Quotients of triangular numbers

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The subject of this paper is the series: 1, 3, 6, 10, 15, 21, 28, 36, ... , the nth term of which is $T_n = \frac{n(n+1)}{2}$. Our question is: Can the quotient $\frac{T_m}{T_n}$ be any given integer? Can it be a square?

We will take the latter first since it turns out to be the simpler case: Can we find integers, m, n and k satisfying the relation $\frac{T_m}{T_n} = k^2$? Of course, if $m = n$ then $k = 1$. More interesting is, for example, the relation $\frac{T_m}{T_n} = 18^2$, which is satisfied by $m = 80, n = 4$, since

$$\frac{80 \times 81}{2} = \frac{4 \times 5}{2} \times 18^2.$$

In Part 1 our intention is to show that there are infinitely many pairs of triangular numbers whose quotient is a square, and to find expressions for them.

Part 1

To solve this problem the following diophantine equation is considered:

$$\frac{m(m+1)}{2} = \frac{n(n+1)}{2} k^2.$$

We will look for some numerical solutions for different n. We fix n and then look for solutions. The purpose is to observe any pattern. In this search, the solutions are computer generated. For $m = 1, 2, 3, \ldots$ the computer tests whether the quotient $q = \frac{m(m+1)}{n(n+1)}$ is a square or not.

Example (i): $n = 1$

The case $n = 1$ gives the equation $m(m+1) = 2k^2$. This is the well-known problem of finding triangular numbers which are also squares (see for example [1,2]). Table 1 shows some numerical findings, a sequence of solutions (m_i, k_i).

We focus here on the 'growth factor' from one to the next. It seems close to 6. More precisely, we observe the recursive relation for (m_i, k_i) for $i \geqslant 3$:

$$m_1 = 1 \qquad k_1 = 1$$
$$m_2 = 8 \qquad k_2 = 6$$
$$m_i = 6m_{i-1} - m_{i-2} + 2 \qquad k_i = 6k_{i-1} - k_{i-2}$$

https://doi.org/10.1017/mag.2015.33 Published online by Cambridge University Press

i	m_i	n_i	k_i
1	1	1	1
2	8	1	6
3	49	1	35
4	288	1	204
5	1681	1	1189
6	9 800	1	6 930

TABLE 1

Example (ii): $n = 2$

Then we move to the case $n = 2$, which gives a sequence of solutions shown in Table 2.

i	m_i	n_i	k_i
1	2	2	1
2	24	2	10
3	242	2	99
4	2400	2	980
5	23 762	2	9 701
6	235 224	2	96 030

TABLE 2

Similarly here the 'growth factor' points to 10. The following recursive relation appears for $i \geqslant 3$:

$$m_1 = 2 \qquad k_1 = 1$$
$$m_2 = 24 \qquad k_2 = 10$$
$$m_i = 10m_{i-1} - m_{i-2} + 4 \qquad k_i = 10k_{i-1} - k_{i-2}$$

In the numerical examples investigated, we recognise a recursive pattern of the form:

$$m_i = tm_{i-1} - m_{i-2} + s$$

$$k_i = tk_{i-1} - k_{i-2}$$

The factor t in both of the equations is of interest, as well as the addend s, and also the two pairs of starting numbers, (m_1, k_1) and (m_2, k_2). Table 3 shows the pattern emerging; from this general expressions may be extracted.

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n	t	s	k_1	k_2	m_1	m_2
1	6	2	1	6	1	8
2	10	4	1	10	2	24
3	14	6	1	14	3	48
4	18	8	1	18	4	80
5	22	10	1	22	5	120
…						
n	$4n + 2$	$2n$	1	$4n + 2$	n	$4n^2 + 4n$

TABLE 3

Our conjecture is then the following, based on investigating numerical solutions:

Theorem 1: Let n be a given integer. All the solutions of the diophantine equation

$$m(m + 1) = n(n + 1)k^2 \tag{1}$$

are given by a sequence

$$(m_1, k_1), (m_2, k_2), (m_3, k_3), \ldots$$

satisfying the following recursive relation for $i = 3, 4, 5, \ldots$

$$m_1 = n \qquad k_1 = 1$$
$$m_2 = 4n^2 + 4n \qquad k_2 = 4n + 2$$
$$m_i = (4n + 2)m_{i-1} - m_{i-2} + 2n \qquad k_i = (4n + 2)k_{i-1} - k_{i-2}$$

Now the issue is to prove Theorem 1. Can we single out a square on the left hand side in (1)? After multiplying by 4 and adding 1 the equation becomes

$$4m^2 + 4m + 1 = 4n(n + 1)k^2 + 1$$

Then, by substituting $u = 2m + 1$ and $D = 4n(n + 1)$, (1) is equivalent to the equation

$$u^2 - Dk^2 = 1. \tag{2}$$

Here D is not a perfect square, since $D + 1$ is a square, $(2n + 1)^2$. Now (2) is a Pell equation, widely studied in number theory! (See for instance [3, Theorem 7.26], or [4, Chap. 32]). According to the theory, if (u_1, k_1) is the least positive integer solution of (2), then all positive solutions are given by (u_i, k_i) where

$$u_i + k_i\sqrt{D} = \left(u_1 + k_1\sqrt{D}\right)^i$$

for $i = 1, 2, 3, \ldots$. Note that, since $\sqrt{D}$ is irrational, if $a + b\sqrt{D} = c + d\sqrt{D}$ then $a = c$ and $b = d$.

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We can find the least integer solution of (2). For a given n, the least positive solution of (1) must be for $k = 1$, and then $m = n$. That means $m_1 = n$, $k_1 = 1$ and $u = 2m + 1 = 2n + 1$. The least integer solution of (2) for each n is $(u_1, k_1) = (2n + 1,1)$. Representing this solution, we set

$$\varepsilon = u_1 + k_1\sqrt{D} = (2n + 1) + \sqrt{D}.$$

The next solution in the sequence is given by ε^2, which gives

$$u_2 + k_2\sqrt{D} = \varepsilon^2 = \left((2n + 1) + \sqrt{4n(n + 1)}\right)^2 = \left(8n^2 + 8n + 1\right) + (4n + 2)\sqrt{D},$$

from which we see that $u_2 = 8n^2 + 8n + 1$, which further means $m_2 = 4n^2 + 4n$, and $k_2 = 4n + 2$.

We have shown so far that the values of m_1, m_2, k_1 and k_2 agree with those which were conjectured in Theorem 1 on the basis of Table 3. We must now find whether the general values of k_i and m_i agree with those given by the reduction formula conjectured in Theorem 1. Suppose that (u_{i-2}, k_{i-2}), (u_{i-1}, k_{i-1}) and (u_i, k_i) are three consecutive solutions of (2) represented by ε^{i-2}, ε^{i-1} and ε^i respectively. We will now show that

$$\varepsilon^i = (4n + 2)\varepsilon^{i-1} - \varepsilon^{i-2} \tag{3}$$

which, by cancelling ε^{i-2}, is equivalent to

$$\varepsilon^2 = (4n + 2)\varepsilon - 1.$$

Here, since $\varepsilon = (2n + 1) + \sqrt{D}$,

$$\varepsilon - (2n + 1) = \sqrt{4n(n + 1)}$$

and by squaring

$$\varepsilon^2 - (4n + 2)\varepsilon + (2n + 1)^2 = 4n(n + 1)$$

or

$$\varepsilon^2 - (4n + 2)\varepsilon + 1 = 0.$$

So (3) is proved.

What we have found is that, if (u_{i-2}, k_{i-2}), (u_{i-1}, k_{i-1}) and (u_i, k_i) are three consecutive solutions of (2), then

$$u_i = (4n + 2)u_{i-1} - u_{i-2} \tag{4}$$

and

$$k_i = (4n + 2)k_{i-1} - k_{i-2}. \tag{5}$$

Equation (4) is an equation in u, which has to be modified to express the recursion of m.

$$2m_i + 1 = (4n + 2)(2m_{i-1} + 1) - (2m_{i-2} + 1)$$

gives the recursion on m

$$m_i = (4n + 2)m_{i-1} - m_{i-2} + 2n. \tag{6}$$

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According to (4) and (6), the recursion formula gives all the solutions of the original equation (1). Theorem 1 is proved.

Part 2

Now we change the problem: Given any non-square integer a, can the quotient of two triangular numbers $\frac{T_m}{T_n}$ equal this a? For example, if $a = 5$, we find that $\frac{14 \times 15}{2} = \frac{6 \times 7}{2} \times 5$ so that $\frac{T_{14}}{T_6} = 5$. But what is the situation in general?

The diophantine equation is now

$$m(m+1) = n(n+1)a \tag{7}$$

Our aim is to show that for each non-square a, there are infinitely many pairs of triangular numbers whose quotient is this given a, and to give expressions for sequences of such pairs.

In (7) we multiply by 4 and add 1, and write it

$$4m^2 + 4m + 1 = a(4n^2 + 4n + 1) + 1 - a$$

which is

$$(2m+1)^2 - a(2n+1)^2 = 1 - a.$$

By substituting $2m + 1 = x$ and $2n + 1 = y$, (7) is equivalent to the diophantine equation

$$x^2 - ay^2 = 1 - a \tag{8}$$

where a is a non-square parameter. For all a, obviously $x = 1, y = 1$ is a solution, representing $m = n = 0$ in (7). But what about the general solutions?

Also for this problem we will do numerical investigations, to see if the findings may indicate any pattern in the solutions.

Example (i): $a = 2$

The simplest case is $a = 2$. The diophantine equation, specialising (8), is now

$$x^2 - 2y^2 = -1.$$

A sequence of (computer generated) solutions is listed in Table 4. Here we also include the related solutions (m, n) of the original equation (7), to indicate the connection. Thus we can see directly for example that $T_{119} = 2T_{84}$.

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i	x_i	y_i	m_i	n_i
1	1	1	0	0
2	7	5	3	2
3	41	29	20	14
4	239	169	119	84
5	1393	985	696	492
6	8119	5741	4059	2870

TABLE 4

From this material, we see a recursive pattern for the solutions $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$, which is also reflected in the solutions (m_i, n_i) of (7).

$x_1 = 1$	$y_1 = 1$	$m_1 = 0$	$n_1 = 0$
$x_2 = 7$	$y_2 = 5$	$m_2 = 3$	$n_2 = 2$
$x_i = 6x_{i-1} - x_{i-2}$	$y_i = 6y_{i-1} - y_{i-2}$	$m_i = 6m_{i-1} - m_{i-2} + 2$	$n_i = 6n_{i-1} - n_{i-2} + 2$

for $i = 3,4,5, \ldots$.

So far this seems very similar to Part 1. Will this pattern continue? Or will new aspects appear?

Example (ii): $a = 10$

In this case the equation (8) becomes

$$x^2 - 10y^2 = -9.$$

A sequence of solutions in (x, y) is listed in Table 5.

i	x_i	y_i
1	1	1
2	9	3
3	41	13
4	79	25
5	351	111
6	1 559	493
7	3 001	949
8	13 329	4 215
9	59 201	18 721
10	113 959	36 037
11	506 151	160 059
12	2 248 079	710 905

TABLE 5

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In Example (i) there was a second order recurrence relation connecting x_i, x_{i-1}, x_{i-2}. However, if we use these first values to find such a relation experimentally, then subsequent values don't fit this. But – by magic! – if we split the sequence *in three*, such a relation suddenly appears! We need a precise term for what we observe. We will call a sequence of solutions with a recursive relation a *string of solutions*. The solutions in Table 5 may be arranged in three strings of solutions, as in Table 6.

Index	*String* 1		*String* 2		*String* 3	
i	x_i	y_i	x_i	y_i	x_i	y_i
1	1	1	9	3	41	13
2	79	25	351	111	1 559	493
3	3 001	949	13 329	4 215	59 201	18 721
4	113 959	36 037	506 151	160 059	2 248 079	710 905

TABLE 6

In each string the recursive connection is described by

$$x_i = 38x_{i-1} - x_{i-2}, \qquad y_i = 38y_{i-1} - y_{i-2}.$$

Example (iii): $a = 13$

Here the equation is $x^2 - 13y^2 = -12$ and some solutions are given in Table 7.

i	x_i	y_i
1	1	1
2	14	4
3	25	7
4	155	43
5	274	76
6	1 691	469
7	2 989	829
8	18 446	5 116

TABLE 7

A recursion relation appears when we arrange the solutions in two sequences, two strings. From Table 7, the relation is

$$x_i = 11x_{i-1} - x_{i-2}, \qquad y_i = 11y_{i-1} - y_{i-2}.$$

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For all a investigated, the set of integer solutions of $x^2 - ay^2 = 1 - a$ may be arranged in s strings, each with recursive relations of the form:

$$x_i = tx_{i-1} - x_{i-2}, \qquad y_i = ty_{i-1} - y_{i-2}.$$

Systematic investigations give the results listed in Table 8. Now we are open to investigate if there might be any interplay with the Pell equation. So in Table 8 we include a list of the least positive solutions of the related Pell equation $x^2 - ay^2 = 1$. We let $(x, y) = (u, v)$ be the least solution of this, so that $u^2 - av^2 = 1$. (For a list of least integer solutions, see [4, p. 254]).

However, for most values of a the Pell equation has solutions which are rational numbers as well as integers. For example, when $a = 10$, the equation $x^2 - 10y^2 = 1$ has a solution $\left(\frac{7}{3}, \frac{2}{3}\right)$ which is smaller than the least integer solution (19, 6); and when $a = 13$, the equation $x^2 - 13y^2 = 1$ has a solution $\left(\frac{11}{2}, \frac{3}{2}\right)$ which is smaller than the least integer solution (649, 180). In Table 8, for those values of a which have a solution in rationals with denominator 2, these values of u and v have been used rather than the values in integers. You will note that this occurs in Table 8 for $a = 5,\ 13$ and 21, and that, for all these values of a, $a \equiv 1 \pmod 4$.

To understand this, it is useful to introduce the idea of an *algebraic integer*. This is a number of the form $p + q\sqrt{a}$ which is a root of a quadratic equation with integer coefficients for which the coefficient of the square term is 1. For example, when $a = 13$ the number $\frac{11}{2} + \frac{3}{2}\sqrt{13}$ is an algebraic integer, because it is a root of the equation $x^2 - 11x + 1 = 0$ in which the coefficient of x^2 is 1. But when $a = 10$, the number $\frac{7}{3} + \frac{2}{3}\sqrt{10}$ is not an algebraic integer, because it is a root of $3x^2 - 14x + 3 = 0$, and the coefficient of x^2 is not 1.

The numerical results in Table 8 suggest the following result.

Theorem 2

For each non-square integer a the diophantine equation

$$x^2 - ay^2 = 1 - a$$

has a sequence of solutions that may be arranged in s strings, each given by two start values (x_1, y_1) and (x_2, y_2), and the recursive equations

$$x_i = 2ux_{i-1} - x_{i-2} \qquad \text{and} \qquad y_i = 2uy_{i-1} - y_{i-2}$$

for $i = 3,\ 4,\ 5,\ \ldots$, where (u, v) is the least algebraic integer solution of the Pell equation $x^2 - ay^2 = 1$. (A proof is given in the Appendix.)

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a	Diophantine equation $x^2 - ay^2 = 1 - a$	Least number of strings s	Recursion factor t	First s solutions (x, y)	Pell equation $x^2 - ay^2 = 1$ least x u	least y v
2	$x^2 - 2y^2 = -1$	1	6	(1,1)	3	2
3	$x^2 - 3y^2 = -2$	1	4	(1,1)	2	1
5	$x^2 - 5y^2 = -4$	1	3	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$
6	$x^2 - 6y^2 = -5$	2	10	(1,1), (7,3)	5	2
7	$x^2 - 7y^2 = -6$	2	16	(1,1), (13,5)	8	3
8	$x^2 - 8y^2 = -7$	2	6	(1,1), (5,2)	3	1
10	$x^2 - 10y^2 = -9$	3	38	(1,1), (9,3), (41,13)	19	6
11	$x^2 - 11y^2 = -10$	2	20	(1,1), (23,7)	10	3
12	$x^2 - 12y^2 = -11$	2	14	(1,1), (17,5)	7	2
13	$x^2 - 13y^2 = -12$	2	11	(1,1), (14,4)	$\frac{11}{2}$	$\frac{3}{2}$
14	$x^2 - 14y^2 = -13$	2	30	(1,1), (41,11)	15	4
15	$x^2 - 15y^2 = -14$	2	8	(1,1), (11,3)	4	1
17	$x^2 - 17y^2 = -16$	3	66	(1,1), (16,4), (103,25)	33	8
18	$x^2 - 18y^2 = -17$	2	34	(1,1), (145,13)	17	4
19	$x^2 - 19y^2 = -18$	3	340	(1,1), (39,9), (571,131)	170	39
20	$x^2 - 20y^2 = -19$	2	18	(1,1), (31,7)	9	2
21	$x^2 - 21y^2 = -20$	2	5	(1,1), (8,2)	$\frac{5}{2}$	$\frac{1}{2}$
22	$x^2 - 22y^2 = -21$	4	394	(1,1), (23,5), (89,19), (727,155)	197	42

TABLE 8

Now how may we find potential start values in each string, (x_1, y_1) and (x_2, y_2)? (1,1) is obviously the start of one string. By observing the values of $u - v$ in Table 8, the following result appears.

Theorem 3

$(x, y) = (av - u, u - v)$ is a solution of (8) for all a.

To prove this, we may insert $(x, y) = (av - u, u - v)$ in the left side of (8) and simplify:

$$x^2 - ay^2 = (av - u)^2 - a(u - v)^2$$
$$= a^2v^2 - 2auv + u^2 - au^2 + 2auv - av^2 = (u + av^2)(1 - a) = 1 - a.$$

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Theorem 3 implies that for $a > 3$ there are at least two distinct strings of solutions, generated by (1,1) and $(av - u, u - v)$ respectively. These two cannot coincide for $a > 3$.

From the numerical examples we observe for $a = 10$ the solution $(9, 3)$, and for $a = 17$ the solution $(16, 4)$. This pattern suggests a conjecture for some special a:

Theorem 4

If $a = k^2 + 1$, then $(x, y) = (k^2, k)$ is a solution of (8).

Proof: Inserting $(x, y) = (k^2, k)$ in the left side of (8) gives

$$x^2 - ay^2 = (k^2)^2 - ak^2 = k^2(k^2 - a) = (a - 1)(-1) = 1 - a,$$

so this result follows.

We don't claim that (1,1) together with Theorems 3 and 4 will give the starting pair for all the strings. One example in Table 9 is for $a = 19$, where the solution $(39, 9)$ is not given by Theorem 3 nor by Theorem 4. As far as I know it is necessary to search numerically to ensure completion of the list of solutions and to find the least number of strings s.

However, having found one solution in a string, Theorem 5 shows how to find the next solution.

Theorem 5

Let (x_i, y_i) be a solution in a string. Then the next solution (x_{i+1}, y_{i+1}) is given by $x_{i+1} = ux_i + avy_i$ and $y_{i+1} = vx_i + uy_i$. (This is proved in the Appendix.)

In particular, for the string starting with $(x_1, y_1) = (1, 1)$, the next solution is $(u + av, u + v)$. Thus this special string of solutions for each a is given by

$$x_1 = 1, \qquad y_1 = 1,$$
$$x_2 = u + av, \qquad y_2 = u + v,$$
$$x_i = 2ux_{i-1} - x_{i-2}, \qquad y_i = 2uy_{i-1} - y_{i-2}.$$

Now the results may be adapted to the original problem about triangular numbers $\frac{m(m+1)}{2} = \frac{n(n+1)}{2}\,a$. Substituting $(x_i, y_i) = (2m_i + 1,\ 2n_i + 1)$ Theorem 2 takes this form:

Equation (7) has solutions that may be arranged in strings, each of which is given by (m_1, n_1) and (m_2, n_2) and for $i = 3, 4, \ldots$

$$m_i = 2um_{i-1} - m_{i-2} - 1, \qquad n_i = 2un_{i-1} - n_{i-2} + u - 1.$$

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And Theorem 5 takes this form: If (m_i, n_i) is one solution of (7) then the next is given by

$$m_{i+1} = um_i + avn_i + \frac{u + av - 1}{2}, \qquad n_{i+1} = vm_i + un_i + \frac{u + v - 1}{2}.$$

Example: When is $\frac{T_m}{T_n} = 10$?

Taking $a = 10$, the substitution $(m_i, n_i) = \left(\frac{x_i - 1}{2}, \frac{y_i - 1}{2}\right)$ can be used to translate Table 6 into Table 9.

Index	*String* 1		*String* 2		*String* 3	
i	m_i	n_i	m_i	n_i	m_i	n_i
1	0	0	4	1	20	6
2	39	12	175	55	779	246
3	1 500	474	6 664	2 107	29 600	9 360
4	56 979	18 018	253 075	80 029	1 124 039	355 452

TABLE 9

Thus for example $T_{39} = \frac{39 \times 40}{2} = 780$ and $T_{12} = \frac{12 \times 13}{2} = 78$.

Appendix

To solve the diophantine equation

$$x^2 - ay^2 = 1 - a \tag{9}$$

we may look at the *field* $Q(\sqrt{a}) = \{x + y\sqrt{a} \mid x, y \in Q\}$ and the *ring* $Z(\sqrt{a}) = \{x + y\sqrt{a} \mid x, y \in Z\}$. The *norm* of a number $\alpha = x + y\sqrt{a}$ in this field is the product $(x + y\sqrt{a})(x - y\sqrt{a})$, and we denote it $N(\alpha)$, so $N(\alpha) = x^2 - ay^2$. The solutions of (9) are represented by numbers $\alpha = x + y\sqrt{a}$ with norm $N(\alpha) = 1 - a$. Now the solutions must be in integers, which is a significant restriction.

A *unit* in the ring of *algebraic integers* is a number $\varepsilon = r + s\sqrt{a}$ whose norm is 1. That means $(r + s\sqrt{a})(r - s\sqrt{a}) = r^2 - as^2 = 1$. According to the Dirichlet theorem of units, the set of units in this ring make a cyclic group. It is generated by a *fundamental unit*, and all units are powers of this one. For all a except for $a \equiv 1 \pmod 4$, all the units are in $Z(\sqrt{a})$. For $a \equiv 1 \pmod 4$ units may have the form $\frac{1}{2}b + \frac{1}{2}c\sqrt{a}$. ([5, Theorem 3.2]).

We let $\varepsilon = u + v\sqrt{a}$ denote the fundamental unit in the ring of algebraic integers. As

$$\varepsilon - u = v\sqrt{a}$$

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this implies

$$(\varepsilon - u)^2 = av^2,$$

$$\varepsilon^2 - 2u\varepsilon + u^2 = av^2,$$

$$\varepsilon^2 - 2u\varepsilon + 1 = 0,$$

$$\varepsilon^2 = 2u\varepsilon - 1.$$

Generally for the norm function, $N(\alpha\beta) = N(\alpha)N(\beta)$.

Thus, if α represents a solution of (9) and has the norm $1 - a$, then so has $\alpha\varepsilon$, $\alpha\varepsilon^2$ and so on, because $N(\varepsilon) = 1$. These will represent solutions of (9).

Now we let $\alpha_i = x_i + y_i\sqrt{a}$ represent a solution, i.e. α_i has norm $1 - a$.

(i) Then so also has $\alpha_i\varepsilon$, the next solution in the string.
Here $\alpha_i\varepsilon = (x_i + y_i\sqrt{a})(u + v\sqrt{a}) = (ux_i + avy_i) + (vx_i + uy_i)\sqrt{a}$. So we have expressions for the next solution in the string,

$$(x_{i+1}, y_{i+1}) = (ux_i + avy_i, vx_i + uy_i),$$

which proves Theorem 5.

(ii) Now since

$$\varepsilon^2 = 2u\varepsilon - 1$$

we may multiply by α_i and get

$$\alpha_{i+2} = 2u\alpha_{i+1} - \alpha_i$$

or

$$(x_{i+2}, y_{i+2}) = (2ux_{i+1} - x_i,\ 2uy_{i+1} - y_i),$$

which proves the recursive relation for three consecutive solutions in Theorem 2.

Example

Let $a = 5$, an example where $a \equiv 1 \pmod 4$.

The first few solutions of $x^2 - 5y^2 = -4$ are (1,1), (4,2), (11,5), (29,13), (76,34), (199,89) and (521,233). Written as quadratic integers $x + y\sqrt{5}$ these are $1 + \sqrt{5}$, $4 + 2\sqrt{5}$, $11 + 5\sqrt{5}$, $29 + 13\sqrt{5}$, $76 + 34\sqrt{5}$, $199 + 89\sqrt{5}$ and $521 + 233\sqrt{5}$. All these have norm -4.

The number $\varepsilon = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ is a root of the equation $x^2 - 3x + 1 = 0$. The leading coefficient in this equation is 1, so ε is an algebraic integer. Further $(\frac{3}{2} + \frac{1}{2}\sqrt{5})(\frac{3}{2} - \frac{1}{2}\sqrt{5}) = \frac{9}{4} - \frac{5}{4} = 1$, so its norm is 1. It represents a solution of the Pell equation $x^2 - 5y^2 = 1$. ε is the least unit, i.e. the fundamental unit in the ring of algebraic integers in $Q(\sqrt{5})$.

https://doi.org/10.1017/mag.2015.33 Published online by Cambridge University Press

Now the product $(1+\sqrt{5})\varepsilon = (1+\sqrt{5})(\frac{3}{2}+\frac{1}{2}\sqrt{5}) = 4+2\sqrt{5}$, while $(1+\sqrt{5})\varepsilon^2 = (1+\sqrt{5})(\frac{3}{2}+\frac{1}{2}\sqrt{5})^2 = 11+5\sqrt{5}$ and so on. The starting number $1+\sqrt{5}$ muliplied by powers of ε will generate all the solutions. This shows that the solutions now, based on the unit ε, are organised in one string. The recursion factor t is $2 \times \frac{3}{2} = 3$.

Contrast this with what occurs if we were to use the fundamental unit in the ring $Z(\sqrt{5})$ which is not ε, but $\gamma = 9+4\sqrt{5}$ which is ε^3. Now the product $(1+\sqrt{5})\gamma = (1+\sqrt{5})(9+4\sqrt{5}) = 29+13\sqrt{5}$, while

$$(1+\sqrt{5})\gamma^2 = (1+\sqrt{5})(9+4\sqrt{5})^2 = 521+233\sqrt{5}.$$

So $(1+\sqrt{5})$ multiplied by powers of the unit γ will generate a string consisting of every third solution.

A second string is generated from the second solution. This consists of $(4+2\sqrt{5})\gamma^i$ for $i = 0, 1, 2, \ldots$.

And a third string similarly comes from the third solution, given by $(11+5\sqrt{5})\gamma^i$ for $i = 0, 1, 2, \ldots$.

Thus, for this value of a, using the fundamental unit in $Z(\sqrt{5})$ results in the solutions organised in three strings, rather than the single string which is obtaines if the fundamental unit of algebraic integers in $Q(\sqrt{5})$ is used.

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doi:10.1017/mag.2015.33

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https://doi.org/10.1017/mag.2015.33 Published online by Cambridge University Press