\alpha}}{\pi(1-r)}\{2^{1+\alpha-\alpha\gamma}\Phi(\gamma,\alpha)+3\}$$

for $\frac{1}{4} \leqslant r < 1$, an inequality which is readily generalized, to give

$$|u(r\,e^{i\theta})| \leqslant \frac{A2^{2+\alpha}}{\pi(1-r)}\{2^{1+\alpha-\alpha\gamma}\Phi(\gamma,\alpha)+3\} \qquad (8\cdot7)$$

for $\frac{1}{4} \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

When $0 \leqslant r < \frac{1}{4}$ we apply Carathéodory's Theorem ((5), pp. 174–5) to obtain

$$|u(r\,e^{i\theta})| \leqslant \frac{2r}{R-r}\operatorname*{Max}_{0\leqslant\phi<2\pi} u(R\,e^{i\phi})$$

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whenever $0 \leqslant r < R < 1$. Letting $r = \frac{1}{4}$, $R = \frac{1}{2}$ this yields

$$|u(r e^{i\theta})| \leqslant 2^{\alpha+1} A \tag{8·8}$$

for $0 \leqslant r < \frac{1}{4}$ by (7·1) and the maximum modulus principle. By definition $\Phi(\gamma, \alpha)$ is positive. Hence (8·8) implies (8·7) which is consequently valid for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

The inequality (7·2) of Theorem 5 follows from (8·7) by application of Theorem 1 with $\alpha = 1$.

9. We deduce Theorem 3 as a corollary of Theorem 5, but first we prove the following lemma which gives bounds to the function $\Phi(\gamma, \alpha)$ defined in this latter theorem.

LEMMA 3. *If* $0 < \gamma < 1$, $0 \leqslant \alpha < 1$, $\alpha(\gamma+1) < 1$ *and* $\alpha(\gamma+2) \neq 1$ *then*

$$\Phi(\gamma, \alpha) = \int_0^{\frac{1}{2}\pi} \{(\cos\phi)^{\gamma+1} \cos(1-\gamma)\phi\}^{-\alpha} d\phi$$
$$< \frac{\pi\gamma(\sin\frac{1}{2}\pi\gamma)^{-\alpha(\gamma+2)}}{2(1-\alpha-\alpha\gamma)} + \frac{\pi(1-\gamma^{1-2\alpha-\alpha\gamma})}{2(1-2\alpha-\alpha\gamma)}.$$

Dividing the range of integration of $\Phi(\gamma, \alpha)$ into two parts we note first that

$$\int_{\frac{1}{2}\pi(1-\gamma)}^{\frac{1}{2}\pi} \{(\cos\phi)^{\gamma+1} \cos(1-\gamma)\phi\}^{-\alpha} d\phi < (\sin\tfrac{1}{2}\pi\gamma)^{-\alpha} \int_{\frac{1}{2}\pi(1-\gamma)}^{\frac{1}{2}\pi} (\cos\phi)^{-\alpha-\alpha\gamma} d\phi$$
$$= (\sin\tfrac{1}{2}\pi\gamma)^{-\alpha} \int_0^{\frac{1}{2}\pi\gamma} (\sin\phi)^{-\alpha-\alpha\gamma} d\phi$$
$$< (\sin\tfrac{1}{2}\pi\gamma)^{-\alpha} \left(\frac{\sin\frac{1}{2}\pi\gamma}{\frac{1}{2}\pi\gamma}\right)^{-\alpha-\alpha\gamma} \int_0^{\frac{1}{2}\pi\gamma} \phi^{-\alpha-\alpha\gamma} d\phi$$
$$= \frac{\pi\gamma(\sin\frac{1}{2}\pi\gamma)^{-\alpha(\gamma+2)}}{2(1-\alpha-\alpha\gamma)}$$

since $\sin\phi/\phi$ is a decreasing function of ϕ for $0 < \phi < \frac{1}{2}\pi$. Also, since $0 < \gamma < 1$, we have

$$\int_0^{\frac{1}{2}\pi(1-\gamma)} \{(\cos\phi)^{\gamma+1} \cos(1-\gamma)\phi\}^{-\alpha} d\phi < \int_0^{\frac{1}{2}\pi(1-\gamma)} (\cos\phi)^{-2\alpha-\alpha\gamma} d\phi$$
$$= \int_{\frac{1}{2}\pi\gamma}^{\frac{1}{2}\pi} (\sin\phi)^{-2\alpha-\alpha\gamma} d\phi$$
$$< \int_{\frac{1}{2}\pi\gamma}^{\frac{1}{2}\pi} \left(\frac{2\phi}{\pi}\right)^{-2\alpha-\alpha\gamma} d\phi$$
$$= \frac{\pi(1-\gamma^{1-2\alpha-\alpha\gamma})}{2(1-2\alpha-\alpha\gamma)},$$

provided that $\alpha(\gamma+2) \neq 1$. Thus we have the inequality asserted in the statement of Lemma 3.

Any function $f(z)$ which satisfies the hypothesis of Theorem 3 must also satisfy the hypothesis of Theorem 5. We apply this latter theorem to $f(z)$ making a convenient choice for the parameter γ.

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When $\frac{5}{12} < \alpha < 1$ we set $\gamma = (1-\alpha)/2\alpha$. Then the hypothesis of Lemma 3 is satisfied and the inequality (7·2) can be written as

$$|f(re^{i\theta})| < \frac{A2^{2+\alpha}}{1-r}\left[2^{\frac{1}{2}(3\alpha-1)}\left\{\frac{1}{\alpha}\left(\sin\frac{\pi(1-\alpha)}{4\alpha}\right)^{-\frac{1}{2}(3\alpha+1)} + \frac{2}{3\alpha-1}\left\{\left(\frac{2\alpha}{1-\alpha}\right)^{\frac{1}{2}(3\alpha-1)} - 1\right\}\right\} + \frac{3}{\pi}\right]\left(\frac{16}{\pi}+1\right)$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. This is the assertion of Theorem 3 (i).

For the proof of Theorem 3 (ii) we set $\gamma = \frac{1}{4}$ when $0 \leqslant \alpha \leqslant \frac{5}{12}$. Then the hypothesis of Lemma 3 is again satisfied and we can write (7·2) as

$$|f(re^{i\theta})| < \frac{A2^{2+\alpha}}{1-r}\left[2^{\frac{3}{4}\alpha}\left\{\frac{(\sin\frac{1}{8}\pi)^{-\frac{2}{4}\alpha}}{4-5\alpha} + \frac{4-2^{\frac{2}{4}\alpha}}{4-9\alpha}\right\} + \frac{3}{\pi}\right]\left(\frac{16}{\pi}+1\right)$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. This completes the proof of Theorem 3.

10. The function $C(\alpha)$ defined in Theorem 1 is $O(\alpha^{-1})$ as $\alpha \to 0+0$. There is a simple counter-example which shows that this order of magnitude is best possible.

THEOREM 6. *In order that Theorem A be valid we must have*

$$C(\alpha) \geqslant \operatorname{Max}\left(1, \frac{2}{\pi\alpha}\right), \tag{10·1}$$

in (1·2).

The inequality $C(\alpha) \geqslant 1$ is immediate. Thus Theorem 6 will be proved if we verify

$$C(\alpha) \geqslant \frac{2}{\pi\alpha}$$

when $0 < \alpha < 2/\pi$.

The function

$$f(z) = \frac{i}{\alpha}\{(1-z)^{-\alpha} - 1\}$$

is regular for $|z| = r < 1$. Let us write

$$1 - z = 1 - re^{i\theta} = \rho e^{i\phi}. \tag{10·2}$$

Then
$$f(z) = u(z) + iv(z) = \frac{i}{\alpha}(\rho^{-\alpha}\cos\phi\alpha - 1 - i\rho^{-\alpha}\sin\phi\alpha)$$

and we have
$$|u(re^{i\theta})| = \frac{|\sin\phi\alpha|}{\alpha}\rho^{-\alpha}.$$

Now by (10·2) we have
$$\rho^2 = 1 - 2r\cos\theta + r^2 \geqslant (1-r)^2 \tag{10·3}$$

and, if $|z| = r < 1$,
$$|\phi| < \tfrac{1}{2}\pi. \tag{10·4}$$

Thus
$$|u(re^{i\theta})| < \frac{1}{\alpha}\sin(\tfrac{1}{2}\pi\alpha)(1-r)^{-\alpha}$$
$$< \tfrac{1}{2}\pi(1-r)^{-\alpha}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$ and $0 < \alpha < 2/\pi$. However

$$\operatorname*{Max}_{0\leqslant\theta<2\pi}|f(re^{i\theta})| \sim \frac{1}{\alpha}(1-r)^{-\alpha}$$

as $r \to 1-0$ and it follows that $C(\alpha) \geqslant 2/\pi\alpha$.

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11. According to the definition of Theorems 2 and 3 we have $K(\alpha) = O\{(\alpha-1)^{-2}\}$ as $\alpha \to 1$. The following theorems show that in general this order of magnitude of $K(\alpha)$ cannot be further reduced.

THEOREM 7. *In order that Theorem B be valid when* $1 < \alpha \leqslant 3$ *we must have*

$$K(\alpha) \geqslant \operatorname{cosec}^2 \tfrac{1}{4}\pi(\alpha-1). \tag{11·1}$$

THEOREM 8. *In order that Theorem C be valid when* $\frac{1}{2} < \alpha < 1$ *we must have*

$$K(\alpha) \geqslant \tfrac{1}{2}(\operatorname{cosec} \pi\alpha)^{\alpha-1} \sec^2 \tfrac{1}{2}\pi\alpha. \tag{11·2}$$

We shall consider the function

$$f(z) = -\{(1-z)^{-\frac{1}{2}} - (1-z)^{-\frac{1}{2}\alpha}\}^2 \tag{11·3}$$

on the circle $|z| = r < 1$. By (10·2) we have

$$u(r e^{i\theta}) = \Re f(r e^{i\theta}) = -\frac{\cos\phi}{\rho} + \frac{2\cos\tfrac{1}{2}\phi(\alpha+1)}{\rho^{\frac{1}{2}(\alpha+1)}} - \frac{\cos\phi\alpha}{\rho^{\alpha}}. \tag{11·4}$$

When $1 < \alpha \leqslant 3$ we write (11·4) as

$$u(r e^{i\theta}) = -\cos\phi\{\rho^{-\frac{1}{2}} - \rho^{-\frac{1}{2}\alpha}\sec\phi\,\cos\tfrac{1}{2}\phi(\alpha+1)\}^2 + \rho^{-\alpha}\sec\phi\{\cos^2\tfrac{1}{2}\phi(\alpha+1) - \cos\phi\,\cos\phi\alpha\}.$$

Further when $|z| < 1$ the inequality (10·4) implies $\cos\phi > 0$ and since

$$\cos^2\tfrac{1}{2}\phi(\alpha+1) - \cos\phi\,\cos\phi\alpha = \sin^2\tfrac{1}{2}\phi(\alpha-1) \tag{11·5}$$

we have

$$u(r e^{i\theta}) < \rho^{-\alpha}\sec\phi\,\sin^2\tfrac{1}{2}\phi(\alpha-1) \tag{11·6}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

Now we have

$$2\rho\cos\phi = 1 - r^2 + \rho^2$$

by (10·2) and therefore

$$2\rho\cos\phi \geqslant 1 - r^2 + (1-r)^2 = 2(1-r) \tag{11·7}$$

by (10·3). Thus

$$\rho^{\alpha}\cos\phi > (\rho\cos\phi)^{\alpha} \geqslant (1-r)^{\alpha}$$

when $\alpha > 1$ and we deduce from (11·6) that

$$u(r e^{i\theta}) < (1-r)^{-\alpha}\sin^2\tfrac{1}{4}\pi(\alpha-1) \tag{11·8}$$

whenever $1 < \alpha \leqslant 3$, $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

If $\alpha > 1$ in (11·3) it is clear that

$$\operatorname*{Max}_{|z|=r} |f(z)| \sim (1-r)^{-\alpha} \tag{11·9}$$

as $r \to 1-0$. This completes the proof of Theorem 7 since (11·8) and (11·9) imply (11·1) for the function $f(z)$ defined by (11·3).

For our proof of Theorem 8 we assume first that $0 < \alpha < 1$ and we write (11·4) as

$$u(r e^{i\theta}) = -\left\{\left(\frac{\cos\phi}{\rho}\right)^{\frac{1}{2}} - \left(\frac{\cos\phi\alpha}{\rho^{\alpha}}\right)^{\frac{1}{2}}\right\}^2 + 2\rho^{-\frac{1}{2}(1+\alpha)}\{\cos\tfrac{1}{2}\phi(\alpha+1) - (\cos\phi\,\cos\phi\alpha)^{\frac{1}{2}}\}. \tag{11·10}$$

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Since $|\phi| < \frac{1}{2}\pi$ when $|z| < 1$ we have

$$\cos\tfrac{1}{2}\phi(\alpha+1) - (\cos\phi\,\cos\phi\alpha)^{\frac{1}{2}} = \frac{\cos^2\tfrac{1}{2}\phi(\alpha+1) - \cos\phi\,\cos\phi\alpha}{\cos\tfrac{1}{2}\phi(\alpha+1) + (\cos\phi\,\cos\phi\alpha)^{\frac{1}{2}}}$$
$$< \frac{\sin^2\tfrac{1}{2}\phi(1-\alpha)}{(\cos\phi\,\cos\phi\alpha)^{\frac{1}{2}}}$$

by (11·5), and it follows from (11·7) and (11·10) that

$$u(r\,e^{i\theta}) < 2\rho^{-\frac{1}{2}(1+\alpha)}\sin^2\tfrac{1}{2}\phi(1-\alpha)\,(\cos\phi\,\cos\phi\alpha)^{-\frac{1}{2}}$$
$$\leqslant 2(1-r)^{-\frac{1}{2}(1+\alpha)}\sin^2\tfrac{1}{2}\phi(1-\alpha)\,(\cos\phi)^{\frac{1}{2}\alpha}\,(\cos\phi\alpha)^{-\frac{1}{2}}.$$

However $$(\cos\phi)^{\frac{1}{2}\alpha}\,(\cos\phi\alpha)^{-\frac{1}{2}} < (\cos\phi\alpha)^{\frac{1}{2}(\alpha-1)} < (\cos\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}(\alpha-1)}$$

for $0 < \alpha < 1$, $|\phi| < \frac{1}{2}\pi$ and we have

$$u(r\,e^{i\theta}) < 2(1-r)^{-\frac{1}{2}(1+\alpha)}\sin^2\tfrac{1}{4}\pi(1-\alpha)\,(\cos\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}(\alpha-1)} \tag{11·11}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$.

When $0 < \alpha < 1$ in (11·3) we have

$$\operatorname*{Max}_{|z|=r}|f(z)| \sim \frac{1}{1-r} \tag{11·12}$$

as $r \to 1-0$. Thus (11·11) and (11·12) imply that

$$K(\tfrac{1}{2}(1+\alpha)) \geqslant \tfrac{1}{2}(\sec\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}(\alpha-1)}\operatorname{cosec}^2\tfrac{1}{4}\pi(1-\alpha)$$

when $0 < \alpha < 1$, that is

$$K(\alpha) \geqslant \tfrac{1}{2}(\operatorname{cosec}\pi\alpha)^{\alpha-1}\sec^2\tfrac{1}{2}\pi\alpha,$$

when $\frac{1}{2} < \alpha < 1$. This is the inequality (11·2) asserted in Theorem 8.

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