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MULTIVARIATE TIME SERIES WITH VARIOUS HIDDEN UNIT ROOTS, PART I

Integral Operator Algebra and Representation Theory

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Following the approach proposed by Gregoir and Laroque (1993, *Econometric Theory* 9, 329–342), we consider a class of multivariate processes that, when differenced enough, yields covariance stationary processes whose determinant of the matrix series associated with their Wold representation has various unit roots with various orders of multiplicity we restrict to be integers. A representation theorem is provided that involves different polynomial error correction terms at each frequency associated with each unit root. An identification criterion for each set of error correction terms is proposed.

1. INTRODUCTION

Applied macroeconometricians are nowadays used in summarizing the dynamic behavior of a set of economic variables with a finite vector autoregressive linear approximation. Nevertheless the Wold theorem that is the keystone of this approach does not ensure that the moving average representation of a covariance stationary multidimensional process is invertible. There may be unit roots in the determinant of the matrix series associated with the Wold representation, which we call hidden unit roots. Working on nonstationary time series, Granger (1981) by the introduction of the idea of cointegration and Granger and Weiss (1983) by their representation theorem showed the method to follow to build up an autoregressive representation when the Wold representation is not invertible: it is necessary to introduce error correction terms among the right-hand-side variables of the vector autoregressive regression.

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But, the Granger representation theorem as stated in Engle and Granger (1987) does not allow the practitioners to deal with all the possible configurations. The residuals of the model of differenced variables including error correction terms may be still correlated with the right-hand-side variables so that the misuse of this representation leads to biased estimates. A general representation theorem has been proposed by Gregoir and Laroque (1993) when the noninvertibility of the Wold representation is due to the presence of the unit root one with a given integer order of multiplicity in the determinant of the Wold representation. This representation has close links with the moving average representation proposed by Stock and Watson (1993), obtained under a priori knowledge on the set of variables present in the error correction terms. It can also be related to the representation theorem stated in Johansen (1992) for VAR processes integrated of order two. This kind of representation must be used when there exist linear combinations of variables of different order of integration that are covariance stationary. This is denoted as polynomially cointegrated time series by Gregoir and Laroque (1993) and was introduced previously in some examples as multicointegrated time series by Yoo (1986) and Granger and Lee (1989a, 1989b). Empirically, this can happen when we work with $I(1)$ variables and some $I(2)$ variables (e.g., prices or wages). There may exist a relation of cointegration between the $I(2)$ variables that is $I(1)$ (e.g., a real wage measure) and is cointegrated with another $I(1)$ variable. All of this work assumes that the root of interest is one and that there are no other unit roots in the system. But on the one hand, when we pile up various covariance stationary time series, any unit root may appear in the determinant of its Wold representation, even if each component independently considered is an independent and identically distributed (i.i.d.) process (see Example 3.1, which follows).

On the other hand, many economic time series have seasonal patterns. They may have unit roots at the seasonal frequencies. Therefore from an economic point of view, there may be cointegration at seasonal frequencies as introduced by Engle, Granger, and Hallman (1989). Similarly, the representation equation presented by Hylleberg, Engle, Granger, and Yoo (1990) does not allow the econometrician to deal with all the possible configurations. The purpose of this work is to provide a representation theorem and statistical tools to analyze multivariate times series that belong to a general class. This class is composed of real multivariate covariance stationary time series with possibly various unit roots with various orders of multiplicity in the determinant of the matrix series associated to their Wold representation. Our only restriction in the definition of this class is that these orders of multiplicity are integers.

The framework we use has been introduced by Gregoir and Laroque (1993). It consists in working on covariance stationary processes and defining integral operators to build up nonstationary time series. The first part of this work is devoted to the representation theorem of a process in the class described previously under a vector autoregressive form with polynomial error correction terms. In the second part (Gregoir, 1999), following Gregoir and Laroque (1994), we will develop

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statistical tools to investigate the structure of such a time series and estimate the error correction terms.

The paper is organized as follows. The integral operator algebra is studied in Section 2. The general representation theorem is stated in Section 3, and identification restrictions are provided for the error correction terms. In Section 4, the particular case of a time series with two conjugate complex unit roots in the determinant of its Wold representation is analyzed to provide a better insight into the representation equation. The arguments of the proof in the general case are presented in Section 5. The main purpose of Section 6 is to extend the results to a more general class of processes. This class is composed of processes that can be decomposed into the sum of a process in the class introduced previously and a deterministic trend that we restrict to be a linear combination of monomials $(t^{d_j}\cos\omega_j t, t^{d_j}\sin\omega_j t)_{(\omega_j\in[0,\pi],d_j\in\mathbb{N})}$. Finally, a short conclusion is given in Section 7, and the proofs of the technical theorems, lemmas, and properties are gathered in the last section.

2. INTEGRAL OPERATOR ALGEBRA

Our representation theorem will be stated for a covariance stationary time series, and we need integral operators to build nonstationary processes at different frequencies. We start by recalling some well-known factors and definitions, and then we define the integral operator for different frequencies and study their algebraic properties.

A covariance stationary purely nondeterministic real random process x_t has a unique Wold representation

$$x_t = c(B)\epsilon_t$$

such that the ϵ_t are zero-mean white noise random variables and $c(0)=1$, $c(u)=\sum_{i=0}^{+\infty} c_i u^i$ satisfies $\sum_{i=0}^{+\infty} c_i^2 < +\infty$ and has all its zeros on or outside the unit circle. Here B is the usual backshift operator $Bx_t = x_{t-1}$. The analysis of such a process is usually carried out with the help of its autocorrelation structure or its spectrum. In the sequel, we use the following notations for the first-difference operator at various frequencies.

$$\Delta_\omega(B) = \begin{cases} 1 - e^{-i\omega}B, \omega \in \{0,\pi\} \\ 1 - 2\cos\omega B + B^2 = (1 - e^{-i\omega}B)(1 - e^{i\omega}B), \omega \in]0,\pi[. \end{cases}$$

When there is no ambiguity, we sometimes replace $\Delta_\omega(B)$ by Δ_ω. We now recall some well-known definitions.

DEFINITION 2.1. *A purely nondeterministic real random process x_t is said to be integrated of order h, h integer, at frequency ω, $\omega \in [0,\pi]$, or $I_\omega(h)$, if $\Delta_\omega(B)^h x_t$ is a covariance stationary process such that the series $c(u)$ associ-*

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ated with its Wold representation satisfies $c(e^{i\omega}) \neq 0$ $\left(\textit{consequently } c(e^{-i\omega}) \neq 0 \textit{ when } \omega \in]0, \pi[\right)$.

DEFINITION 2.2. *A zero-mean stationary real random process* x_t *is said to be integrated of order* $-h$, h *integer, at frequency* ω, $\omega \in [0, \pi]$, *or* $I_\omega(-h)$, *if the series* $c(u)$ *associated with its Wold representation satisfies*: $c(u) = \Delta_\omega(u)^h \tilde{c}(u)$ *where* $\tilde{c}(u)$ *is well defined on and inside the unit circle and different from zero for all* u *inside the unit circle and* $u = e^{i\omega}$ $\left(\textit{and consequently also for } u = e^{-i\omega} \textit{ when } \omega \in]0, \pi[\right)$.

These definitions are similar to those that can be found in the literature. For instance, Engle and Granger (1987) similarly defined $I_0(h)$ processes. Hylleberg et al. (1990) defined $I_\omega(h)$ processes by the form of their spectrum in the neighborhood of ω. By symmetry with respect to the y axis, we have the same expansion in the neighborhood of $-\omega$, and both definitions coincide for a real process. A real $I_\omega(h)$ process with $\omega \in]0, \pi[$ has $2h$ unit roots. Last, Gregoir and Laroque (1993) defined $I_0(-h)$ processes. Of course, a real process can be simultaneously integrated of different orders at different frequencies; the preceding definitions can be easily modified to describe these situations. Nevertheless, note that there exists a discontinuity in the end points of the interval $]0, \pi[$. The order of integration doubles when ω tends toward 0 or π because the process has the same unit root with the order of multiplicity $2h$. This is the consequence of the fact that we only consider real processes. In the sequel, we will separately study the properties related to a pair of complex roots and those of the unit roots 1 and -1. We now define the classes of processes on which we work.

DEFINITION 2.3. *A process* $y_t, y_t \in \mathbb{R}^N$, *belongs to* C_Ω *where* Ω *is a set of cardinal number* K *of couples of different values* $\omega_j \in [0, \pi]$ *and non-null integers* $k_j, j = 1, \ldots, K$, *if it is covariance stationary, has a Wold representation*

$$y_t = C(B)\epsilon_t,$$

and is such that the determinant of the matrix series associated with the multivariate Wold representation, $\det C(u)$, *satisfies*

$$\det C(u) = \mathcal{D}(u)d(u),$$

where

$$\mathcal{D}(u) = \prod_{j=1}^{j=K} \Delta_{\omega_j}(u)^{k_j}.$$

Here $d(u)$ *is well defined and different from zero for all* u, $\|u\| \leq \rho$, *for some* $\rho > 1$.

The integer k_j is different from zero whenever there exists some linear combination of the components of y_t that is $I_{\omega_j}(-h)$, for $h > 0$. We only consider situ-

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ations in which h is an integer. This will correspond to the cointegration at frequency ω_j between some variables of y_t. As did Gregoir and Laroque (1993), we can also find polynomial cointegration at various frequencies in such a multivariate time series. To construct these terms, we need to define an integral operator at each frequency.

DEFINITION 2.4. *The integral operator S_ω associates to any sequence $y_t = (y_t, t = \ldots, -1, 0, 1, \ldots)$ of $\mathbb{R}^N$, a sequence $S_\omega y$ defined by*

$$S_\omega y_t = \begin{cases} \sum_{\tau=1}^{t} y_\tau e^{-i\omega(t-\tau)} & \text{for } t \geq 1 \\ 0 & \text{for } t = 0 \\ -\sum_{\tau=0}^{t+1} y_\tau e^{-i\omega(t-\tau)} & \text{for } t < 0. \end{cases}$$

To understand the role played by these integral operators, apply one of them to a real white noise ϵ_τ. This can be decomposed into three steps. The first one corresponds to the definition of a complex white noise $e^{i\tau\omega}\epsilon_\tau$, the second one to the construction of a $I_0(1)$ complex valued process: $\sum_{\tau=1}^{t} e^{i\omega\tau}\epsilon_\tau$ for $t \geq 1$ (and the appropriate operation for $t \leq 0$). Last, a demodulation operation as described by Granger and Hatanaka (1964) (the multiplication by $e^{-i\omega t}$) shifts the peak in the spectrum of the $I_0(1)$ process to the frequency ω. This therefore gives an $I_\omega(1)$ complex valued process. Applied to the same white noise, different integral operators give asymptotically orthogonal processes. This property will be explicitly used in the statistical analysis of such processes.

In our manipulation of real time series, we want to avoid complex numbers. The right way to build an $I_\omega(1)$ time series when $\omega \in]0, \pi[$ is to apply to an $I(0)$ series the operator $S_\omega S_{-\omega}$. Simple algebraic computations give

$$S_\omega S_{-\omega} y_t = \begin{cases} \sum_{\tau=1}^{t} y_\tau \dfrac{\sin \omega(t+1-\tau)}{\sin \omega} & \text{for } t \geq 1 \\ 0 & \text{for } t = 0 \\ -\sum_{\tau=0}^{t+1} y_\tau \dfrac{\sin \omega(t+1-\tau)}{\sin \omega} & \text{for } t < 0. \end{cases}$$

In the sequel, for sake of simplicity, we use the following notations for the integral operators:

$$G_\omega = \begin{cases} S_\omega, \omega \in \{0, \pi\} \\ S_\omega S_{-\omega}, \omega \in]0, \pi[. \end{cases}$$

Remark 2.1. The restriction of considering only integral cointegration can be removed, but with some technical difficulties. We have to extend the preceding

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definitions to the fractional integration (see Granger and Joyeux, 1980) and therefore to fractional cointegration. It is then necessary to define a fractional integral operator for each fraction with the help of the coefficients of the expansion of $\Delta_{\omega_j}(B)^d$ where $\frac{1}{2} \leq d < 1$. When $0 < d < \frac{1}{2}$, the operator $\Delta_\omega(B)^d$ is invertible. The following example illustrates the situation of fractional cointegration of degree $\frac{1}{2}$. Consider the covariance stationary time series y defined by the following Wold representation:

$$y_t = \begin{pmatrix} (1-B)^{1/2} & -2^{-1/2}B(1-B)^{1/2} & 0 \\ 2^{-1/2}B(1-B)^{1/2} & (1-B)^{1/2}(1-B^2) & -2^{-1/2}B \\ 0 & 2^{-1/2}B(1-B)^{1/2} & 1 \end{pmatrix} \epsilon_t.$$

The determinant of this matrix is $1-B$, but the rank of $C(1)$ is one. There exist two relations of cointegration at frequency 0 but of order $\frac{1}{2}$.

We now state some algebraic properties of the integral operators.

Property 2.1. $\forall \omega \in [-\pi, \pi]$,

$$\Delta_\omega G_\omega = I,$$

where I stands for the identity operator.

Property 2.2. $\forall y_t \in \mathbb{R}^N$

$$\omega \in \{0, \pi\}, G_\omega \Delta_\omega y_t = y_t - y_0 e^{-i\omega t}$$

$$\omega \in]0, \pi[, G_\omega \Delta_\omega y_t = y_t - y_0 \frac{\sin \omega(t+1)}{\sin \omega} + y_{-1} \frac{\sin \omega t}{\sin \omega}.$$

Property 2.3 (Commutativity). $\forall (\omega_1, \omega_2) \in [0, \pi]^2$,

$$G_{\omega_1} G_{\omega_2} = G_{\omega_2} G_{\omega_1}.$$

Property 2.4 (Associativity). $\forall (\omega_1, \omega_2, \omega_3) \in [0, \pi]^3$,

$$(G_{\omega_1} G_{\omega_2}) G_{\omega_3} = G_{\omega_1} (G_{\omega_2} G_{\omega_3}).$$

Property 2.5.

$$G_0 G_\pi = \tfrac{1}{2} G_0 + \tfrac{1}{2} G_\pi$$

$\forall \omega \in]0, \pi[$,

$$G_0 G_\omega = \frac{1}{2(1-\cos\omega)} G_0 + \frac{1 - 2\cos\omega + B}{2(1-\cos\omega)} G_\omega$$

$$G_\pi G_\omega = \frac{1}{2(1+\cos\omega)} G_\pi + \frac{1 + 2\cos\omega - B}{2(1+\cos\omega)} G_\omega$$

$\forall (\omega_1, \omega_2) \in]0, \pi[^2$, $\omega_1 \neq \omega_2$,

$$G_{\omega_1} G_{\omega_2} = \frac{2\cos\omega_1 - B}{2(\cos\omega_1 - \cos\omega_2)} G_{\omega_1} + \frac{2\cos\omega_2 - B}{2(\cos\omega_2 - \cos\omega_1)} G_{\omega_2}.$$

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As noticed by Gregoir and Laroque (1993), the algebraic properties of G_ω are different from the usual properties of the backshift operator. Particularly, the introduction of y_0 (and y_{-1}) in Property 2.2 is due to the definition of G_ω. In our approach, an integrated process at any frequency ω goes through 0 at $t = 0$. This corresponds to the idea that there exists an infinity of integrated processes that have the same Wold representation; they differ by a constant random term that is in the kernel of the operator $\Delta_\omega(B)$. For instance, when $\omega = 0$, the time series have different levels but follow parallel trajectories. We simply select one integrated process by fixing its value in $t = 0$. The analysis of processes in $\mathcal{C}_\Omega$ is not affected by such a convention. If we change it, it only induces some modifications of the constant random terms that will appear in the polynomial cointegrated terms in the representation theorem.

Similarly, in Property 2.5, we have to introduce a lag operator to transform a product into a sum. But as Δ_ω and G_ω do not commute (a consequence of Property 2.2), we have to be precise in their order of appearance.

We denote G_ω^k the kth iterate of G_ω and $\mathcal{T}_\omega^k$ the vector space of deterministic polynomials $\mathcal{P}(t)$ solutions of the equation $\Delta_\omega^k(B)\mathcal{P}(t) = 0$. A basis of such a vector space is given by $\{(e^{-i\omega})^t, t(e^{-i\omega})^t, \ldots, t^{k-1}(e^{-i\omega})^t\}$ when $\omega \in \{0, \pi\}$. For $\omega \in]0, \pi[$, a real basis of this vector space of deterministic polynomials is given by $\{\cos \omega t, \sin \omega t, t \cos \omega t, t \sin \omega t, \ldots, t^{k-1} \cos \omega t, t^{k-1} \sin \omega t\}$. The basic argument used in the proof of the representation theorem relies on the possibility of expanding any product of various real integral operators raised at various powers under the form of a sum of polynomials in each integral operator. We have the following lemma.

LEMMA 2.1. $\forall (l_1, l_2) \in (\mathbb{N}^*)^2$, $\forall (\omega_1, \omega_2) \in [0, \pi]^2$, $\omega_1 \neq \omega_2$,

$$G_{\omega_1}^{l_1} G_{\omega_2}^{l_2} = P_{\omega_1}^{1,(l_1,l_2)}(G_{\omega_1}) + BP_{\omega_1}^{2,(l_1,l_2)}(G_{\omega_1}) + P_{\omega_2}^{1,(l_1,l_2)}(G_{\omega_2}) + BP_{\omega_2}^{2,(l_1,l_2)}(G_{\omega_2}),$$

where $P_{\omega_1}^{g,(l_1,l_2)}$ *and* $P_{\omega_2}^{g,(l_1,l_2)}$, $g = 1, 2$, *are real polynomials without constant term of respective degree* l_1 *and* l_2 *and such that when* $\omega_j \in \{0, \pi\}$, $P_{\omega_j}^{2,(l_1,l_2)} = 0$.

Remark 2.2. Similarly, the deterministic polynomials that are solutions of the following equation:

$$\prod_{j=1}^{n} \Delta_{\omega_j}^{l_j} \mathcal{P}(t) = 0$$

are elements of the vector space $\oplus_{j=1}^{n} \mathcal{T}_{\omega_j}^{l_j}$, where $\oplus$ stands for the direct sum of vector spaces.

3. THE REPRESENTATION THEOREM

Before stating the general representation theorem, we must define what it means to state that the components of a multidimensional time series are polynomially cointegrated.

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DEFINITION 3.1. *A purely nondeterministic real N-dimensional time series y_t is said to be polynomially cointegrated of degree m, m integer, at frequency ω, $\omega \in [0, \pi]$, if there exist a $(1 \times 2N)$ polynomial vector $P_\omega(u)$ of degree m and a deterministic polynomial $\mathcal{P}_\omega(t)$ in $\mathcal{T}_\omega^m$ such that*

$$\left(P_\omega(G_\omega)\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \mathcal{P}_\omega(t)\right)$$

is a covariance stationary process. When $\omega \in \{0, \pi\}$, then the last N elements of $P_\omega(u)$ are equal to 0.

Definition 3.1 is a generalization of the situation studied in Gregoir and Laroque (1993). It defines the general form of the cointegration corresponding to the existence of two complex conjugate unit roots. The cointegration relation involves the current and the once lagged observations of the vector y_t to obtain real polynomial cointegration. This form has been obtained by Hylleberg et al. (1990) for the monomial cointegration. The simple following example will illustrate this feature.

Example 3.1

Let us consider the bivariate time series $y_t = (y_{1,t}, y_{2,t})'$ defined by its Wold representation:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & B \\ -B & 1 \end{pmatrix}\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

The determinant of the Wold representation of y_t is equal to $1 + B^2$. Therefore there exist linear relations of the component of y_t that are $I_{\pi/2}(-1)$. If first we consider each complex root separately, we have

$$(1 \quad -i)\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = (1 + iB)(1 \quad -i)\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

and

$$(1 \quad i)\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = (1 - iB)(1 \quad i)\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

so that $S_{\pi/2}y_{1,t} - iS_{\pi/2}y_{2,t}$ and $S_{-\pi/2}y_{1,t} + iS_{-\pi/2}y_{2,t}$ are two covariance stationary complex processes. We want to work with real linear combinations. We need to extract simultaneously the unit roots i and $-i$, and we obtain

$$(1 \quad -B)\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = (1 + B^2)(1 \quad 0)\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

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and

$$(B \quad 1)\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = (1+B^2)(0 \quad 1)\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

so that $G_{\pi/2}y_{1,t} - G_{\pi/2}y_{2,t-1}$ and $G_{\pi/2}y_{1,t-1} + G_{\pi/2}y_{2,t}$ are covariance stationary processes. We can notice that in $\mathbb{R}^4$, we have two independent relations of cointegration:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} = (1+B^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix},$$

which means that we have two relations of cointegration, one for each unit root. Last notice that if $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are two Gaussian i.i.d. processes, the times series y_t is such that each of its components individually considered is an i.i.d. process, but once piled up into a vector, the autoregressive representation of the process requires the introduction of a error correction term at the frequency $\pi/2$.

We introduce a piece of notation that will be used in the sequel to simplify the statements.

DEFINITION 3.2. *Let a_1 and a_2 be two $p \times q$ matrices. We define a $2p \times 2q$ matrix denoted by $(a_1, a_2)^{*\omega}$ through the equation*

$$(a_1, a_2)^{*\omega} = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 + 2\cos\omega a_2 \end{pmatrix}.$$

We are now in a position to state the general representation theorem.

THEOREM 3.1. *If a process y_t belongs to $\mathcal{C}_\Omega$ where $\Omega = \{(\omega_j, k_j), j = 1, \ldots, K\}$, it satisfies an equation of the following form*:

$$\Phi(B)y_t - \epsilon_t = B\sum_{j=1}^{j=K}\{\Theta^1_{\omega_j}(\Delta_{\omega_j}) + B\Theta^2_{\omega_j}(\Delta_{\omega_j})\} \times \{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)\}, \quad \textbf{(1)}$$

where

(i) *$\Phi(u) = \sum_{l=0}^{+\infty}\phi_l u^l$ is an absolutely summable matrix series of dimension $N \times N$ such that $\Phi(0) = I_N$, and ϵ_t is the innovation of y_t from the Wold representation.*

(ii) *For $g = 1, 2$, $\alpha^g_{\omega_j}(u) = \sum_{l=1}^{m_{\omega_j}}\alpha^g_{\omega_j,l}u^l$ is a real polynomial matrix of dimension $r_{\omega_j} \times N$ and $\beta_{\omega_j}(t)$ is an $r_{\omega_j} \times 1$ real vector whose components belong to $\mathcal{T}^{m_{\omega_j}}_{\omega_j}$. Any linear*

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combination of the r_{ω_j} polynomial error correction terms in $\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)$ is covariance stationary, $r_{\omega_j} \leq N$.

(iii) (*Row decomposition*) *For $g = 1,2$, the $(r_{\omega_j} \times N)$ polynomial matrix $\alpha^g_{\omega_j}(u)$ is made of $m_{\omega_j}(r_{\omega_j,h} \times N)$ submatrices $\alpha^{g,h}_{\omega_j}(u) = \sum_{l=1}^{h} \alpha^{g,h}_{\omega_j,l} u^l$ of degree h, $h = 1,\ldots,m_{\omega_j}$; furthermore $\sum_{j=1}^{m} r_{\omega_j,h} = r_{\omega_j}$. The $(2r_{\omega_j} \times 2N)$ matrix obtained by stacking up the matrices $(\alpha^{1,h}_{\omega_j,h}, \alpha^{2,h}_{\omega_j,h})^{*\omega_j}$ is of full rank $2r_{\omega_j}$.*
(*Column decomposition*) *For $g = 1,2$, $\Theta^g_{\omega_j}(u) = \sum_{l=0}^{m_{\omega_j}-1} \theta^g_{\omega_j,l} u^l$ is a matrix polynomial of dimension $N \times r_{\omega_j}$, with the matrix obtained by stacking up the matrices $(\Theta^1_{\omega_j}(0), \Theta^2_{\omega_j}(0))^{*\omega_j}$ of full rank. The degree of the polynomial submatrices $\Theta^{g,h}_{\omega_j}(u)$ of dimension $N \times r_{\omega_j,h}$ corresponding to $\alpha^{g,h}_{\omega_j}$ is at most equal to $h-1$. Furthermore $\beta^h_{\omega_j}(t)$ of dimension $r_{\omega_j,h} \times 1$ is in $\mathcal{T}^h_{\omega_j}(\beta^h_{\omega_j}(t) = \sum_{l=0}^{h-1} \beta^h_{\omega_j,l} t^l)$.*

(iv) *When $\omega_j \in \{0,\pi\}$, $\alpha^2_{\omega_j}(u)$, and $\Theta^2_{\omega_j}(u)$ are equal to 0.*

(v) *The order of multiplicity k_j of the unit root $e^{-i\omega_j}$ in the determinant of the Wold representation of y_t is equal to $\sum_{h=1}^{m_{\omega_j}} h r_{\omega_j,h}$.*

Theorem 3.1 is an extension of Theorem 1.1 in Gregoir and Laroque (1993). The existence of various unit roots leads to the introduction of polynomial error correction terms in each corresponding integral operator into the equation. A similar representation has been proposed by Hylleberg et al. (1990) for a time series in $\mathcal{C}_\Omega$ with $\Omega = \{(0,r_0),(\pi,r_\pi),(\pi/2,r_{\pi/2})\}$ where r_0, r_π, and $r_{\pi/2}$ are, respectively, the dimensions of the kernel of the matrices $C(1)'$, $C(-1)'$, and $C(i)'$. In equation (1), the variables $(\alpha^1_{\omega_j}(G_{\omega_j}) + B\alpha^2_{\omega_j}(G_{\omega_j}))y_t + \beta_{\omega_j}(t)$, for all j, are covariance stationary, and standard econometric techniques can be used to handle the model once the shape of each error correction term has been determined.

Remark 3.1. In equation (1) are implicitly present two different sets of r_{ω_j} polynomial relations of cointegration, one for each pair of conjugate complex unit roots. Indeed, using the identity

$$B^2 = \Delta_{\omega_j} - (1 - 2\cos\omega_j B) \qquad (J_{\omega_j})$$

and plugging it into the expressions of the r_{ω_j} lagged polynomial relations of cointegration, we obtain that any linear combination of the r_{ω_j} following polynomial error terms $\tilde{\alpha}^2_{\omega_j}(G_{\omega_j})y_t + (\alpha^1_{\omega_j} + 2\cos\omega_j \alpha^2_{\omega_j})(G_{\omega_j})y_{t-1} + \tilde{\beta}_{\omega_j}(t)$ is covariance stationary where $\tilde{\alpha}^2_{\omega_j}(G_{\omega_j})$ is equal to $[-\alpha^2_{\omega_j} + \Delta_{\omega_j}(B)\alpha^2_{\omega_j}](G_{\omega_j})$ after having removed the constant terms, and $\tilde{\beta}_{\omega_j}(t)$ is built from $\beta_{\omega_j}(t-1)$ and terms introduced by the commutation of the operators B and G_ω. There exist complex linear relations between these two sets of r_{ω_j} polynomial vectors. We notice that the terms of highest degree in each pair, namely, the matrices $(\alpha^{1,h}_{\omega_j,h} \quad \alpha^{2,h}_{\omega_j,h})$ and $(-\alpha^{2,h}_{\omega_j,h} \quad \alpha^{1,h}_{\omega_j,h} + 2\cos\omega_j \alpha^{2,h}_{\omega_j,h})$ for $h = 1,\ldots,m_{\omega_j}$, once piled up give the matrices $(\alpha^{1,h}_{\omega_j,h}, \alpha^{2,h}_{\omega_j,h})^{*\omega_j}$.

Remark 3.2. The representation (1) has for each unit root ω_j, r_{ω_j} polynomial cointegration terms appearing at various lags on its right-hand side. It can alternatively be rewritten with k_j contemporaneous polynomial cointegration terms

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when $\omega_j \in \{0, \pi\}$ and $2k_j$ terms when $\omega_j \in]0, \pi[$. To obtain this form, we expand the products

$$\Theta^1_{\omega_j}(\Delta_{\omega_j})\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \beta_{\omega_j}(t)\}$$

when $j \in \{0, \pi\}$ using Property 2.1 and the row decomposition described in (iii) and (iv). It involves covariance stationary terms of the form

$$\Delta^s_{\omega_j}\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \beta_{\omega_j}(t)\}$$

with $s < h$ for $h = 1, \ldots, m_{\omega_j}$. We then move the nonintegrated terms from the right-hand side to the left-hand side of the equation. When $\omega_j \in]0, \pi[$, we first expand the product

$$\Theta^1_{\omega_j}(\Delta_{\omega_j})\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)\}$$

using Property 2.1 and the row decomposition described in (iii). It involves covariance stationary processes of the form

$$\Delta^s_{\omega_j}\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)\}$$

with $s < h$ for $h = 1, \ldots, m_{\omega_j}$. Then according to the preceding remark, we use (J_{ω_j}) to rewrite

$$B\Theta^2_{\omega_j}(\Delta_{\omega_j})\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)\}$$

in the form

$$\Theta^2_{\omega_j}(\Delta_{\omega_j})\{-\alpha^2_{\omega_j}(G_{\omega_j})y_t + (\alpha^1_{\omega_j} + 2\cos\omega_j\alpha^2_{\omega_j})(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t-1) + \Delta_{\omega_j}\alpha^2_{\omega_j}(G_{\omega_j})y_t\}$$

and expand this product using 2.1 and the row decomposition of the polynomial matrices $\alpha^g_{\omega_j}(u)$. At last we move the nonintegrated terms to the other side of the equation. As noted in Gregoir and Laroque (1993), there exist many cross restrictions between the coefficients of these relations because of the way these terms are derived from the set of r_{ω_j} initial cointegrating relations for each ω_j.

Remark 3.3. If we apply the representation theorem to univariate time series with various unit roots, we obtain a similar expansion to that introduced by Hylleberg et al. (1990) to test simultaneously for the presence of different unit roots. This equation can be the basis of a statistical procedure to test for the presence of various unit roots with various orders of multiplicity. The idea consists of testing the nullity of the loading factor of the different integrated processes built from the $I(0)$ covariance stationary representation of the time series under study.

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Example 3.1 (continued)

We can now give the representation under the error correction form of the process introduced previously. We have

$$\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = B\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\left\{G_{\pi/2}y_{2,t} + G_{\pi/2}y_{1,t-1} + (y_{1,0} + \epsilon_{2,0})\cos\left(\frac{\pi}{2}t\right) - (y_{1,2} - \epsilon_{2,1})\sin\left(\frac{\pi}{2}t\right)\right\} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

According to Remark 3.2, we can rewrite this equation in the form

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left\{G_{\pi/2}y_{2,t} + G_{\pi/2}y_{1,t-1} + (y_{1,0} + \epsilon_{2,0})\cos\left(\frac{\pi}{2}t\right) - (y_{2,1} - \epsilon_{2,1})\sin\left(\frac{\pi}{2}t\right)\right\} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix}\left\{G_{\pi/2}y_{1,t} - G_{\pi/2}y_{2,t-1} - (y_{1,1} - \epsilon_{1,1})\sin\left(\frac{\pi}{2}t\right) - (y_{2,0} + \epsilon_{1,0})\sin\left(\frac{\pi}{2}t\right)\right\} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

Example 3.2

The following example illustrates the situation of polynomial cointegration. Let us consider the process $y_t = (y_{1,t}y_{2,t})'$ defined by the following Wold representation:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 2B - B^3 \\ B & 1 \end{pmatrix}\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} = C(B)\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

The determinant of the Wold representation is equal to $(1 + B)^2(1 - B)^2$, which means that because the process is bivariate and we cannot factorize the operator $1 + B$ or $1 - B$ in each univariate component, there exist two polynomial cointegration relations $z_{1,t}$ and $z_{2,t}$. A simple computation shows that the matrix $C(B)$ is such that $(1 \quad -1)C(1) = 0$ and $(1 \quad 1)C(-1) = 0$, which means that $G_0y_{1,t} - G_0y_{2,t}$ and $G_\pi y_{1,t} + G_\pi y_{2,t}$ are stationary processes. Furthermore, if we consider the bivariate process obtained by piling up both cointegration relationships, we can derive from the preceding Wold representation the following equation:

$$\begin{pmatrix} 1 - B & 0 \\ 0 & 1 + B \end{pmatrix}\left\{\begin{pmatrix} G_0y_{1,t} - G_0y_{2,t} \\ G_\pi y_{1,t} + G_\pi y_{2,t} \end{pmatrix} - \begin{pmatrix} 1 & -1 + B + B^2 \\ 1 & 1 + B - B^2 \end{pmatrix}\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}\right\} = 0,$$

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where the second polynomial matrix in B is again not of full rank in 1 and -1. This means that there exist supplementary cointegrating relationships between the integrated processes at frequency 0 and π. We compute these cointegrating directions. Using Property 2.2 and the fact that $G_0 G_\pi = \frac{1}{2}G_0 + \frac{1}{2}G_\pi$, we can define two polynomial relationships at frequency 0 and π, namely, $z_{1,t}$:

$$-G_0^2 y_{1,t} + G_0^2 y_{2,t} + \tfrac{1}{2}G_0 y_{1,t} + \tfrac{1}{2}G_0 y_{2,t} + \beta_{0,1} t + \beta_{0,0}$$
$$= -\tfrac{1}{2}\epsilon_{1,t} + \tfrac{1}{2}(3 + 3B + B^2)\epsilon_{2,t}$$

and $z_{2,t}$:

$$G_\pi^2 y_{1,t} + G_\pi^2 y_{2,t} - \tfrac{1}{2}G_\pi y_{1,t} + \tfrac{1}{2}G_\pi y_{2,t} + (\beta_{\pi,1} t + \beta_{\pi,0})(-1)^t$$
$$= \tfrac{1}{2}\epsilon_{1,t} + \tfrac{1}{2}(3 - 3B + B^2)\epsilon_{2,t},$$

where $\beta_{0,1}, \beta_{\pi,0}, \beta_{\pi,1}, \beta_{\pi,0}$ are linear combinations of $y_0, \epsilon_{.,0}, \epsilon_{.,1}$. An error correction representation of this process is given by

$$\begin{pmatrix} 1 & \frac{5}{8}B \\ -\frac{3}{8}B & 1 \end{pmatrix}\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = B\left\{\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} + (1-B)\begin{pmatrix} \frac{5}{8} \\ -\frac{3}{8} \end{pmatrix}\right\} z_{1,t}$$
$$+ B\left\{\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} + (1+B)\begin{pmatrix} \frac{5}{8} \\ \frac{3}{8} \end{pmatrix}\right\} z_{2,t} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

The coefficients in representation equation (1) are not identified. Equation (1) is not changed if we premultiply $\alpha^1_{\omega_j}(G_{\omega_j})$ and $\alpha^2_{\omega_j}(G_{\omega_j})$ by a lower triangular block diagonal $(r_{\omega_j} \times r_{\omega_j})$ constant matrix with full rank blocks of dimension $(r_{\omega_j,h} \times r_{\omega_j,h})$, $h = 1, \ldots, m_{\omega_j}$ on the diagonal and simultaneously multiply the corresponding $\Theta^1_{\omega_j}(\Delta_{\omega_j})$ and $\Theta^2_{\omega_j}(\Delta_{\omega_j})$ from the right by its inverse. Similarly, we can multiply the polynomial matrix $\alpha^g_{\omega_j}$ by an upper triangular polynomial matrix in $u = \Delta_{\omega_j}(B)$, whose (l_1, l_2) blocks have $u^{l_2 - l_1}$ in factor and rearrange terms to get another form of (1). We therefore need for estimation purposes an identification rule that reduces the degree of indetermination on each $\alpha^g_{\omega_j}$.

The identification rule we will consider may appear complicated. It relies on the estimation procedure that we describe in Part II (Gregoir, 1999) and is easily satisfied in practice. For $\omega_j \in \{0, \pi\}$, it is similar to the choice made in Gregoir and Laroque (1994), and we state a theorem without proof. For $\omega_j \in]0, \pi[$, the identification rule is more complicated. For each h, we have to make a careful choice of $r_{\omega_j,h}$ polynomials between $2r_{\omega_j,h}$ candidates that are related two by two (cf. Remark 3.1). We have to pick up one polynomial in each couple.

THEOREM 3.2 (Identification). *For any process $y_t \in \mathcal{C}_\Omega$, the representation* (1) *of Theorem* (3.1) *can be chosen with the following properties, when $\omega_j \in \{0, \pi\}$.*

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1. *There exists an* $(N - r_{\omega_j} \times N)$ *matrix* $\alpha^0_{\omega_j}$ *such that*

$$\begin{pmatrix} \alpha^0_{\omega_j} \\ \alpha^{1,1}_{\omega_j}(G_{\omega_j}) \\ \vdots \\ \alpha^{1,m_{\omega_j}}_{\omega_j}(G_{\omega_j}) \end{pmatrix} y_t + \begin{pmatrix} 0 \\ \beta^1_{\omega_j}(t) \\ \vdots \\ \beta^{m_{\omega_j}}_{\omega_j}(t) \end{pmatrix} = \begin{pmatrix} E^0_{\omega_j}(B) \\ E^1_{\omega_j}(B) \\ \vdots \\ E^{m_{\omega_j}}_{\omega_j}(B) \end{pmatrix} \varepsilon_t$$

$$= E_{\omega_j}(B)\varepsilon_t, \tag{2}$$

   where $E_{\omega_j}(e^{-i\omega_j})$ *is of full rank.*
2. *The* $(N \times N)$ *matrix*

$$\begin{pmatrix} \alpha^0_{\omega_j} \\ \alpha^{1,1}_{\omega_j,1} \\ \vdots \\ \alpha^{1,m_{\omega_j}}_{\omega_j,m_{\omega_j}} \end{pmatrix}$$

   is unitary.
3. *The variance covariance matrix of* $E_{\omega_j}(B)\varepsilon_t$ *is diagonal. For all* $h = 0,1,\ldots,m_{\omega_j}$, *the variance covariance matrix of* $\alpha^h_{\omega_j}(G_{\omega_j})y_t + \beta^h_{\omega_j}(t) = E^h_{\omega_j}(B)\varepsilon_t$ *has nonincreasing diagonal terms. It is a minimum of the variance covariance matrices of the processes* $a^h(G_{\omega_j})y_t + b^h_{\omega_j}(t)$, *over all polynomials* $a^h(G_{\omega_j})$ *of degree* h (*without constant term*) *but with deterministic trends in* $\mathcal{T}^h_{\omega_j}$, *such that* $a^h_h(a^h_h)'$ *is equal to the identity matrix of dimension* $r_{\omega_j,h}$ *and* a^h_h *is orthogonal to* α^ℓ_ℓ *for all* ℓ, $h < \ell \leq m_{\omega_j}$.

The preceding conditions 1–3 *generically determine the error correction terms uniquely, up to a sign transformation* $(+ \text{ or } -)$.

The identification rule described previously does not allow us to identify some cointegration directions when the minimum considered at each step h in condition 3 is not unique, i.e., when there are some eigenvalues with an order of multiplicity larger than 1. Then, we only identify a subspace of polynomial cointegration of degree h. If it seems sensible to disregard this problem when $\omega_j \in \{0,\pi\}$, the situation is different when $\omega_j \in]0,\pi[$, simply because it may correspond to a richer informational content. We have to stress that the error correction terms may have two different forms. Generally integrated processes at two consecutive dates are necessary to build up a covariance stationary process, but for some directions and some degrees, they may involve only contemporaneous variables ($\alpha^{1,h}_{\omega_j}(G_\omega)$ or $\alpha^{2,h}_{\omega_j}(G_\omega)$ may be of rank smaller than $r_{\omega_j,h}$ for some $h \in \{1,\ldots,m_{\omega_j}\}$). If we look for a similar identification rule when $\omega_j \in]0,\pi[$, the cases of nonidentification may correspond to a direction of polynomial cointegration that involves only contemporaneous variables. For such a cointegrating direction any linear combination of $(\alpha^{1,h}_{\omega_j}(G_{\omega_j}) \quad 0)$ and $(0 \quad \alpha^{1,h}_{\omega_j}(G_{\omega_j}))$ is a good candidate to characterize the stationary direction. A constraint on the terms of higher degree in the polynomial such as that proposed in Theorem 3.1 is not sufficient. It may be of interest to know the di-

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mension of the space of contemporaneous polynomial cointegration for each degree, but this means more information available or assumed, more complicated null hypothesis and identification rule. For sake of simplicity, we omit this situation in the following theorem.[1]

THEOREM 3.3 (Identification). *For any process* $y_t \in \mathcal{C}_\Omega$, *the representation* (1) *of Theorem* 3.1 *can be chosen with the following properties when* $\omega_j \in]0, \pi[$.

1. *There exists an* $(N - r_{\omega_j} \times 2N)$ *matrix* $(\alpha^{1,0}_{\omega_j} \quad \alpha^{2,0}_{\omega_j})$ *such that*

$$\begin{pmatrix} \alpha^{1,0}_{\omega_j} & \alpha^{2,0}_{\omega_j} \\ \alpha^{1,1}_{\omega_j}(G_{\omega_j}) & \alpha^{2,1}_{\omega_j}(G_{\omega_j}) \\ \vdots & \\ \alpha^{(1,m_{\omega_j}}_{\omega_j}(G_{\omega_j}) & \alpha^{2,m_{\omega_j}}_{\omega_j}(G_{\omega_j}) \end{pmatrix} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \beta^1_{\omega_j}(t) \\ \vdots \\ \beta^{m_{\omega_j}}_{\omega_j}(t) \end{pmatrix}$$

$$= \begin{pmatrix} E^0_{\omega_j}(B) \\ E^1_{\omega_j}(B) \\ \vdots \\ E^{m_{\omega_j}}_{\omega_j}(B) \end{pmatrix} \varepsilon_t = E_{\omega_j}(B)\varepsilon_t, \tag{3}$$

   where $E_{\omega_j}(e^{-i\omega_j})$ *is a complex matrix of full rank.*
2. *The* $(N \times 2N)$ *matrix*

$$\begin{pmatrix} \alpha^{1,0}_{\omega_j} & \alpha^{2,0}_{\omega_j} \\ \alpha^{1,1}_{\omega_j,1} & \alpha^{2,1}_{\omega_j,1} \\ \vdots & \vdots \\ \alpha^{1,m_{\omega_j}}_{\omega_j,m_{\omega_j}} & \alpha^{2,m_{\omega_j}}_{\omega_j,m_{\omega_j}} \end{pmatrix}$$

   is composed of N orthonormal vectors.
3. *The variance covariance matrix of* $E_{\omega_j}(B)\varepsilon_t$ *is diagonal. For all* $h = 0,1,\ldots,m_{\omega_j}$, *the variance covariance matrix of* $E^h_{\omega_j}(B)\varepsilon_t$ *has nonincreasing diagonal terms. It is a minimum of the variance covariance matrices of the processes* $a^{1,h}(G_{\omega_j})y_t + a^{2,h}(G_{\omega_j})y_{t-1} + b^h_{\omega_j}(t)$, *over all polynomials* $(r_{\omega_j,h} \times 2N)$ $a^h(G_{\omega_j})$ *of degree h, without constant term but with deterministic trends in* $\mathcal{T}^h_{\omega_j}$, *under the following constraints*: $a^h_h(a^h_h)'$ *is equal to the identity matrix of dimension* $r_{\omega_j,h}$ *and* $a^h_{h(q)}$, *the qth vector of* a^h_h, *is orthogonal to* $(-a^{2,h}_{h(q')} \quad a^{1,h}_{h(q')} + 2\cos\omega_j a^{2,h}_{h(q')})$ $\forall 1 \le q' < q$, *and* $\forall \ell$, $h < \ell \le m_{\omega_j}$ *to* $(\alpha^{1,\ell}_\ell, \alpha^{2,\ell}_\ell)^{*\omega_j}$.

The preceding conditions 1–3 *generically determine the error correction terms uniquely, up to a sign transformation* (+ *or* −).

In the following section, to provide a better insight into the structure of the representation theorem, we give the arguments of its proof in a particular case—the case of multivariate time series that have only two conjugate complex unit

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roots in the determinant of their Wold representation. The arguments of the proof in the general case are in Section 5.

4. PARTICULAR CASE OF TWO COMPLEX CONJUGATE UNIT ROOTS

We consider in this part an N-dimensional covariance stationary process in $\mathcal{C}_\Omega$ where $\Omega = \{(\omega, k)\}, \omega \in]0, \pi[$. We first describe the algebraic properties of the matrix of the Wold representation of such a process in the same spirit as that of Lemma 3.1 in Gregoir and Laroque (1993). Then we use this structure to derive the representation theorem. In the sequel, we use implicitly the assumptions we made about the matrix of the Wold representation in Definition 2.3, particularly that this matrix is an analytic function in the disk of radius $\rho, \rho > 1$. Therefore, it can be expanded around any point on the unit circle; the same holds for its derivatives.[2]

The basic argument on which the representation theorem relies in this case is the following one: If an $N \times N$ polynomial matrix $C(B)$ is such that its determinant is equal to $\Delta_\omega(B)^k d(B)$ with $d(e^{-i\omega}) \neq 0$ then there exist two $N \times N$ polynomial matrices $R(B)$ (with finite degree) and $D(B)$ such that $C(B) = R(B)D(B)$, $\det R(B) = \Delta_\omega(B)^k$ and $\det D(B) = d(B)$. The row structure of $R(B)$ is particular (cf. Lemma 4.3), and we can find two $N \times N$ polynomial matrices of finite degree $A^1(B)$ and $A^2(B)$ and a diagonal polynomial matrix $\delta_\omega(B)$ whose diagonal terms are monomial terms such that

$$\delta_\omega(\Delta_\omega(B))\{(A^1(G_\omega) + BA^2(G_\omega))R(B) - I_N\} = 0.$$

Here $A^1(G_\omega) + BA^2(G_\omega)$ plays the role of a generalized inverse of $R(B)$. Its row structure depends heavily on that of $R(B)$. Once we have got this result, we can apply the preceding equation to $D(B)\epsilon_t$ to get the result we are looking for. Structures and rank conditions are by-products of the structure of $R(B)$.

We first present the algebraic properties of a real matrix series $C(B)$ such that its determinant is equal to $\Delta_\omega(B)^k d(B)$ with $d(e^{-i\omega}) \neq 0$. Then we know that the complex matrix $C(e^{-i\omega})$ is not of full rank. If a is a full rank complex matrix such that $aC(e^{-i\omega}) = 0$ then $\bar{a}$ is such that $\bar{a}C(e^{i\omega}) = 0$ ($\bar{z}$ is the conjugate of z). We have the following result on the form of the linear combinations that allows us to extract two complex conjugate unit roots.

LEMMA 4.1. *Let $C(B)$ be a real matrix series of dimension $N \times N$ such that $C(e^{-i\omega})$ is of rank $N - r$ and let a' be an $N \times r$ complex matrix of full rank such that $aC(e^{-i\omega}) = 0$. Then there exists a space of $(r \times N)$ real polynomial matrices $\mathcal{K}_a = \{P_a(\phi)(B) = \phi a(1 - e^{-i\omega}B) + \bar{\phi}\bar{a}(1 - e^{i\omega}B) = 2\Re(\phi a) - 2\Re(\phi a e^{-i\omega})B$, where ϕ is an $r \times r$ complex matrix of full rank$\}$ such that*

$$\forall P_a(\phi) \in \mathcal{K}_a, \qquad P_a(\phi)(B)C(B) = \Delta_\omega(B)D_\phi(B),$$

where $D_\phi(B)$ is a matrix series of dimension $r \times N$ such that $D_\phi(e^{-i\omega})$ is well defined.

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We can thus only extract r couples of conjugate complex roots when the rank of the matrix $C(e^{-i\omega})$ is equal to $N-r$. A careful choice of the matrix ϕ allows us to build an $N \times N$ invertible polynomial matrix that isolates r couples of unit roots.

LEMMA 4.2. *Let $C(B)$ be a real matrix series of dimension $N \times N$ such that $C(e^{-i\omega})$ is of rank $N-r$. There exists on $N \times N$ real polynomial matrix $A(B)$ whose rows can be partitioned into two subgroups: an $N-r \times N$ real constant matrix and an $r \times N$ real polynomial of degree one such that*:

(i) $A(B)C(B) = \delta_\omega(\Delta_\omega)D(B)$ *where*
  (a) $\delta_\omega(\Delta_\omega)$ *is a diagonal matrix whose $N-r$ first elements are equal to one and the r remaining ones to $\Delta_\omega(B)$*;
  (b) $D(B)$ *is an $N \times N$ real matrix series such that $D(e^{-i\omega})$ is well defined*;
(ii) $\det A(B) = 1$.

Remark 4.1. When $\ker C(e^{-i\omega})^1$ contains a real direction, the corresponding polynomial in $A(B)$ is a constant term.

Remark 4.2. The matrix $A(e^{-i\omega})$ is a full rank complex matrix.

As remarked in Example 3.1 and Remark 3.1, it is possible to build in $\mathbb{R}^{2N}$ $2r$ linearly independent $2N \times 1$ vectors such that once stacked up in a $2r \times 2N$ matrix premultiplying $(C(B)' \quad BC(B)')'$, it allows us to factorize $2r$ polynomials $\Delta_\omega(B)$. There exist as many independent relations of cointegration as unit roots. These matrices are derived from the complex matrix $A(e^{-i\omega})$. The degree of the polynomial matrix $A(B)$ is one, i.e., $A(B) = A_0 + BA_1$. Remark 4.2 and Property 4.1 hereinafter allow us to build with the help of submatrices of the real matrices A_0, A_1 a $2r \times 2N$ real matrix of full rank, rows of which capture the complete set of independent cointegration directions.

Property 4.1. Let $\vartheta^1 + e^{i\omega}\vartheta^2$ be an $r \times N$ complex matrix of full rank r where ϑ^1 and ϑ^2 are real matrices. Then

$$(\vartheta^1, \vartheta^2)^{*\omega} = \begin{pmatrix} \vartheta^1 & \vartheta^2 \\ -\vartheta^2 & \vartheta^1 + 2\cos\omega\vartheta^2 \end{pmatrix}$$

is a real matrix of full rank $2r$.

We want to eliminate all the couples of conjugate complex roots by using the integral operator. We have to use iteratively Lemma 4.1 and apply it to the matrix $D(B)$ itself when it is such that $D(e^{-i\omega})$ is not of full rank, and so on. To keep the combination of these matrices manageable in the sequel, we have to be more precise on the form of the matrix $A(B)$. We get the following corollary.

COROLLARY 4.1. *If in Lemma* 4.2, *$C(B)$ is such that the r' first rows of $C(e^{-i\omega})$ are of full rank for some $r' \leq N-r$, then $A(B)$ can be chosen with the following form*:

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$$\begin{pmatrix} I_{r'} & 0 \\ 0 & A_{22}^0 \\ A_{31}^0 + A_{31}^1 B & A_{32}^0 + A_{32}^1 B \end{pmatrix} = A(B),$$

where A_{22}^0, A_{31}^j, *and* A_{32}^j, $j \in \{0,1\}$ *are, respectively, an* $N - r' - r \times N - r'$, *an* $r \times r'$, *and an* $r \times N - r'$ *real matrix, and* $\det A(B) = 1$.

We build an induction proof to extract the k couples and obtain the following lemma.

LEMMA 4.3. *Let* $C(B)$ *be a real matrix series of dimension* $N \times N$. *Then* $\det C(B) = (\Delta_\omega(B))^k d(B)$ *with* $d(e^{-i\omega}) \neq 0$ *if and only if there exist an* $N \times N$ *polynomial matrix* $R(B)$, *an* $N \times N$ *matrix series* $D(B)$, *and a set of* m *ordered nonincreasing integers* $(s_h)_h$ *such that*

(i) $C(B) = R(B)D(B)$;
(ii) $D(e^{-i\omega})$ *is of full rank and* $\det D(B) = d(B)$;
(iii) $R(B) = \prod_{h=1}^m \{A_h^{-1}(B)\delta_{\omega,h}(\Delta_\omega)\}$ *where*
  (a) $A_1(B)$ *has the form described in Lemma* 4.2 *with* $r = r_\omega = s_1$;
  (b) $A_h(B), h = 2,\ldots,m$ *has the form described in Corollary* 4.1 *with* $r' = N - s_{h-1}, r = s_h$;
  (c) $\delta_{\omega,h}(\Delta_\omega), h = 1,2,\ldots,m$ *has the form described in Lemma* 4.2 *with* $r = s_h$;
  (d) $\sum_{j=h}^m s_h = k$.

Remark 4.3. The sequence of integers $(s_h)_h$ is nonincreasing because $\forall h$, the matrix structure described in Lemma 4.2 is such that $r' = N - s_{h-1} \leq N - r = N - s_h$.

We now use this lemma to get a moving average invertible representation of some well-chosen linear polynomial combinations of the time series y_t. We have

$$y_t = A_1^{-1}(B)\delta_{\omega,1}(\Delta_\omega) \prod_{h=2}^m (A_h^{-1}(B)\delta_{\omega,h}(\Delta_\omega))D(B)\epsilon_t.$$

Using Property 2.1, we get

$$\delta_{\omega,1}(\Delta_\omega)\left\{\delta_{\omega,1}(G_\omega)A_1(B)y_t - \prod_{j=2}^m (A_j^{-1}(B)\delta_{\omega,j}(\Delta_\omega))D(B)\epsilon_t\right\} = 0.$$

This implies that there exists a deterministic vector $b_{\omega,1}(t)$ whose $N - r$ first components are equal to zero and the following r ones belong to $\mathcal{T}_\omega^1$ such that

$$\delta_{\omega,1}(G_\omega)A_1(B)y_t + b_{\omega,1}(t) = \prod_{j=2}^m (A_j^{-1}(B)\delta_{\omega,j}(B))D(B)\epsilon_t.$$

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We repeat this computation using the structure of $A_j(B), j = 2, \ldots, m$. The second step gives, using $B^2 G_\omega^2 = G_\omega - G_\omega^2 + 2\cos\omega B G_\omega^2$,

$$\begin{pmatrix} A^0 y_t \\ \alpha_1^{1,1} G_\omega y_t + \alpha_1^{2,1} G_\omega y_{t-1} + \beta_1(t) \\ \sum_{l=1}^{2} a_l^{1,2} G_\omega^l y_t + a_l^{2,2} G_\omega^l y_{t-1} + b_2(t) \end{pmatrix} = \prod_{h=3}^{m} (A_h^{-1}(B)\delta_{\omega,h}(\Delta_\omega)) D(B)\epsilon_t,$$

where $\alpha_1^{1,1}$ and $\alpha_1^{2,1}$ are $((s_1 - s_2 = r_{\omega,1}) \times N)$ real matrix. The matrix obtained by stacking up $(\alpha_1^{1,1}, \alpha_1^{2,1})^{*\omega}$ and $(a_2^{1,2}, a_2^{2,2})^{*\omega}$ is a matrix of full rank. Indeed, we first have to notice that

$$A_2(B)A_1(B) = A_{2,0}A_{1,0} - A_{2,1}A_{1,1} + B(A_{2,1}A_{1,0} + A_{2,0}A_{1,1} + 2\cos\omega A_{2,1}A_{1,1}) + \Delta_\omega(B)A_{2,1}A_{1,1},$$

second that the matrix constant term and the matrix coefficient of degree one in this polynomial correspond to the highest degree terms in G_ω obtained in the preceding equation, and third, that we can apply Property 4.1 to

$$A_{2,0}A_{1,0} - A_{2,1}A_{1,1} + e^{-i\omega}(A_{2,1}A_{1,0} + A_{2,0}A_{1,1} + 2\cos\omega A_{2,1}A_{1,1}),$$

which is a full rank matrix as the product of two full rank matrices (cf. Remark 4.2). At last, $b_2(t) \in \mathcal{T}_\omega^2$. When the m operations are completed, we obtain an invertible moving average representation of

$$\begin{pmatrix} A^0 y_t \\ \alpha_1^{1,1} G_\omega y_t + \alpha_1^{2,1} G_\omega y_{t-1} + \beta_1(t) \\ \vdots \\ \sum_{l=1}^{m} \alpha_l^{1,m} G_\omega^l y_t + \alpha_l^{2,m} G_\omega^l y_{t-1} + \beta_m(t) \end{pmatrix} = D(B)\epsilon_t,$$

where $\forall h = 1, \ldots, m$, $\forall l = 1, \ldots, h$, $g = 1, 2, \alpha_l^{g,h}$ are $((s_h - s_{h+1} = r_{\omega,h}) \times N)$, the matrix obtained by stacking up the terms $(\alpha_h^{1,h}, \alpha_h^{2,h})^{*\omega}$ for $h = 1, \ldots, m$ is a $2r_\omega \times 2N$ matrix of full rank, and each $\beta_h(t) \in \mathcal{T}_\omega^h$. Last notice that $\sum_{h=1}^{m} h r_{\omega,h} = \sum_{h=1}^{m-1} h(s_h - s_{h+1}) + m s_m = \sum_{h=1}^{m} s_h = k$.

To get the representation theorem, we first rewrite the preceding equation by premultiplying by $\Pi_{h=1}^{m} \delta_{\omega,h}(\Delta_\omega)$ each side of the equation and using Property 2.1. We get $A_\omega(B)C(B) = (\Pi_{h=1}^{m} \delta_{\omega,h}(\Delta_\omega))D(B)$ where

$$A_\omega(B)y_t = \prod_{h=1}^{m} \delta_{\omega,h}(\Delta_\omega) \begin{pmatrix} A^0 y_t \\ \alpha^1(G_\omega)y_t + \alpha^2(G_\omega)y_{t-1} + \beta(t) \end{pmatrix},$$

whence we get $\det A_\omega(B) = 1$ and $A_\omega(0) = D(0)$. This result corresponds to Lemma 3.1 in Gregoir and Laroque (1993).

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We inverse $D(B)$ and partition its columns into $m+1$ subgroups $D^{-1,h}(B)$ of respective dimensions $N \times N - r_\omega$ for $h = 0$ and $N \times r_{\omega,h}$ for $h = 1, \ldots, m$. Then using the factorization $1 - u^h = (1-u)\vartheta_h(u)$ for $u = \Delta_\omega$, we write

$$D^{-1,h}(B) = D^{-1,h}(B)\Delta_\omega^h + B(2\cos\omega - B)\vartheta_h(\Delta_\omega)D^{-1,h}(B).$$

We expand each element of the second term of the right-hand side of the previous equation (it is a well-defined matrix series inside and on the unit circle) and rewrite it under the form

$$B\tilde{D}^{-1,h}(B)\Delta_\omega^h + B\mathcal{R}_h(B),$$

where $\mathcal{R}_h(B)$ plays the role of a remainder of degree strictly less than $2h$.[3] If we notice that $\{\Delta_\omega^l, B\Delta_\omega^l, l = 0, \ldots, h-1\}$ is a basis of the vector space of the polynomials of degree at most $2h-1$ in B, we conclude that

$$D^{-1}(B) = (D^{-1}(B) + B\tilde{D}^{-1}(B))\prod_{j=1}^{m}\delta_{\omega,j}(\Delta_\omega) + B(0 \quad \Theta^1(\Delta_\omega) + B\Theta^2(\Delta_\omega)),$$

where $\Theta^1(u)$ and $\Theta^2(u)$ are as described in (iii) in Theorem 3.1. The rank properties are proven subsequently. We thus have obtained the representation (1) of Theorem 3.1 with $\Phi(B) = (D^{-1}(B) + B\tilde{D}^{-1}(B))A(B)$. The representation equation can be rewritten under an algebraic form by identifying the matrices after replacing y by $C(B)\epsilon$:

$$I_N = \Phi(B)C(B) + B(0 \quad \Theta^1(\Delta_\omega) + B\Theta^2(\Delta_\omega))D(B)$$

so that if we consider the value $B = e^{i\omega}$, because $C(e^{i\omega})$ is of rank $N - r_\omega$ and $D(e^{i\omega})$ is of full rank, we conclude that $\Theta^1(0) + e^{i\omega}\Theta^2(0)$ is a complex matrix of full rank r_ω. The rank properties in Theorem 3.1 (iii) are derived from Property 4.1.

5. GENERAL CASE

The proof of Theorem 3.1 is different from that of the representation theorem proved in Gregoir and Laroque (1993) that can be used when there exists only one unit root (1 or -1). When there are at least two different unit roots that are not conjugate as in the special case considered previously, we would like to extract the various unit roots simultaneously, but we cannot always obtain a neat algebraic property similar to Lemma 3.1 in Gregoir and Laroque (1993) or to the algebraic representation obtained in the preceding section, which are the basis of the proofs. For instance, one reason why such a property is not available can be that we have introduced a new unit root by extracting the previous ones simultaneously and thus obtained a noninvertible moving average representation.

The basic arguments used in the general proof are the following ones: First, as we noted in Lemma 2.1 and Remark 2.2, we can replace a product of integral

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operators by a sum of polynomials in each operator and have a similar property for the deterministic polynomials we consider. Second, when a linear combination of polynomials in integral operators at different frequencies applied to the process under study is covariance stationary then each polynomial in each integral operator applied to the process gives a covariance stationary process (hereinafter Property 5.1). Third, these polynomials can be rewritten as the product of a polynomial in the lag operator and some well-chosen polynomials in the integral operator that play the role of the basis of the cointegration space (hereinafter Lemma 5.1). We can then iteratively use the type of computation we made in the preceding section. We deal with the various linear combinations of products in different integral operators to derive a decomposition of the covariance stationary terms into a sum of cointegration sets at each frequency.

We thus start by stating a simple statistical property.

Property 5.1. Let $(\omega_1,\dots,\omega_n)$ be an n-tuple of different values of $[0,\pi]$, $(P_1(u),\dots,P_n(u))$ be n $(1\times 2N)$ matrix polynomials of respective degree $m_1,\dots,m_n$, and $(\mathcal{P}_1(t),\dots,\mathcal{P}_n(t))$ be n deterministic polynomials, respectively, elements of $\mathcal{T}_{\omega_1}^{m_1},\dots,\mathcal{T}_{\omega_n}^{m_n}$, such that the N last terms of $P_j(u)$ are equal to 0 when $\omega_j\in\{0,\pi\}$.

Let y_t be an N-dimensional purely nondeterministic covariance stationary process.

If there exists $(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$ such that

$$\sum_{j=1}^{n}\lambda_j\left(P_j(G_{\omega_j})\begin{pmatrix}y_t\\ y_{t-1}\end{pmatrix}+\mathcal{P}_j(t)\right)$$

is a nondeterministic covariance stationary process then $\forall j\in\{1,\dots,n\}$,

$$\lambda_j\left(P_j(G_{\omega_j})\begin{pmatrix}y_t\\ y_{t-1}\end{pmatrix}+\mathcal{P}_j(t)\right)$$

are nondeterministic covariance stationary processes.

The following lemma whose first statement sums up some results from Gregoir and Laroque (1993) and from the preceding section introduces notations.

LEMMA 5.1. *Let y_t be an N-dimensional covariance stationary time series in $\mathcal{C}_\Omega$ where $\Omega=\{(\omega_j,k_j),j=1,\dots,K\}$. Then*

(i) *if we denote $N-r_{\omega_j}$ the rank of the matrix $C(e^{i\omega_j})$, there exist an $N-r_{\omega_j}\times N$ real matrix of full rank $A_{\omega_j}^0$ and $2m_{\omega_j}$ polynomial matrices $\alpha_{\omega_j}^{g,h}(G_{\omega_j})$ $h=1,\dots,m_{\omega_j}$, $g=1,2$ of respective dimension $r_{\omega_j,h}\times N$ and degree h and m_{ω_j} deterministic trend vector $\beta_{\omega_j,h}(t)$ of respective dimension $r_{\omega_j,h}\times 1$ element of $\mathcal{T}_{\omega_j}^h$ such that*

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$$\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha^{1,1}_{\omega_j}(G_{\omega_j}) y_t + \alpha^{2,1}_{\omega_j}(G_{\omega_j}) y_{t-1} + \beta_{\omega_j,1}(t) \\ \vdots \\ \alpha^{1,m_{\omega_j}}_{\omega_j}(G_{\omega_j}) y_t + \alpha^{2,m_{\omega_j}}_{\omega_j}(G_{\omega_j}) y_{t-1} + \beta_{\omega_j,m_{\omega_j}}(t) \end{pmatrix} = D_{\omega_j}(B)\epsilon_t, \tag{4}$$

where $D_{\omega_j}(e^{i\omega_j})$ is a complex matrix of full rank. The matrix $A_{\omega_j}(B)$ obtained by premultiplying the left-hand side of equation (4) *by a diagonal matrix $\delta_{\omega_j}(\Delta_\omega)$ whose diagonal element on the rows corresponding to $\alpha^h_{\omega_j}(G_{\omega_j})$ is equal to $\Delta^h_{\omega_j}$ has a determinant equal to* 1. *Furthermore $\sum_{h=1}^{m_{\omega_j}} r_{\omega_j,j} h = k_{\omega_j}$. When $\omega_j \in \{0,\pi\}$, $\alpha^{2,\cdot}_{\omega_j}(u) = 0$.*

(ii) *When $\omega_j \in \{0,\pi\}$, if $P(G_{\omega_j})y_t + b(t)$ is a nondeterministic covariance stationary process where $P(u)$ is a polynomial vector of degree n then*
   (a) $n \leq m_{\omega_j}$;
   (b) *there exist*
      a real polynomial vector $\rho_{\omega_j}(B)$ of degree at most $m_{\omega_j} - 1$,
      a real polynomial vector $\lambda_{\omega_j}(B)$ whose columns can be partitioned into $m_{\omega_j} + 1$ subgroups according to the subdivision of (i) *where the polynomials in the hth subgroup are of degree at most h,*
      such that

$$P(G_{\omega_j})y_t + b(t) = \lambda_{\omega_j}(B)\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha_{\omega_j}(G_{\omega_j}) y_t + \beta_{\omega_j}(t) \end{pmatrix} + \Delta_{\omega_j}\rho_{\omega_j}(B) y_t. \tag{5}$$

(iii) *When $\omega_j \in]0,\pi[$, if $P^1(G_{\omega_j})y_t + P^2(G_{\omega_j})y_{t-1} + b(t)$ is a nondeterministic covariance stationary process where $P^g(u)$ $(g = 1,2)$ are polynomial vectors of degree at most n then*
   (a) $n \leq m_{\omega_j}$;
   (b) *there exist*
      a real polynomial vector $\rho_{\omega_j}(B)$ of degree at most $m_{\omega_j} - 1$,
      two real polynomial vectors $\lambda^g_{\omega_j}(B)$ whose columns can be partitioned into $m_{\omega_j} + 1$ subgroups according to the subdivision of (i) *where the polynomials in the hth subgroup are of degree at most h,*
      such that

$$\begin{aligned} &P^1(G_{\omega_j})y_t + P^2(G_{\omega_j})y_{t-1} + b(t) \\ &\quad = (\lambda^1_{\omega_j}(B) + B\lambda^2_{\omega_j}(B))\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha^1_{\omega_j}(G_{\omega_j}) y_t + \alpha^2_{\omega_j}(G_{\omega_j}) y_{t-1} + \beta_{\omega_j}(t) \end{pmatrix} \\ &\qquad + \Delta_{\omega_j}\rho_{\omega_j}(B) y_t. \end{aligned} \tag{6}$$

The first step of the proof consists of extracting each unit root from the matrix of the Wold representation $C(B)$. To do so, start for instance from the representation (4) according to the value of ω_1. This defines a new nondeterministic covariance stationary process $y_{1,t}$. If Ω is a singleton, the proof stops. Otherwise, we apply the same reasoning to $y_{1,t}$ for the following unit root ω_2.[4] Then use Property

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2.5 and Lemma 2.1 to expand these polynomials in both integral operators in ω_1 and ω_2 into a sum of polynomials in each integral operator. Separate the deterministic polynomial in a sum of two polynomials, one in each corresponding vectors space $\mathcal{T}_{\omega_j}$ whose degree corresponds to the highest degree of the integral operator in ω_j in each row according to Remark 2.2. Finally, we get a sum of two terms, a polynomial matrix in G_{ω_1} and the deterministic vector in the corresponding vector space and a similar object for the second unit root ω_2. If these were the only unit roots of the time series y_t, Property 5.1 says that each term of this sum is a nondeterministic covariance stationary time process. (ii) or (iii) in Lemma 5.1 allows us to rewrite the two last terms under a matrix form involving the matrices defined in Lemma 5.1 plus a polynomial matrix in y_t. If there exists another unit root perform the same operation and expand similarly the new polynomials in integral operators corresponding to two unit roots in a sum of polynomials in each integral operator. After K operations, the following equation is obtained:

$$D(B)\epsilon_t = CR(B)y_t + \sum_{\omega_j \in \{0,\pi\}} \Lambda_{\omega_j}(B)\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha^1_{\omega_j}(G_{\omega_j})y_t + \beta_{\omega_j}(t) \end{pmatrix} + \sum_{\omega_j \in]0,\pi[} (\Lambda^1_{\omega_j}(B) + B\Lambda^2_{\omega_j}(B))\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t) \end{pmatrix},$$

where $D(B)$ is an invertible matrix and $CR(B)$ is a polynomial matrix that is built from the polynomials $\rho(u)$ of (ii) and (iii) in Lemma 5.1. Notice that because $A_{\omega_j}(0) = D_{\omega_j}(0)$, the following equation is satisfied:

$$D(0) = \sum_{\omega_j \in \{0,\pi\}} \Lambda_{\omega_j}(0)A_{\omega_j}(0) + \sum_{\omega_j \in]0,\pi[} \Lambda^1_{\omega_j}(0)A_{\omega_j}(0) + CR(0).$$

Then rewrite each element of this sum under the following form when $\omega_j \in \{0,\pi\}$:

$$\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha_{\omega_j}(G_{\omega_j})y_t + \beta_{\omega_j}(t) \end{pmatrix} = A_{\omega_j}(B)y_t + B\Psi_{\omega_j}(B)(\alpha_{\omega_j}(G_{\omega_j})y_t + \beta_{\omega_j}(t))$$

and when $\omega_j \in]0,\pi[$

$$\begin{pmatrix} A^0_{\omega_j} y_t \\ \alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t) \end{pmatrix} = A_{\omega_j}(B)y_t + B\Psi_{\omega_j}(B)(\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)),$$

where $\Psi_{\omega_j}(B)$ is an $N \times r_{\omega_j}$ matrix equal to the last r_{ω_j} columns of $I_N - \delta_{\omega_j}(B)$. After this, use the fact that, when $\omega_j \in \{0,\pi\}$, $D^{-1}(B)\Lambda_{\omega_j}(B)\Psi_{\omega_j}(B)$ is a matrix series whose elements and their derivatives can be expanded around $e^{i\omega_j}$. Rewrite it under the form $\tilde{D}_{\omega_j}(B)\tilde{\delta}_{\omega_j}(B) + \Theta_{\omega_j}(\Delta_{\omega_j})$ where $\Theta_{\omega_j}(B)$ has the form described

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in (iii) in Theorem 3.1 and $\tilde{\delta}_{\omega_j}(B)$ is an $N \times r_{\omega_j}$ matrix whose columns are equal to the r_{ω_j} last columns of $\delta_{\omega_j}(B)$. When $\omega_j \in]0, \pi[$, the same kind of computation has been presented in the preceding section and gives

$$D^{-1}(B)(\Lambda^1_{\omega_j}(B) + B\Lambda^2_{\omega_j}(B))\Psi_{\omega_j}(B) = \tilde{D}_{\omega_j}(B)\tilde{\delta}_{\omega_j}(B) + \Theta^1_{\omega_j}(\Delta_{\omega_j}) + B\Theta^2_{\omega_j}(\Delta_{\omega_j}).$$

Rearranging the terms of the sum once premultiplied by the inverse of $D(B)$, the representation equation is proved with

$$\Phi(B) = D^{-1}(B)\left(\sum_{\omega_j \in \{0,\pi\}} \Lambda_{\omega_j}(B)A_{\omega_j}(B) + \sum_{\omega_j \in]0,\pi[} (\Lambda^1_{\omega_j}(B) + B\Lambda^2_{\omega_j}(B))A_{\omega_j}(B) + CR(B)\right)$$
$$+ B\sum \tilde{D}_{\omega_j}(B)A_{\omega_j}(B)$$

so that $\Phi(0) = I_N$. To prove the rank properties stated in Theorem 3.1, rewrite this equation under an algebraic form by identifying the matrices after replacing y by $C(B)\epsilon$:

$$I_N = \Phi(B)C(B) + B\sum_{\omega_j \in \{0,\pi\}} (0 \quad \Theta_{\omega_j}(\Delta_{\omega_j}))D_{\omega_j}(B)$$
$$+ B\sum_{\omega_j \in]0,\pi[} (0 \quad \Theta^1_{\omega_j}(\Delta_{\omega_j}) + B\Theta^2_{\omega_j}(\Delta_{\omega_j}))D_{\omega_j}(B).$$

Use the fact that if $\omega_j \neq \omega_l$,

$$A_{\omega_l}(e^{i\omega_j})C(e^{i\omega_j}) = \delta_{\omega_l}(e^{i\omega_j})D_{\omega_l}(e^{i\omega_j})$$

can be rewritten under the following form because $\delta_{\omega_l}(e^{i\omega_j})$ is a matrix of full rank,

$$D_{\omega_l}(e^{i\omega_j}) = \delta^{-1}_{\omega_l}(e^{i\omega_j})A_{\omega_l}(e^{i\omega_j})C(e^{i\omega_j}).$$

This allows us to rewrite the evaluation of the previous algebraic equation in $e^{i\omega_j}$ when $\omega_j \in \{0, \pi\}$ as

$$e^{i\omega_j}(0 \quad \Theta_{\omega_j}(0))D_{\omega_j}(e^{i\omega_j}) + \zeta_{\omega_j}(e^{i\omega_j})C(e^{i\omega_j}) = I_N, \tag{7}$$

where

$$\zeta_{\omega_j}(e^{i\omega_j}) = \Phi(e^{i\omega_j}) + e^{i\omega_j}(0 \quad \Theta_{\omega_l}(\Delta_{\omega_l}(e^{i\omega_j})))\delta^{-1}_{\omega_l}(e^{i\omega_j})A_{\omega_l}(e^{i\omega_j})$$
$$+ e^{i\omega_j}\sum_{\omega_l \in]0,\pi[} (0 \quad \Theta^1_{\omega_l}(\Delta_{\omega_l}(e^{i\omega_j})) + e^{i\omega_j}\Theta^2_{\omega_l}(\Delta_{\omega_l}(e^{i\omega_j})))\delta^{-1}_{\omega_l}(e^{i\omega_j})A_{\omega_l}(e^{i\omega_j}).$$

Here $D_{\omega_j}(e^{i\omega_j})$ is of full rank and $C(e^{i\omega_j})$ is of rank $N - r_{\omega_j}$, thus $\Theta_{\omega_j}(0)$ is a matrix of full rank r_{ω_j}. Furthermore $\zeta_{\omega_j}(e^{i\omega_j})$ is necessarily of full rank. The same reasoning applies to $\omega_j \in]0, \pi[$ with the additional property (Property 4.1) that if $\Theta^1_{\omega_j}(0) + e^{i\omega_j}\Theta^2_{\omega_j}(0)$ is a complex matrix of full rank then $(\Theta^1_{\omega_j}(0), \Theta^2_{\omega_j}(0))^{*\omega_j}$ is a real matrix of full rank.

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6. DETERMINISTIC TRENDS

The subject of this section is to extend the results obtained in the preceding sections for multivariate time series such that the determinant of the matrix series associated to their Wold representation has various unit roots with various orders of multiplicity to the larger class of similar processes with deterministic trends. We describe the modifications we have to introduce into the representation equation when we assume the presence of a nonzero deterministic trend we restrict to be a linear combination of various elements in some vector spaces $\mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}}$.

We first define a more general class of processes $\mathcal{C}_{\Omega,\Gamma}$ on which we work in the sequel of this section.

DEFINITION 6.1. *A process $y_t, y_t \in \mathbb{R}^N$ belongs to $\mathcal{C}_{\Omega,\Gamma}$ where Ω is a set of cardinal number K of couples of different values $\omega_j \in [0,\pi]$ and non-null positive integers k_j, $j = 1,\dots,K$, and Γ is a set of cardinal number D of couples of different values $\bar{\omega}_{j'} \in [0,\pi]$ and non-null positive integers $d_{j'}$, $j' = 1,\dots,D$, if it is covariance stationary, has a Wold representation*

$$y_t = \nu_t + C(B)\epsilon_t$$

and is such that the determinant of the matrix series associated with the multivariate Wold representation, $\det C(u)$, satisfies

$$\det C(u) = \mathcal{D}(u)d(u),$$

where

$$\mathcal{D}(u) = \prod_{j=1}^{j=K} \Delta(u)^{k_j},$$

$d(u)$ is well defined and different from zero for all u, $\|u\| \le \rho$, for some $\rho > 1$, and where

$$\nu_t \in \bigoplus_{j'=1}^{j'=K} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} = \mathcal{T}_\nu.$$

We now describe the modifications we have to introduce into the representation equation to consider the presence of the deterministic trends. Roughly speaking, deriving the new representation equation can be performed by replacing y_t by $y_t - \nu_t$. We nevertheless need some assumptions to obtain well-defined quantities by this operation. Two modifications are introduced in the equation. On the one hand, a constant term built from the matrix series $\Phi(B)$ and the deterministic trend appears in the right-hand side of the equation. On the other hand, the deterministic trends that appear in each error correction term are modified to take into account ν_t.

The constant term ν_t in the moving average representation of the process becomes $\Phi_\nu(t) = \sum_{j=0}^{+\infty} \Phi_j \nu_{t-j}$ in the autoregressive representation equation. When ν_t is some deterministic function, $\Phi_\nu(t)$ is not well defined without some

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restrictions; similarly its parameterization does not allow in general a simple estimation. Two simple cases can be considered. The first one corresponds to a situation in which $\Phi(B)$ is of finite order. In the second one on which we focus, ν_t is an element of a vector space $\oplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}}$. Then, under the assumption that $\max_{(j_1,j_2)\in\{1,\ldots,N\}^2} |\sum_{l=0}^{+\infty} \Phi_{(j_1,j_2),l}\, l^{d_{j'}} e^{i\bar{\omega}_{j'} l}| < +\infty$, $\forall(\bar{\omega}_{j'}, d_{j'}) \in \Gamma$, where $\Phi_{(j_1,j_2),l}$ corresponds to the element of Φ_l in the j_1th row and j_2th column,[5] we have the following lemma.

LEMMA 6.1.

$$\nu_t \in \bigoplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} \Rightarrow \Phi(B)\nu_t \in \bigoplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}}.$$

We then have to solve the linear equation that gives us the deterministic trend that appears in each error correction term. The polynomial cointegration relations include the polynomials $\beta_{\omega_j}^h(\nu)(t)$ of dimension $r_{\omega_j,h} \times 1$ that are in $\mathcal{T}_{\omega_j}^h(\nu_t)$ affine space of the polynomials $\mathcal{P}(t)$ solutions of the equation

$$\Delta_{\omega_j}(B)^h \mathcal{P}(t) = \sum_{l=1}^{h} \alpha_{\omega_j,h}^{1,l}(\Delta_{\omega_j})^{h-l}\nu_t + \sum_{l=1}^{h} \alpha_{\omega_j,h}^{2,l}(\Delta_{\omega_j})^{h-l}\nu_{t-1}$$

with $\alpha_{\omega_j,h}^{2,l} = 0$ when $\omega_j \in \{0,\pi\}$. Using Property 2.2, it is possible to be more precise on the shape of $\beta_{\omega_j}^h(\nu)$. We first need to describe the image space of $\mathcal{T}_{\bar{\omega}_{j'}}^h$ by any integral operator. This is given by the following lemma.

LEMMA 6.2. $(\omega_1,\omega_2) \in [0,\pi]^2$, $\omega_1 \neq \omega_2$, $d \geq 1$, $h \geq 1$. *We have*

(i) $\tilde{\nu}_t \in \mathcal{T}_{\omega_1}^d \Rightarrow G_{\omega_1}^h \tilde{\nu}_t \in \mathcal{T}_{\omega_1}^{d+h}$
(ii) $\tilde{\nu}_t \in \mathcal{T}_{\omega_1}^d \Rightarrow G_{\omega_2}^h \tilde{\nu}_t \in \mathcal{T}_{\omega_1}^d \oplus \mathcal{T}_{\omega_2}^h$.

From this result, it is easy to conclude that if $\nu_t \in \oplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}}$, then $\beta_{\omega_j}^h(\nu)_t$ is

(a) in $\oplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} \oplus \mathcal{T}_{\omega_j}^h$, when ω_j is not in any couple of Γ;
(b) in $\oplus_{\bar{\omega}_{j'}\neq\omega_j} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} \oplus \mathcal{T}_{\omega_j}^{d_j+h}$, when ω_j is an element of one couple in Γ.

We can now give the description of the representation equation of a process in $\mathcal{C}_{\Omega,\Gamma}$.

THEOREM 6.1. *If a process y_t belongs to $\mathcal{C}_{\Omega,\Gamma}$ where $\Omega = \{(\omega_j, k_j), j = 1,\ldots,K\}$, and $\Gamma = \{(\bar{\omega}_{j'}, d_{j'}), j' = 1,\ldots,D\}$, it satisfies an equation of the following form*:

$$\Phi(B)y_t - \epsilon_t = \Phi_\nu(t) + B\sum_{\omega_j} \{\Theta_{\omega_j}^1(\Delta_{\omega_j}) + B\Theta_{\omega_j}^2(\Delta_{\omega_j})\} \times \{\alpha_{\omega_j}^1(G_{\omega_j})y_t + \alpha_{\omega_j}^2(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(\nu)(t)\}, \quad \textbf{(8)}$$

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where $\alpha_{\cdot}(\cdot)$ *and* $\Theta_{\cdot}(\cdot)$ *satisfy the properties described in Theorem* 3.1 *and*

(i) $\Phi_{\nu}(t) \in \oplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}}$.

(ii) (*Row decomposition*) *The* $(r_{\omega_j} \times 1)$ *vector* $\beta_{\omega_j}(\nu)(t)$ *is made of* m_{ω_j} *subvectors* $\beta_{\omega_j}^{h}(\nu)(t)$ *of dimension* $r_{\omega_j,h} \times 1$ *in* $\oplus_{\bar{\omega}_{j'}} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} \oplus \mathcal{T}_{\omega_j}^{h}$, *when* ω_j *is not in any couple of* Γ, *or in* $\oplus_{\bar{\omega}_{j'} \neq \omega_j} \mathcal{T}_{\bar{\omega}_{j'}}^{d_{j'}} \oplus \mathcal{T}_{\omega_j}^{d_j+h}$, *when* ω_j *is an element of one couple in* Γ.

7. CONCLUSION

In this paper, we proved a representation theorem that allows econometricians to specify the form of the VECM representation for any multivariate real covariance stationary process whose determinant of the matrix expansion related to its Wold representation has a finite number of unit roots at various frequencies (and possibly a deterministic part we restrict to be a finite linear combination of terms in $(t^{k_j} \cos \omega_j t, t^{d_j} \sin \omega_j t)_{(\omega_j \in [0,\pi], k_j \in \mathbb{N})}$). This kind of specification may occur when we work with some non–seasonally adjusted nominal macroeconomic aggregate or price indices that may be well described with time series models containing more than one unit root at frequency 0 and some seasonal unit roots. In practice, this multivariate covariance stationary time series is composed of the time series under study on which first-difference operators $(1 - B, 1 + B, 1 - 2\cos\omega_j B + B^2, \omega \in]0, \pi[)$ have been appropriately applied. The error correction terms in this VECM representation correspond to polynomial cointegrating relationships at each frequency. One appealing property of the representation is that it allows us to deal separately with the cointegratedness at each frequency. In practice, starting from a set of simultaneously integrated univariate processes at n frequencies, we have to appropriately apply the n operators composed of the product of $n - 1$ different first-difference operators to build n set of processes integrated at the n different frequencies and build for each set the polynomial error correction terms.

8. PROOFS

We use in the sequel the following algebraic result.

Property 8.1. Let U, V, W, X be four matrices of respective dimensions $p \times q$, $p \times s$, $q \times r$, and $s \times r$, such that $p \leq s$ and $q \leq r$; $(U \quad V)$ and W are matrices of full rank.

$UW + VX = 0$ $(P1) \Rightarrow V$ is a matrix of full rank.

Proof of Property 8.1. Assume that V is not of full rank; rewrite it with a null row by premultiplying it by a well-chosen matrix. But because W is a matrix of full rank q, according to $(P1)$, the corresponding row of U premultiplied by the same matrix must be null, which is in contradiction with the assumption of full rank for $(U \quad V)$. ■

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Proofs of Properties 2.1–2.4 are omitted; they are simple algebraic properties. Property 2.5 is derived from the following identities and Property 2.1:

$$1 = \frac{1}{2(1-\cos\omega)}\Delta_\omega + \frac{1-2\cos\omega+B}{2(1-\cos\omega)}\Delta_0$$

$$1 = \frac{1}{2(1+\cos\omega)}\Delta_\omega + \frac{1+2\cos\omega-B}{2(1+\cos\omega)}\Delta_\pi$$

$$1 = \frac{2\cos\omega_1 - B}{2(\cos\omega_1-\cos\omega_2)}\Delta_{\omega_2} + \frac{2\cos\omega_2 - B}{2(\cos\omega_2-\cos\omega_1)}\Delta_{\omega_1}.$$

We begin with the proof of Lemma 2.1.

Proof of Lemma 2.1. We consider different cases.

(i) First, we consider the case when $\omega_1 = 0$ and $\omega_2 = \pi$. The proof is by induction; the property holds for $(l_0, l_\pi) = (1,1)$ according to Property 2.5. Let us assume that it holds for all (l_0', l_π') such that $l_0' \le l_0$ and $l_\pi' \le l_\pi$. Then as

$$\begin{aligned}(G_0^{l_0}G_\pi^{l_\pi})G_0 &= P_0^{1,(l_0,l_\pi)}(G_0)G_0 + P_\pi^{1,(l_0,l_\pi)}(G_\pi)G_0\\ &= P_0^{1,(l_0,l_\pi)}(G_0)G_0 + \sum_{h=1}^{l_\pi} P_{\pi,h}^{1,(l_0,l_\pi)}G_\pi^h G_0\\ &= P_0^{1,(l_0,l_\pi)}(G_0)G_0 + \sum_{h=1}^{l_\pi} P_{\pi,h}^{1,(l_0,l_\pi)}(P_0^{1,(1,h)}(G_0) + P_\pi^{1,(1,h)}(G_\pi))\end{aligned}$$

it holds for (l_0+1, l_π). A similar computation for $(l_0, l_\pi+1)$ allows us to conclude the proof.

(ii) We now consider the couple $(\omega_1,\omega_2) \in \{0,\pi\}\times]0,\pi[$. First remark that from the identities $B = e^{i\omega_1} - e^{i\omega_1}(1-e^{-i\omega_1}B)$ and (J_{ω_2}), we get

$$BG_{\omega_2}G_{\omega_1} = \frac{e^{i\omega_1}}{2(1-e^{i\omega_1}\cos\omega_2)}G_{\omega_1} + \frac{-1+(1-2(e^{i\omega_1}-1)\cos\omega_2)B}{2(1-e^{i\omega_1}\cos\omega_2)}G_{\omega_2}.$$

Then if we assume that for all $l_1' \le l_1$

$$BG_{\omega_2}G_{\omega_1}^{l_1'} = \tilde{P}_{\omega_1}^{1,(l_1',1)}(G_{\omega_1}) + \tilde{P}_{\omega_2,1}^{1,(l_1',1)}G_{\omega_2} + B\tilde{P}_{\omega_2,1}^{2,(l_1',1)}G_{\omega_2},$$

where $\tilde{P}_{\omega_1}^{1,(l_1',1)}$ is a real polynomial of degree l_1' without constant term, then as

$$BG_{\omega_2}G_{\omega_1}^{l_1+1} = \tilde{P}_{\omega_1}^{(l_1,1)}(G_{\omega_1})G_{\omega_1} + \tilde{P}_{\omega_2,1}^{1,(l_1,1)}G_{\omega_2}G_{\omega_1} + \tilde{P}_{\omega_2,1}^{2,(l_1,1)}BG_{\omega_2}G_{\omega_1}$$

the property holds for l_1+1. A similar property holds for the products of the form $BG_{\omega_2}^{l_2}G_{\omega_1}$.

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According to Property 2.5, the expansion we want to prove holds for $(l_1, l_2) = (1,1)$. Let us assume that it holds for all (l_1', l_2') such that $l_1' \leq l_1$ and $l_2' \leq l_2$. Then as

$$G_{\omega_2}^{l_2} G_{\omega_1}^{l_1+1} = P_{\omega_1}^{1,(l_1,l_2)}(G_{\omega_1}) G_{\omega_1} + \sum_{h=1}^{l_2} P_{\omega_2,h}^{1,(l_1,l_2)}(G_{\omega_2})^h G_{\omega_1} + P_{\omega_2,h}^{2,(l_1,l_2)} B G_{\omega_2}^h G_{\omega_1}$$

according to the preceding results, it is satisfied for $(l_1 + 1, l_2)$. On the other hand, it trivially holds for $(l_1, l_2 + 1)$.

(iii) At last, we consider $(\omega_1, \omega_2) \in]0, \pi[^2$. The proof is close to the previous one; we have to note that

$$BG_{\omega_1} G_{\omega_2} = \frac{1}{2(\cos\omega_1 - \cos\omega_2)} G_{\omega_1} + \frac{1}{2(\cos\omega_2 - \cos\omega_1)} G_{\omega_2}$$

and use the same kind of arguments. ■

Proof of Lemma 4.1. It can easily be shown that we can rewrite $C(B)$ under the following form:

$$C(B) = C(e^{i\omega}) \frac{e^{-i\omega} - B}{e^{-i\omega} - e^{i\omega}} + C(e^{-i\omega}) \frac{e^{i\omega} - B}{e^{i\omega} - e^{-i\omega}} + \Delta_\omega \tilde{C}(B),$$

where $\tilde{C}(e^{-i\omega})$ is well defined. Then $\forall P_a(\phi) \in \mathcal{K}_a$, we have

$$P_a(\phi)(B)C(B) = \Delta_\omega \left\{ \frac{e^{-i\omega}}{e^{-i\omega} - e^{-i\omega}} \phi a C(e^{i\omega}) + \frac{e^{i\omega}}{e^{i\omega} - e^{-i\omega}} \bar{\phi}\bar{a} C(e^{-i\omega}) + P_a(\phi)(B)\tilde{C}(B) \right\}.$$ ■

Proof of Lemma 4.2. a is an $r \times N$ complex matrix of full rank such that $aC(e^{-i\omega}) = 0$. Rearrange its N columns with an $N \times N$ real matrix ψ so that the last r columns form an $r \times r$ matrix a_2 of full rank and denote the $N - r \times r$ matrix of the remaining columns a_1. Choose $\phi = -ie^{i\omega}\phi_0 a_2^{-1}$ with ϕ_0 a real $r \times r$ matrix of full rank. Then

$$P_a(-ie^{i\omega}\phi_0 a_2^{-1})(B) = 2\phi_0(\Im(a_2^{-1}a_1 e^{i\omega}) - \Im(a_2^{-1}a_1)B \quad \sin\omega I_r)\psi^{-1}.$$

To build $A(B)\psi$, complete $P_a(-ie^{i\omega}\phi_0 a_2^{-1})(0)\psi$ with an $N - r \times N$ real matrix of full rank whose last r columns are equal to zero. In this way, $\det A(B)$ is a non-null constant that can be put to 1 by an appropriate choice of ϕ_0. ■

Proof of Corollary 4.1. Let $C_1(e^{-i\omega})$ be the $r' \times N$ matrix of full rank obtained by stacking up the r' first rows of $C(e^{-i\omega})$ and $C_2(e^{-i\omega})$ be the $N - r' \times N$ matrix obtained by stacking up the $N - r'$ remaining ones. Partition a into a_{11} and a_{12} of respective dimension $r \times r'$ and $r \times N - r'$. By definition the following equation is satisfied:

$$a_{11}C_1(e^{-i\omega}) + a_{12}C_2(e^{-i\omega}) = 0.$$

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From Property 8.1, a_{12} is an $r \times N - r'$ matrix of full rank. Apply the same kind of operation described in Lemma 4.2 to this matrix, i.e., rearrange its $N - r'$ columns with an $N - r' \times N - r'$ real matrix ψ so that the last r columns form an $r \times r$ matrix $a_{12,2}$ of full rank. Choose $\phi = -ie^{i\omega}\phi_0 a_{12,2}^{-1}$. To build up the matrix $A(B)$, complete the $r \times N$ matrix in two steps. First, introduce an $N - r - r' \times N$ matrix as in Lemma 4.2 and second complete by $(I_{r'} \quad 0_{r' \times N - r'})$. ■

Proof of Lemma 4.3. The proof is by induction. It consists of applying Lemma 4.2 and Corollary 4.1 to the matrix $D(B)$ obtained in the preceding steps and repeating this construction to the matrix that plays its role, and so on. The only point we have to prove is the fact that the $N - r_\omega$ first rows of $D(e^{i\omega})$ are of full rank. This is a consequence of the way the matrix $A(B)$ is built up in Lemma 4.2. ■

Proof of Property 4.1. We simply have to note that $\forall (N \times r)$ complex matrix $\mathcal{U}$,

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \otimes I_N \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \overline{\mathcal{U}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \otimes I_r = \begin{pmatrix} \Re\mathcal{U} & -\Im\mathcal{U} \\ \Im\mathcal{U} & \Re\mathcal{U} \end{pmatrix}$$

and that

$$Q(\omega)^{-1} \otimes I_N \begin{pmatrix} \vartheta^1 + \cos\omega\vartheta^2 & -\sin\omega\vartheta^2 \\ \sin\omega\vartheta^2 & \vartheta^1 + \cos\omega\vartheta^2 \end{pmatrix} Q(\omega) \otimes I_r$$
$$= \begin{pmatrix} \vartheta^1 & \vartheta^2 \\ -\vartheta^2 & \vartheta^1 + 2\cos\omega\vartheta^2 \end{pmatrix},$$

where

$$Q(\omega) = \begin{pmatrix} 1 & -\cos\omega \\ 0 & -\sin\omega \end{pmatrix}.$$

The matrix we consider is thus of full rank because it is the product of full rank matrices. ■

Proof of Property 5.1. Note that $(\Delta_{\omega_j})_{j=1,\ldots,n}$ is a set of prime real polynomials in B. Then $\forall \omega_j$ elements of $(\omega_1, \ldots, \omega_n)$, $\Delta_{\omega_j}^{m_j}$ and $P_{-j}(B) = \Pi_{l \neq j} \Delta_{\omega_l}^{m_l}$ are prime. Using Bezout's lemma (see, e.g., Bouvier and George, 1974), we know that there exist two polynomials, say, $U_j(B)$ and $V_j(B)$ such that

$$U_j(B)\Delta_{\omega_j}^{m_j} + V_j(B)P_{-j}(B) = 1.$$

If we premultiply the linear combination of Property 5.1 by $V_j(B)P_{-j}(B)$ for each $\omega_l, l \neq j$ and replace it by its equivalent $1 - U_j(B)\Delta_{\omega_j}^{m_j}$ for ω_j, then using Property 2.1, all the nonstationary terms are removed except the terms in G_ω. We thus obtain a sum of a polynomial in B applied to y_t and $\lambda_j \left(P_j(G_\omega) \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \mathcal{P}_j(t) \right)$ that is covariance stationary. This last term is therefore covariance stationary. ■

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Proof of Lemma 5.1(ii) and (iii).

(ii) Let us introduce a particular basis $\mathcal{A}_{\omega_j}(G_{\omega_j})$ of the $N(m_{\omega_j}+1)$ space of polynomial N-vectors in G_{ω_j} of degree inferior or equal to m_{ω_j}. First take the k_j terms described in Remark 3.2 without constant term that applied to y_t yield stationary processes with possibly deterministic trend. The matrix obtained by stacking up $A^0_{\omega_j}$ and the terms of highest degree $\alpha^h_{\omega_j,h}$, $h=1,\ldots,m_{\omega_j}$ plays the role of the constant terms. Then consider the $N - r_{\omega_j,m_{\omega_j}}$ terms

$$G_{\omega_j}(A^{0\prime}_{\omega_j} \quad \alpha^1_{\omega_j}(G_{\omega_j})'\ldots\alpha^{m_{\omega_j}-1}_{\omega_j}(G_{\omega_j})')'$$

that applied to y_t yield $I(1)$ processes with possibly deterministic trends. Iterating the procedure, at stage h, select the $N - \sum_{l=0}^{h-1} r_{\omega_j,m_{\omega_j}-l}$ terms

$$G^h_{\omega_j}(A^{0\prime}_{\omega_j} \quad \alpha^1_{\omega_j}(G_{\omega_j})'\ldots\alpha^{m_{\omega_j}-h}_{\omega_j}(G_{\omega_j})')'$$

that applied to y_t yield $I(h)$ processes, and so on, until $A^0_{m_j}G^{m_{\omega_j}}_{\omega_j}$. The fact that the matrix obtained by stacking up $A^0_{\omega_j}$ and the terms of highest degree is of full rank implies that this $N(m_{\omega_j}+1)\times N$ matrix is a basis of the N-vector polynomial in G_{ω_j}. Any polynomial of degree at most m_{ω_j} is equal to a linear combination of the vectors of this basis. If $n > m_{\omega_j}$, we have to build similarly a basis of the space of N-vector polynomials of degree inferior or equal to n. By construction, a nonzero weight on a vector of degree n in this basis implies that the resulting time series is nonstationary, which is in contradiction to our assumption. Similarly, a nonzero weight of $P(G_{\omega_j})$ on a vector of $\mathcal{A}_{\omega_j}$ that corresponds to a nonstationary direction would be in contradiction to our assumption of stationarity. Thus $P(G_{\omega_j})$ is equal to a linear combination of the $k_j + N$ first considered vectors in the construction of $\mathcal{A}_{\omega_j}$. But except $A^0_{\omega_j}$, each of these vectors is equal to one of the r_{ω_j} polynomial vectors in (4) premultiplied by Δ_{ω_j} raised to an appropriate power, minus a polynomial in B of degree at most $m_{\omega_j}-1$. Consider now the deterministic trends; introduce the corresponding deterministic polynomials $\beta_{\omega_j,h}(t)$ or their transformed form by an operator $\Delta^h_{\omega_j}$ in each term of the preceding linear relation and subtract them. Necessarily the same linear combination of these deterministic trends holds for $b(t)$ (use, e.g., the mean operator and the natural basis of $\mathcal{T}^{m_{\omega_j}}_{\omega_j}$).

(iii) First using the identity (J_{ω_j}) rewrite (6) under the form

$$P^1(G_{\omega_j})y_t + P^2(G_{\omega_j})y_{t-1} + b(t)$$

$$= (\lambda^1_{\omega_j}(B) \quad \lambda^2_{\omega_j}(B))\begin{pmatrix} A^0_{\omega_j}y_t \\ \alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t) \\ A^0_{\omega_j}y_{t-1} \\ \tilde{\alpha}^2_{\omega_j}(G_{\omega_j})y_t + (\alpha^1_{\omega_j} + 2\cos\omega_j\alpha^2_{\omega_j})(G_{\omega_j})y_{t-1} + \tilde{\beta}_{\omega_j}(t) \end{pmatrix}$$

$$+ \lambda^2_{\omega_j}(B)(\mu^1_{\omega_j} \quad \mu^2_{\omega_j})\,\aleph_{\omega_j}\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} + \Delta_{\omega_j}\rho_{\omega_j}(B)y_t,$$

where $\tilde{\alpha}^2_{\omega_j}$ and $\tilde{\beta}_{\omega_j}$ are defined in Remark 3.1 and $\mu^1_{\omega_j}$ and $\mu^2_{\omega_j}$ are the components in the $2N\times 2N$ basis $\aleph_{\omega_j}$ obtained by stacking up $(A^0_{\omega_j},0)^{*\omega}$, and the terms

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$(\alpha^1_{\omega_j,h}, \alpha^2_{\omega_j,h})^{*\omega_j}$ for $h = 1,\ldots,m_{\omega_j}$ of the $2N$-dimensional vector whose N first columns are equal to zero and the N remaining ones equal to the term of degree one. The terms $\mu^1_{\omega_j}$ and $\mu^2_{\omega_j}$ are uniquely defined in this basis so that we can go from one representation to the other one without any loss of information. The idea of the proof is similar to the previous one. We first construct a particular basis $\{\mathcal{A}_{\omega_j}(G_{\omega_j})\}$ of the $N(m_{\omega_j}+1)$ space of the polynomial $2N$-vectors in G_{ω_j} of degree inferior or equal to m_{ω_j}. First we take the $2k_j$ terms described in Remark 3.2 without a constant term that applied to $(y_t' \quad y_{t-1}')'$ yield stationary processes with possibly deterministic trends and the $2N \times 2N$ matrix $\aleph_{\omega_j}$ described previously that plays the role of the constant term. Then we select the $2(N - r_{\omega_j,m_{\omega_j}})$ terms

$$G_{\omega_j}\begin{pmatrix} (A^0_{\omega_j},0)^{*\omega_j} \\ (\alpha^{1,1}_{\omega_j}(G_{\omega_j}), \alpha^{2,1}_{\omega_j}(G_{\omega_j}))^{*\omega_j} \\ \vdots \\ (\alpha^{1,m\omega_j-1}_{\omega_j}(G_{\omega_j}), \alpha^{2,m_{\omega_j}-1}_{\omega_j}(G_{\omega_j}))^{*\omega_j} \end{pmatrix}$$

that applied to $(y_t' \quad y_{t-1}')'$ yield $I(1)$ times series, and so on. The fact that the matrix obtained by stacking up $(A^0_{\omega_j},0)^{*\omega_j}$ and the terms $(\alpha^1_{\omega_j,h}, \alpha^2_{\omega_j,h})^{*\omega_j}$ for $h = 1,\ldots,m_{\omega_j}$ is $2N \times 2N$ matrix of full rank according to Property 4.1 ensures that the $2N(m_{\omega_j}+1) \times 2N$ matrix built earlier is a basis of the $2N$-vector polynomials in G_{ω_j}. Similar arguments to the preceding proof are used in this basis to get the stated result. ■

Proof of Theorem 3.3. We assume in this proof that there do not exist eigenvalues with an order of multiplicity larger than one in the symmetric matrices we implicitly consider. Start from (4) where $D_{\omega_j}(e^{i\omega_j})$ is a complex matrix of full rank. Similarly, the matrix $A_{\omega_j}(e^{i\omega_j})$ is also of full rank. According to Property 4.1 the $2N \times 2N$ matrix obtained by stacking up $(A^0_{\omega_j},0)^{*\omega_j}$, $(\alpha^{1,h}_{\omega_j,h}, \alpha^{2,h}_{\omega_j,h})^{*\omega_j}$, $h = 1,\ldots,m_{\omega_j}$ is of full rank. We construct the new $\alpha^1_{\omega_j}(\cdot)$ and $\alpha^2_{\omega_j}(\cdot)$ from those in (4) using ideas of principal component analysis that will be the basis of the estimation procedure. We know that there exist $2r_{\omega_j,m_{\omega_j}}$ polynomial error correction terms of degree m_{ω_j} with the corresponding deterministic trends. Any of these terms can be built as a linear combination of the $2k_j$ stationary error correction terms described in Remark 3.2 with a nonzero weight on the vector of degree m_{ω_j}. So, in this space, we look for the $2N$-polynomial of degree m_{ω_j}, whose vector of coefficients of the term of degree m_{ω_j} is normalized, of smallest variance. From this term, we build up another error correction term of the same degree by the transformation described in Remark 3.1. This second vector is associated to the first one in the second set of $r_{\omega_j,m_{\omega_j}}$ polynomial error correction terms of degree m_{ω_j}. Under the null that the first $2Nm_{\omega_j} \times 1$ vector is a polynomial cointegrating vector, the second one belongs to the space spanned by the $2k_j$ stationary directions. Indeed, it can easily be shown but with additional notational costs that this space is invariant by the transformation described in Remark

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3.1. This result holds for each subspace of polynomials of degree at most h, $h = 1,\ldots,m_{\omega_j}$ in the $2k_j$-dimensional space. We then build up the $r_{\omega_j} - 1$ remaining polynomial terms and associate vector sequentially: at each step, we look for a $2N$-vector polynomial in G_{ω_j} of $(y_t' \quad y_{t-1}')'$ with deterministic trends in the space spanned by the $2k_j$ initial terms, of the appropriate degree, of the smallest variance under the constraint that its coefficients of highest degree are normalized and orthogonal to the $2N$-vectors of coefficients of highest degree of each couple of polynomial determined at the previous stages. Finally the $N - r_{\omega_j}$ constant terms are also computed sequentially in a similar way, minimizing the variance of $2N$-dimensional time series without deterministic trends under the normalization and orthogonality constraint.

In this way, we obtain a generically unique representation up to a change of sign of each term. We now check that it satisfies conditions 1–3 of the theorem.

The constructed error correction terms are stationary and linear combinations of the $2k_j$ stationary error correction terms described in Remark 3.2. To prove that $\det E_{\omega_j}(e^{i\omega_j})$ is different from zero, we multiply both sides of (3) by $\delta_{\omega_j}(B)$ defined in Lemma 5.1(i), using Property 2.1 and identifying the matrices after replacing y by $C(B)\epsilon$. We get

$$\det A^*_{\omega_j}(B) \det C(B) = (\Delta_{\omega_j}(B))^{k_j} \det E_{\omega_j}(B),$$

where $A^*_{\omega_j}(B)$ is the matrix corresponding to $A_{\omega_j}(B)$ in Lemma 5.1(ii) built with the new $\alpha^g_{\omega_j}(\cdot)$. By construction the matrix obtained by stacking up $(\alpha^{1,0}_{\omega_j}, \alpha^{2,0}_{\omega_j})^{*\omega_j}$, and for $h = 1,\ldots,m_{\omega_j}$, $(\alpha^{1,h}_{\omega_j h}, \alpha^{2,h}_{\omega_j,h})^{*\omega_j}$, is of full rank. Then from Property 4.1, $A^*(e^{i\omega_j})$ is of full rank, and thus $E_{\omega_j}(e^{i\omega_j})$ is of full rank.

Condition 2 follows directly from the construction. Condition 3 is a property of principal component analysis. At each step of the procedure, the new component that is being computed is uncorrelated with those built with the previous vectors $(\alpha^{1,h}_{\omega_j,h(q)} \quad \alpha^{2,h}_{\omega_j,h(q)})$ for the same value of h and with the complete set of vectors $(\alpha^{1,l}_{\omega_j,l} \quad \alpha^{2,l}_{\omega_j,l})$ for $l > h$. Otherwise, it would have been possible to decrease the variance of the previous components by choosing the suitable linear combination with the current one, whence a contradiction. But there may exist some correlation between the second term of each couple and the previously computed terms. ■

NOTES

1. Theorem 3.3 was analyzed in a preceding version.

2. For more details on this side of the analysis see Remark 1.1 and note 3 in Gregoir and Laroque (1993).

3. It is equal to the remainder of the Euclidean division when the polynomial that is divided is of finite degree.

4. Notice that the polynomial matrix that allows us to extract the k_2 unit roots ω_2 that are present in $D_{\omega_i}(B)$ is different from that which allows us to extract the k_2 unit roots in $C(B)$.

5. This condition is satisfied by a process in $\mathcal{C}_{\Omega,\Gamma}$ when each element of the matrix series associated to the Wold representation is well defined on a disk whose radius is strictly larger than one.

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