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Arcs with increasing chords

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Let $f: [0, 1] \to R^2$ be a Jordan arc, and for $t, u \in [0, 1]$ let $d(t, u) = d(f(t), f(u))$ denote the Euclidean length of the chord between $f(t)$ and $f(u)$, and $l(t, u) = l(f(t), f(u))$ the corresponding arc-length, when this is defined. We say that f has the *increasing chord property* if $d(t_2, t_3) \leqslant d(t_1, t_4)$ whenever $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4 \leqslant 1$. In connexion with a problem in complex analysis, K. Binmore has asked (private communication, see (1)) whether there exists an absolute constant K such that

$$l(0, 1) \leqslant Kd(0, 1).$$

It is the purpose of this paper to show that such a constant exists. We shall prove

THEOREM. *Let $f: [0, 1] \to R^2$ be a Jordan arc with the increasing chord property. Then f is rectifiable, and*

$$l(0, 1) \leqslant 2\sqrt{3}.d(0, 1).$$

It is clear from the definition that f lies in the area enclosed by the two circular arcs with centres $f(0)$ and $f(1)$ and radius $d(0, 1)$. Thus if f is a convex arc, then

$$l(0, 1) \leqslant \tfrac{2}{3}\pi d(0, 1),$$

with equality for the arc consisting of two sides of a Reuleaux triangle. It may be conjectured that the constant $\frac{2}{3}\pi$ is best possible for all arcs, whether convex or not.

We prove the theorem in several steps. We first observe that f is necessarily a simple curve, and so we may assume that $f(t) \neq f(u)$ if $t \neq u$. Further, there is no loss in generality in supposing that

$$f(0) = (0, 0), \quad f(1) = (1, 0).$$

Let us write $$f(t) = (x(t), y(t)) \quad (t \in [0, 1]).$$

We first show

LEMMA 1. *If $0 \leqslant t < u \leqslant 1$, then $x(t) < x(u)$.*

For, if not, since f is continuous we can find $t, u \in [0, 1]$, with $t < u$, such that $x(t) = x(u)$. If $|y(t)| < |y(u)|$ then $d(t, 1) < d(u, 1)$, while if $|y(t)| > |y(u)|$, then $d(0, t) > d(0, u)$, in either case contradicting the assumption that f has the increasing chord property. Thus $|y(t)| = |y(u)|$ and, since $f(t) \neq f(u)$, we must have

$$y(t) = -y(u) \neq 0.$$

But then the arc of f from $f(t)$ to $f(u)$ must consist simultaneously of the arcs of the circles with centres $(0, 0)$ and $(1, 0)$ through $f(t)$ and $f(u)$, which is impossible. Hence $x(t) < x(u)$, as claimed.

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In the proof of the theorem, the chords of f which make an angle of $\frac{1}{3}\pi$ with the x-axis play an important rôle. Concerning such chords, we have

LEMMA 2. *Let C_1 and C_2 be two chords of f, and let θ be the angle between them. If $\theta < \frac{1}{3}\pi$, then C_1 and C_2 do not meet, while if $\theta = \frac{1}{3}\pi$, then C_1 and C_2 can only meet at a common end-point.*

Note. If two chords $C_1 = f(t_1)f(t_2)$, $C_2 = f(t_3)f(t_4)$ meet at ω say, with $t_1 < t_3 \leqslant t_2 < t_4$ then the angle between C_1 and C_2 is defined to be the angle at ω in the triangle with vertices $f(t_1)\,\omega f(t_2)$, or π if the triangle is degenerate.

For, suppose that $C_1 = f(t_1)f(t_2)$, $C_2 = f(t_3)f(t_4)$, and $\theta < \frac{1}{3}\pi$. If C_1 and C_2 meet, then we may assume $t_1 < t_3 \leqslant t_2 < t_4$, and consider the chord $C_2' = f(t_2)f(t_4)$ instead of C_2, since the angle between C_1 and C_2' is at most θ. At least one of the remaining two angles of the triangle $f(t_1)f(t_2)f(t_4)$ is greater than θ; if it is that at $f(t_1)$, then $d(t_1, t_4) < d(t_2, t_4)$, and if it is that at $f(t_4)$, then $d(t_1, t_4) < d(t_1, t_2)$. In either case, this contradicts the assumption that f has the increasing chord property.

In case $\theta = \frac{1}{3}\pi$, if C_1 and C_2 meet otherwise than at an end-point of each, then in the notation used above, we have $t_1 < t_3 < t_2 < t_4$. Hence the angle between one of the pairs of chords $f(t_1)f(t_3)$, C_2 and C_1, $f(t_2)f(t_4)$ is less than $\frac{1}{3}\pi$, which is impossible by the first part. This proves the lemma.

We now prove the theorem. Let $\{t_0, t_1, \ldots, t_n\}$ be any partition of $[0, 1]$. The chords $C_i = f(t_{i-1})f(t_i)$ of f are of two types, according as the angles θ_i they make with the x-axis satisfy (1) $0 \leqslant \theta_i \leqslant \frac{1}{3}\pi$, or (2) $\frac{1}{3}\pi < \theta_i < \frac{1}{2}\pi$. We construct a refinement of our given partition as follows. Suppose that, for some $j < i \leqslant k$, $\theta_i > \frac{1}{3}\pi$, while $\theta_j, \theta_{k+1} \leqslant \frac{1}{3}\pi$. From Lemma 2 we conclude that the angles θ_i have the same sense, so that

$$y(t_j) < y(t_{j+1}) < \ldots < y(t_k), \quad \text{or} \quad y(t_j) > y(t_{j+1}) > \ldots > y(t_k).$$

It is clear that at least one of the lines through $f(t_j)$ and $f(t_k)$ making an angle of $\frac{1}{3}\pi$ with the x-axis (in the same sense) will meet the arc again. For example, in the case where $y(t_i)$ increases, if $y(t_j) \geqslant 0$ we may take the line through $f(t_j)$, if $y(t_k) \leqslant 0$ we may take that through $f(t_k)$, and if $y(t_j) < 0 < y(t_k)$, both the lines are suitable. If the chosen line next meets the arc in $f(s)$, we introduce (if necessary) the point s into our partition.

It may happen that there are several successive such sets of chords of type (2), whose angles with the x-axis all have the same sense. In such a case, we first introduce the new point of partition associated with the set of chords which straddles the x-axis (if one exists), and then successively work away from the x-axis on each side. In doing this, we may in fact associate with a new chord of angle $\frac{1}{3}\pi$ more than one set of chords of type (2); if this happens we need only introduce the extreme new partition point. In any case, with each chord C_i of type (2) we have associated a chord C of angle $\frac{1}{3}\pi$, such that the subarc of f corresponding to C_i is contained in the subarc of f corresponding to C. (Notice that chords of type (1) will also be associated with C in this way.) These chords C do not meet, except possibly at end-points. Using Lemma 2 again, we see that the new chords of f, corresponding to the new partition points s, must actually be of type (1).

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Let us now change notation, and suppose that $\{t_0, t_1, \ldots, t_n\}$ is our partition of $[0, 1]$, refined as above if necessary. The chords $C_i = f(t_{i-1})f(t_i)$ of type (2) are associated with chords C of angle $\frac{1}{3}\pi$; these chords, and hence their projections on to the x-axis, meet at most at their end-points, and so the total length of the projections is (say) $B \leqslant 1$. If C_i is one of the chords (of types (1) or (2)) associated with C, then C_i makes an angle with C lying between $\frac{1}{6}\pi$ in one sense (since by Lemma 1, C_i makes an angle of at most $\frac{1}{2}\pi$ with the x-axis), and $\frac{1}{2}\pi$ in the other sense (by Lemma 1, with C playing the rôle of the x-axis). If we now rearrange the chords C_i associated with C according to increasing angle θ_i, they form a convex polygonal arc whose end-points are those of C, which lies within a triangle with base C and base angles $\frac{1}{6}\pi$ and $\frac{1}{2}\pi$. Thus the polygonal arc has length at most

$$\frac{2}{\sqrt{3}}L(C)+\frac{1}{\sqrt{3}}L(C) = \sqrt{3}\,.L(C),$$

when $L(C)$ is the length of C. Since $L(C)$ is twice the length of its projection on to the x-axis, the total length of the chords C_i associated with the chords C of angle $\frac{1}{3}\pi$ is at most $2\sqrt{3}\,B$.

The remaining chords C_i are all of type (1), and so their total length is clearly at most $2(1-B)$. We conclude that

$$\begin{aligned}\sum_{i=1}^{n} d(t_{i-1}, t_i) &\leqslant 2\sqrt{3}\,B + 2(1-B)\\ &\leqslant 2\sqrt{3},\end{aligned}$$

which leads at once to the assertion of the theorem.

We remark, in conclusion, that our result is not the best possible. For example, we could have related chords corresponding to a general partition to chords of a fixed angle ϕ, the factor $2\sqrt{3}$ then being replaced by

$$\sec\phi\,(\operatorname{cosec}\phi + \cot\phi).$$

The optimal value of ϕ, for our purposes, is then $\arccos\tau^{-1}$, where $\tau = \frac{1}{2}(\sqrt{5}+1)$. However, we have added complications from the fact that chords of f making this angle with the x-axis may meet at other than end-points. In any case, the resulting factor $\tau^{\frac{5}{2}} = 3{\cdot}33$ is little improvement over $2\sqrt{3} = 3{\cdot}46$, in comparison with the conjectured factor $\frac{2}{3}\pi$.

The problem has a natural extension to $n \geqslant 3$ dimensions, and a plausible guess is that an arc in R^n with the increasing chord property is rectifiable, and the ratio of its length to its diameter is bounded above by the length of the arc consisting of n successive sides of a Reuleaux simplex of diameter one. This length is

$$n\sqrt{\frac{2n}{n-1}}\arcsin\sqrt{\frac{n-1}{2n}}.$$

However, we have not investigated this problem, and so we are not prepared to propose this as a conjecture.

REFERENCE

(1) Binmore, K. On Turan's lemma. *Bull. London Math. Soc.* **3** (1971), 313–317.

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