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Modular spin representations of the symmetric group

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1. Let Γ_n be the *representation group* or *spin group* (4, 9) of S_n. Then the irreducible representations of Γ_n are of two distinct types. These are (*a*) *ordinary* representations, which are the irreducible representations of the symmetric group and (*b*) *spin* or *projective* representations. Corresponding to every partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n with $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$ there is an irreducible spin representation $\langle\lambda\rangle$ of Γ_n.

Any partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n may be associated with a *graph* $\langle\lambda\rangle$ consisting of rows of symbols called *nodes*. λ_1 nodes in the first row, λ_2 in the second row, ..., λ_m in the mth row. The node in the ith row and jth column of $\langle\lambda\rangle$ is called its (ij)-node.†

In this paper, results corresponding to those of Nakayama(6) are proved for irreducible spin characters with the aid of the concept of a *bar graph*.

We make the following definitions.

Definition 1. There are two kinds of bars; viz.

(i) An $(i,1)_k$-bar $(k = i+1, i+2, \ldots, m)$ consisting of the $\lambda_i + \lambda_k$ nodes in the ith and kth rows of $\langle\lambda\rangle$.

(ii) An $(i,j)_{(i)}$-bar consisting of the (i,j)-node together with the $\lambda_i - j$ nodes to the right of it.

Definition 2. The length of a bar is defined as

(i) $S_{i1}^{(k)} = \lambda_i + \lambda_k \quad (k = i+1, \ldots, m)$,

(ii) $S_{ij}^{(i)} = \lambda_i - j + 1$.

λ_k is called the *arm length* of the $(i,1)_{(k)}$-bar.

Now let
$$S_{i1} = \prod_{k=i}^{m} S_{i1}^{(k)}$$

and
$$S_{ij} = S_{ij}^{(i)} \quad \text{if} \quad j > 1 \quad \text{and} \quad j-1 \neq \lambda_t \quad \text{for any } t.$$
$$1 \quad \text{if} \quad j > 1 \quad \text{and} \quad j-1 = \lambda_t \quad \text{for some } t.$$

Definition 3. The graph obtained by replacing the (i,j)-node of $\langle\lambda\rangle$ by S_{ij} is called the *bar graph* $S\langle\lambda\rangle$. The bar product S^λ is the product of all the S_{ij}'s in $S\langle\lambda\rangle$, i.e.
$$S^\lambda = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} S_{ij}.$$

† $\langle\lambda\rangle$ will be used to denote the graph and spin representation corresponding to the partition (λ).

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2. *The removal of bars from a graph.* The removal of a single bar is treated in section 3 of (5) and it is clear that the graph obtained by removing two distinct bars from $\langle\lambda\rangle$ is independent of the order of removal of the bars.

We now prove

LEMMA 1. *Let B be a bar of length l which may be removed from $\langle\lambda\rangle$ and suppose $l = l_1 + l_2$, where l_1, l_2 are positive integers. Then there exist bars B_1, B_2, of length l_1, l_2 respectively such that at least one of $(\langle\lambda\rangle - B_1) - B_2$ and $(\langle\lambda\rangle - B_2) - B_1$ exists and is equivalent to $\langle\lambda\rangle - B$.*

N.B. (*a*) Two graphs $\langle\lambda\rangle$ by $\langle\lambda\rangle'$ are equivalent if $\langle\lambda\rangle'$ may be obtained from $\langle\lambda\rangle$ by an interchange of rows.

(*b*) $(\langle\lambda\rangle - B_1) - B_2$ denotes the graph obtained by first removing B_1 followed by the removal of B_2.

Proof. Case 1. B is a bar of type (i).

Let B be the $(i, 1)_{(k)}$-bar of $\langle\lambda\rangle$. Then $l = \lambda_i + \lambda_k$. Suppose $\lambda_i = l_1 + \alpha$, then $\lambda_k = l_2 - \alpha$. If $\alpha \neq \lambda_q$ for any q we may remove a bar B_1 of length l_1 from the ith row of $\langle\lambda\rangle$ leaving $\langle\lambda\rangle - B_1 = \langle\lambda_1, \ldots, \lambda_{i-1}, \alpha, \lambda_{i+1}, \ldots, \lambda_m\rangle$.

We may then remove a bar B_2 of length l_2 from $\langle\lambda\rangle - B_1$ consisting of the ith and kth rows. Thus $(\langle\lambda\rangle - B_1) - B_2 = \langle\lambda_1, \ldots, \lambda_{i-1}, \lambda_{j+1}, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m\rangle$ which is identical with $\langle\lambda\rangle - B$.

If now $\lambda_t = \alpha$, we may remove a bar B_2 of length l_2 from $\langle\lambda\rangle$, i.e. the $(k, 1)_{(t)}$-bar and $\langle\lambda\rangle - B_2 = \langle\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{t-1}, \lambda_{t+1}, \ldots, \lambda_m\rangle$. We may now remove a bar of length l_1 from the ith row of $\langle\lambda\rangle - B_2$ giving rise to

$$(\langle\lambda\rangle - B_2) - B_1 = \langle\lambda_1, \ldots, \lambda_{i-1}, \alpha, \lambda_{i+1}, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{t-1}, \lambda_{t+1}, \ldots, \lambda_m\rangle$$

and, since $\lambda_t = \alpha$, this is equivalent to $\langle\lambda\rangle - B$.

Case 2. A similar argument is used if B is a bar of type (ii).

Remark. In particular, it follows from the above that a bar of length $r.l$ can always be decomposed into r successive bars of length l.

3. *Power of a prime dividing the degree of an irreducible spin representation of Γ_n.* Hereinafter, p will denote an odd prime and by an l-bar we will mean a bar of length l.

3·1. *Degree of an irreducible representation.* As before, let $\langle\lambda\rangle = \langle\lambda_1, \lambda_2, \ldots, \lambda_m\rangle$ be a graph on n nodes. Then we have the following formulae (5, 9) for the degree f^λ of $\langle\lambda\rangle$:

$$f = \frac{2^{[\frac{1}{2}(n-m)]}.n!}{\lambda_1!\lambda_2!\ldots\lambda_m!} \prod_{1 \leqslant r < s \leqslant m} \frac{\lambda_r - \lambda_s}{\lambda_r + \lambda_s} \tag{1}$$

$$= \frac{2^{[\frac{1}{2}(n-m)]}n!}{S^\lambda} \tag{2}$$

where $[x]$ means the greatest integer $\leqslant x$.

If α is a positive integer, we denote by $e(\alpha)$ the index of the highest power of p dividing α, then (2) enables us to determine those spin characters of Γ_n whose degrees are divisible by the full power $p^{e(n!)}$ of p which divides the order of Γ_n, i.e. $2(n!)$.

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Clearly, $p^{e(n!)} | f^\lambda$ if and only if $p \nmid S^\lambda$. Now

$$S = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} S_{ij},$$

and thus $p \nmid S^\lambda$ if and only if no p-bar may be removed from $\langle\lambda\rangle$. Such a representation is said to be of defect $O \pmod p$. See (1).

DEFINITION 4. A graph $\langle\tilde{\lambda}\rangle$ from which no p-bar may be removed is called a *p-core*.

If all possible p-bars are removed from a graph $\langle\lambda\rangle$, the result is a p-core $\langle\tilde{\lambda}\rangle$. If two p-bars have nodes in common they must be identical, and it follows that every graph $\langle\lambda\rangle$ has a unique p-core $\langle\tilde{\lambda}\rangle$.

DEFINITION 5. The *p-sequence* of $\langle\lambda\rangle$ is a finite sequence $\{e_1 = e, e_2, e_3, \ldots\}$ of strictly positive integers defined inductively as follows: for $i = 1, 2, \ldots$; e_i is the largest integer such that $e_i \leqslant e_{i-1}$ and for which there exists a bar B_i of length p^{e_i} in

$$(\ldots((\langle\lambda\rangle - B_1) - B_2) - \ldots - B_{i-2}) - B_{i-1}.$$

The corresponding polynomial $g\lambda(p) = \sum_j p^{e_j}$ is called the p-series of $\langle\lambda\rangle$.

It is clear that the p-sequence (and hence the p-series) is independent of the choice of B_i at each stage and that the sequence terminates when we arrive at $\langle\tilde{\lambda}\rangle$.

If $\langle\lambda\rangle$ contains at least one bar of length p^u, but no bar of length p^t with $t > u$, we call $\langle\lambda\rangle$ a *p^{u+1}-core*. Let $\langle\lambda\rangle - B_1 = \langle\lambda\rangle'$. We find the connexion between the p-series of $\langle\lambda\rangle$ and $\langle\lambda\rangle'$. Two cases are considered.

Case 1. B_1 is the $(i, 1)_{(k)}$-bar of $\langle\lambda\rangle$, where $\lambda_i + \lambda_k = p^e$. It may be possible to remove bars of length $\mu . p^{e+1} (u > 0)$ from $\langle\lambda\rangle'$. Since $\langle\lambda\rangle$ is a p^{e+1}-core, such bars must be of type (ii) and may only be removed from rows of length $\mu(i) p^{e+1} + \lambda_i$ and $\mu(k) p^{e+1} + \lambda_k$ in $\langle\lambda\rangle$. Let $x = i$, k and $u(x)$ be the greatest positive integer for which there exists a row of length $\mu(x) . p^{e+1} + \lambda_x$ in $\langle\lambda\rangle$. Then, since $\langle\lambda\rangle$ is a p^{e+1}-core, there exist rows of length $\mu(x) . p^{e+1} + \lambda_x$ in $\langle\lambda\rangle$ for $\mu(x) = 1, 2, \ldots, u(x)$.

Let

$$u(x) = \gamma_0(x) + \gamma_1(x) . p + \gamma_2(x) . p^2 + \ldots + \gamma_{a_x}(x) . p^{a_x + e + 1},$$

where $0 \leqslant \gamma_j(x) \leqslant p - 1$. Then the p-sequence of $\langle\lambda\rangle'$ will contain $\gamma_j(i) + \gamma_j(k)$ copies of each $e + j + 1$.

Further, let $\langle\lambda\rangle''$ be the diagram obtained from $\langle\lambda\rangle'$ by removing all bars of length p^t with $t > e$. We have $g\lambda(p) = \sum_j (\gamma_j(i) + \gamma_j(k) . p^{e+j+1} + g\lambda''(p))$, which, since $g\lambda(p)$ cannot include any terms p^t with $t > e$, we write in the form

$$g\lambda'(p) = \{p . (u(i) + u(k))\} . p^e + g\lambda''(p).$$

Thus

$$\begin{aligned} g\lambda(p) &= p^e + g\lambda'(p) \\ &= \{p . (u(i) + u(k)) + 1\} . p^e + g\lambda''(p), \end{aligned} \tag{3}$$

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where $\{p(u(i)+u(k))+1\}.p^e$ is to be interpreted as

$$p^e+p^e+\ldots+p^e[\{p(u(i)+u(k))+1\} \text{ terms}].$$

Case 2. B_1 is the $(i, \beta+1)_{(i)}$-bar of $\langle\lambda\rangle$, where $\lambda_i = p^e+\beta$ and $\beta \neq \lambda_q$ for any q.

In the same way as the above we will have $V(i)$ rows of length $\nu(i).p^{e+1}+\lambda_i$. $(\nu(i) = 1, 2, \ldots, V(i))$ from which we may remove a bar (of type (ii)) of length $\nu(i).p^{e+1}$. In addition we will have $W(i)$ rows of length $w(i).p^{e+1}-\beta$ and removal of such a row together with the ith row of $\langle\lambda\rangle'$ gives rise to a bar (of type (i)) of length $w(i).p^{e+1}$ $(w(i) = 1, 2, \ldots, W(i))$.

Then if $\langle\lambda\rangle''$ is defined in the same way as in Case 1 above, we have

$$g\lambda(p) = \{p(V(i)+W(i))+1\}.p^e+g\lambda''(p). \tag{4}$$

Hence we have proved

LEMMA 2. *With the notation of Definition* 5 *let* $\langle\lambda\rangle - B_1 = \langle\lambda\rangle'$. *Then the connexion between the p-series of* $\langle\lambda\rangle$ *and* $\langle\lambda\rangle'$ *is given by* (3) *if* B_1 *is a bar of type* (i) *and by* (4) *if* B_1 *is a bar of type* (ii).

We now proceed to prove

LEMMA 3. *Let* $\langle\lambda\rangle$, $\langle\lambda\rangle'$ *with degrees* $f^\lambda, f^{\lambda'}$ *respectively be defined as in Lemma* 2. *Then*

$$e(f^\lambda/f^{\lambda'}) = e(n!)-e((n-p^e)!)-e(p^e!)+e((p.u(i))!)+e((p.u(k))!) \tag{5}$$

if B_1 *is a bar of type* (i),

$$e(f^\lambda/f^{\lambda'}) = e(n!)-e((n-p^e)!)-e(p^e!)+e((p.W(i))!)+e((p.V(i))!) \tag{6}$$

if B_1 *is a bar of type* (ii).

Proof. Case 1. Let B_1 be the $(i, 1)_{(k)}$-bar of $\langle\lambda\rangle$, where $\lambda_i+\lambda_k = p^e$. Let $\lambda_i = p^r+\alpha$, then $\lambda_k = p^e-p^r-\alpha$.

By (1), we have
$$f^\lambda = \frac{2^{[\frac{1}{2}(n-m)]}.n!}{\lambda_1!\lambda_2!\ldots\lambda_m!}\prod_{1\leqslant r<s\leqslant m}\frac{\lambda_r-\lambda_s}{\lambda_r+\lambda_s}$$

and
$$f^{\lambda'} = \frac{2^{[\frac{1}{2}(n-p^e-(m-2))]}.(n-p^e)!}{\lambda_1!\lambda_2!\ldots\lambda_{i-1}!\lambda_{i+1}!\ldots\lambda_{k-1}!\lambda_{k+1}!\ldots\lambda_m!}\prod_{1\leqslant r<s\leqslant m,\ \{r\neq i,k}_{s}\frac{\lambda_r-\lambda_s}{\lambda_r+\lambda_s}.$$

Then, it follows that

$$f^\lambda/f^{\lambda'} = \frac{2^{\{[\frac{1}{2}(n-m)]-[\frac{1}{2}(n-p^e-(m-2))]\}}n!}{(n-p^e)!\,\lambda_i!\,\lambda_k!} \times\frac{(\lambda_1-\lambda_i)(\lambda_2-\lambda_i)\ldots(\lambda_i-\lambda_m)}{(\lambda_1+\lambda_i)(\lambda_2+\lambda_i)\ldots(\lambda_i+\lambda_m)}\times\frac{(\lambda_1-\lambda_k)(\lambda_2-\lambda_k)\ldots(\lambda_k-\lambda_m)}{(\lambda_1+\lambda_k)(\lambda_2+\lambda_k)\ldots(k_k+\lambda_m)}.$$

Denote those rows of length $\mu(x).p^{e+1}+\lambda_x$ by $\lambda_x^{\mu(x)}$ $(x = i, k)$. We have

$$\frac{(\lambda_i^{\mu(i)}-\lambda_i)(\lambda_i^{\mu(i)}-\lambda_k)}{(\lambda_i^{\mu(i)}+\lambda_i)(\lambda_i^{\mu(i)}+\lambda_k)} = \left(\frac{\mu(i)\,p^{e+1}}{\mu(i)\,p^{e+1}+2p^r+2\alpha}\right)\left(\frac{\mu(i)\,p^{e+1}-p^e+2p^r+2\alpha}{\mu(i)\,p^{e+1}+p^e}\right).$$

Since
$$e\left(\frac{\mu(i)\,p^{e+1}-p^e+2p^r+2\alpha}{\mu(i)\,p^{e+1}+2p^r+2\alpha}\right) = 0 \quad \text{for} \quad \mu(i) = 1, 2, \ldots, u(i)$$

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it follows that

$$e\left(\prod_{\mu(i)=1}^{u(i)} \frac{(\lambda_i^{\mu(i)}-\lambda_i)(\lambda_i^{\mu(i)}-\lambda_k)}{(\lambda_i^{\mu(i)}+\lambda_i)(\lambda_i^{\mu(i)}+\lambda_k)}\right) = e\left(\prod_{\mu(i)=1}^{u(i)} \frac{\mu(i)\,p}{\mu(i)\,p+1}\right) = e((p\,.\,u(i))!).$$

Similarly
$$e\left(\prod_{\mu(k)=1}^{u(k)} \frac{(\lambda_k^{\mu(k)}-\lambda_i)(\lambda_k^{\mu(k)}-\lambda_k)}{(\lambda_k^{\mu(k)}+\lambda_i)(\lambda_k^{\mu(k)}+\lambda_k)}\right) = e((p\,.\,u(k))!).$$

Now
$$e\left(\frac{\lambda_i-\lambda_k}{\lambda_i+\lambda_k}\right) = e\left(\frac{2p^r-p^e+\alpha}{p^e}\right),$$

and if λ_t is not one of the rows $\lambda_x^{\mu(x)}$ we see that

$$e\left(\frac{(\lambda_t-\lambda_i)(\lambda_t-\lambda_k)}{(\lambda_t+\lambda_i)(\lambda_t+\lambda_k)}\right) = 0.$$

Hence
$$\begin{aligned} e(f^\lambda/f^{\lambda'}) = {} & e(n)!-e((n-p^e)!)-e((p^r+\alpha)!) \\ & -e((p^e-p^r-\alpha)!)+e((p\,.\,u(i))!)+e((p\,.\,u(k))!) \\ & +e(2p^r-p^e+\alpha)-e(p^e). \end{aligned}$$

Further, since

$$\begin{aligned} & e(2p^r-p^e+\alpha)-e(p^e)-e((p^r+\alpha)!)-e((p^e-p^r-\alpha)!) \\ & \quad = e(2p^r-p^e+\alpha)-e((p^r+\alpha)!)-e(p^e!)+e((p^r+\alpha-1)!) \\ & \quad = -e(p^e!), \end{aligned}$$

it follows that

$$e(f^\lambda/f^{\lambda'}) = e(n!)-e((n-p^e)!)-e(p^e!)+e((p\,.\,u(i))!)+e((p\,.\,u(k))!).$$

Case 2. Let B_1 be the $(i,\beta+1)_{(i)}$-bar of $\langle\lambda\rangle$. The proof is similar to the above and will be omitted.

We are now able to prove the main result.

THEOREM 1. *Let $\langle\lambda\rangle = \langle\lambda_1,\lambda_2,\ldots,\lambda_m\rangle$ be a graph on n-nodes. Then if the p-sequence of $\langle\lambda\rangle$ is $\{e_1,e_2,e_3,\ldots\}$, the exact exponent of the power of p dividing f^λ is*

$$\begin{aligned} e(f^\lambda) &= e(n!)-\sum_{t=1,2,\ldots} e(p^{e_t}!) \\ &= e(n!)-\sum_{t=1,2,\ldots}(p^{e_t-1}+p^{e_t-2}+\ldots+1). \end{aligned}$$

Proof. This is by induction on n.

The theorem is clearly true for $n=1$. Assume therefore that it is true for all groups Γ_r with $1\leqslant r\leqslant n-1$. Then if $\{e_1',e_2',e_3',\ldots\}$ is the p-sequence of $\langle\lambda\rangle' = \langle\lambda\rangle - B_1$ (see Lemma 2) we have that

$$e(f^{\lambda'}) = e((n-p^{e_1})!)-\sum_{t=1,2,\ldots}(p^{e_t'-1}+p^{e_t'-2}+\ldots+1). \tag{7}$$

Two cases are again considered.

Case 1. Let B_1 be the $(i,1)_{(k)}$-bar of $\langle\lambda\rangle$. By (3) we know the first $\sum_j(\gamma_j(i)+\gamma_j(k))$ terms of the p-series of $\langle\lambda\rangle'$. Let $A = \sum_j(\gamma_j(i)+\gamma_j(k))$.

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Then

$$\sum_{t=1}^{A}(p^{e_t'-1}+p^{e_t'-2}+\ldots+1)$$
$$=\sum_j \gamma_j(i)\,(p^{j+e_1}+p^{j+e_1-1}+\ldots+1)+\sum_j \gamma_j(k)\,(p^{j+e_1}+p^{j+e_1-1}+\ldots+1)$$
$$=\sum_j \gamma_j(i)\,p^{j+1}(p^{e_1-1}+p^{e_1-2}+\ldots+1)+\sum_j \gamma_j(k)\,p^{j+1}(p^{e_1-1}+^{e_1-2}+\ldots+1)$$
$$+\sum_j \gamma_j(i)\,(p^j+p^{j-1}+\ldots+)+\sum_j \gamma_j(k)\,(p^j+p^{j-1}+\ldots+1)$$
$$=p\,.\,u(i)\,(p^{e_1-1}+p^{e_1-2}+\ldots+1)+p\,.\,u(k)\,(p^{e_1-1}+p^{e_1-2}+\ldots+1)$$
$$+\Sigma(\gamma_j(i)+\gamma_{j+1}(i)\,.\,p+\ldots+\gamma_{a_i}(i)\,.\,p^{a_i-j})$$
$$+\Sigma(\gamma_j(k)+\gamma_{j+1}(k)\,p+\ldots+\gamma_{a_k}(k)\,.\,p^{a_k-j})$$
$$=p(u(i)+u(k))\,(p^{e_1-1}+p^{e_1-2}+\ldots+1)+e((p\,.\,u(i))!)+e((p\,.\,u(k))!).$$

Furthermore, $e'_{A+1}, e'_{A+2}, \ldots$ coincide with $e_{p(u(i)+u(k))+2}, e_{p(u(i)+u(k)+3}, \ldots$ as in (3). Therefore

$$e(f^{\lambda'}) = e((n-p^e)!)-p(u(i)+u(k))\,(p^{e_1-1}+p^{e_1-2}+\ldots+1)$$
$$-e((p\,.\,u(i))!)-e((p\,.\,u(k))!)-\sum_{t\geqslant p\,.\,(u(i)+u(k))+2}(p^{e_t-1}+p^{e_t-2}+\ldots+1)$$

and substituting for $e(f^{\lambda'})$ from (5) yields

$$e(f^{\lambda}) = e(n!)-(p\,.\,u(i)+u(k)+1)\,(p^{e_1-1}+p^{e_1-2}+\ldots+1)$$
$$-\sum_{t\geqslant p\,.\,(u(i)+u(k))+2}(p^{e_t-1}+p^{e_t-2}+\ldots+1)$$

and by (3), the theorem is proved.

Case 2. B_1 the $(i,\beta+1)_{(i)}$-bar of $\langle\lambda\rangle$. A similar argument shows that the theorem holds in this case also.

3·2. *Further developments.* We now turn to the problem of the generalization of the p-quotient (or star) diagram $[\lambda]_p$ (or $[\lambda]^*$) of $[\lambda]$ (1, 2, 8).

Suppose that we have a p-core $\langle\tilde{\lambda}\rangle$ on $n-p$ nodes. It is important to know how many different graphs $\langle\lambda\rangle$ exist on n nodes such that $\langle\lambda\rangle$ − one p-bar $= \langle\tilde{\lambda}\rangle$ (see (7), p. 287). Before proceeding we make the following definition.

DEFINITION 6. Let $\langle\lambda\rangle = \langle\lambda_1, \lambda_2, \ldots, \lambda_m\rangle$ be a graph on n nodes. Suppose that we may remove B, the $(i,\beta+1)_{(i)}$-bar of $\langle\lambda\rangle$ leaving $\langle\lambda\rangle - B = \langle\lambda_1, \ldots, \lambda_{i-1}, \beta, \lambda_{i+1}, \ldots, \lambda_m\rangle$. If when we rearrange the λ's to make them read in descending order, we have

$$\langle\lambda\rangle - B = \langle\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{t-1}, \beta, \lambda_t, \ldots, \lambda_m\rangle$$

then the *leg length* of B is defined to be $t-i-1$, i.e. $t-i-1$ is the number of rows λ_α such that $\lambda_i > \lambda_\alpha > \beta$. Leg length may also be defined for a bar of type (i) but will not be needed in the following.

Returning to the problem stated above, we see that the following conditions are necessary and sufficient for $\langle\tilde{\lambda}\rangle$ to be a p-core.

(1) No row is of length rp, or otherwise removal of this row would be equivalent to the removal of r p-bars. Thus every row is of length $rp+\alpha$ where $p\nmid\alpha$.

(2) If a row of length $sp+\alpha$ exists, then rows of length $tp+\alpha$ $(t = 0, 1, 2, \ldots, s-1)$ must also exist, since if there were no row of length $kp+\alpha$ $(0 \leqslant k \leqslant s-1)$ a p-bar could be removed from the row of length $(k+1)p+\alpha$.

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(3) If a row of length $sp+\alpha$ exists then no row of length $s'p+(p-\alpha) = (s'+1)p-\alpha$ can exist for any s', or otherwise the removal of both rows would give a bar of length $(s+s'+1)p$.

Let the rows of length less than p in $\langle\lambda\rangle$ be $\alpha_1, \alpha_2, \ldots, \alpha_a$ where $\alpha_1 > \alpha_2 > \ldots > \alpha_a$. Then $a \leqslant \frac{1}{2}(p-1)$.

We consider, first, the number of bars with non-zero arm length which may be added to $\langle\tilde{\lambda}\rangle$. These bars are inserted amongst the rows $\alpha_1, \alpha_2, \ldots, \alpha_a$, the graph $\langle\lambda\rangle$ having two rows more than $\langle\tilde{\lambda}\rangle$. It is clear that we can add $\frac{1}{2}(p-1)-a$ such bars to $\langle\tilde{\lambda}\rangle$, each having a different arm length.

Let the longest row in $\langle\tilde{\lambda}\rangle$ be of length $s_j p+\alpha_j$. Then, we may add a p-bar to this row, i.e. increase the number of the nodes in the row by p, making it of length

$$(s_j+1)+\alpha_j.$$

This p-bar has leg length 0.

Now, let the longest row in $\langle\tilde{\lambda}\rangle$ involving a different 'α' be of length $s_k p+\alpha_k$. Then, we may add a p-bar to this row making it of length $(s_k+1)p+\alpha_k$. If $j<k$, then

$$\ldots < s_k p+\alpha_k < s_k p+\alpha_j < (s_k+1)p+\alpha_k < (s_k+1)p+\alpha_j < \ldots$$

and if $j>k$, then

$$\ldots < s_k p+\alpha_j < s_k p+\alpha_k < (s_k+1)p+\alpha_j < (s_k+1)p+a_k < \ldots.$$

Thus, in either case, the p-bar added has leg length 1. We can clearly now proceed in exactly the same way until all the α_i $(i = 1, \ldots, a)$ have been used up, the p-bar added being of length $0, 1, \ldots, a-1$ respectively. Finally, we may add a p-bar to $\langle\tilde{\lambda}\rangle$ consisting of a row of length p inserted immediately before the row of length α_1. This p-bar is clearly of leg length a.

Gathering these results together, we find that the number of graphs with p-core $\langle\tilde{\lambda}\rangle$ is $\frac{1}{2}(p-1)-a+(a+1) = \frac{1}{2}(p+1)$. We have proved

THEOREM 2. *Let $\langle\tilde{\lambda}\rangle$ be a p-core on $n-p$ nodes. Then there are $\frac{1}{2}(p+1)$ graphs $\langle\lambda\rangle$ on n nodes such that $\langle\lambda\rangle - B = \langle\tilde{\lambda}\rangle$, where B is a p-bar.*

In order to obtain a generalization of the p-quotient diagram it therefore seems reasonable to attempt to construct a diagram $\langle\lambda\rangle_p$ from $\langle\lambda\rangle$, whose construction is similar to that of $[\lambda]_p$ from $[\lambda]$, having at most $\frac{1}{2}(p+1)$ disjoint constituents. This problem will be considered in a future publication.

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