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Weakly wandering vectors for Abelian unitary actions

BORIS BEGUN

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel
(e-mail: begun@math.huji.ac.il)

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Abstract. A result of Krengel on the density of the set of vectors with infinitely many pairwise orthogonal iterates under powers of a unitary operator with continuous spectrum is generalized to actions of countable Abelian groups.

Introduction

In the early 1970s, Krengel investigated the phenomenon of so-called weak wandering of vectors under iterates of a weakly mixing unitary operator. A unitary operator U acting on a Hilbert space $\mathcal{H}$ is called *weakly mixing* if it has no eigenvectors. A non-zero vector $f \in \mathcal{H}$ is called *weakly wandering* with respect to U (or, equivalently, with respect to the $\mathbb{Z}$-action $\{U^n\}_{n\in\mathbb{Z}}$) if there exists an increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that the vectors $\{U^{n_i} f\}_{i=1}^{\infty}$ are pairwise orthogonal. Krengel has proved the following theorem.

THEOREM. ($\mathbb{Z}$-action, [**Kre**]) *The set of vectors that are weakly wandering with respect to a weakly mixing $\mathbb{Z}$-action is dense in $\mathcal{H}$.*

Later on, numerous refinements and generalizations of this result were obtained for unitary actions (= representations) of more general Abelian groups [**Gra, BKM1, BKM2, delJ**] and recently for finitely-generated nilpotent groups [**BKLM**]. The notion of weak mixing for an Abelian group action can be defined in the following natural way.

Definition. Suppose that G is an Abelian group, $\mathcal{H}$ is a Hilbert space, and $\{U_g\}_{g\in G}$ is a unitary action of G on $\mathcal{H}$. The action $\{U_g\}_{g\in G}$ is called *weakly mixing* if it has no eigenvectors, i.e. if there is no $f \in \mathcal{H}$, $f \neq 0$ such that $U_g f = \lambda(g) f$ ($\lambda(g) \in \mathbb{C}$) for all $g \in G$.

(More generally, for an amenable group G, not necessarily Abelian, one should require the absence of finite-dimensional G-invariant subspaces—see [**Dye**].) The definition of a weak wandering vector for a general group action is straightforward.

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Definition. A non-zero vector $f \in \mathcal{H}$ is called *weakly wandering* (with respect to $\{U_g\}_{g\in G}$) if its orbit $\{U_g f \mid g \in G\}$ contains an infinite subset of pairwise orthogonal vectors.

Shortly after the publication of Krengel's work, Graham proved a similar theorem for $\mathbb{Z}^d$ (in fact, in [**Gra**] the result is obtained for all compactly generated Abelian groups, i.e. for compact Abelian groups, $\mathbb{Z}^d$, $\mathbb{R}^d$ and their direct sums). Then, in 1993–1994, Bergelson *et al* introduced a new method, which allowed them to extend Krengel's theorem to wider classes of Abelian groups, in particular to groups of the form $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{n_i}$ ($\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$), $\sup(n_i) < \infty$. The core of this new approach was the use of the fixed point theorem for a specially constructed nonlinear contraction of a ball in some Banach space.

An additional feature of the method used in [**BKM1, BKM2**] is that it allows one to force weak wandering along *IP-sets*, which play an important role in ergodic theory and its applications to combinatorics (see, for example, [**FW1, FW2, Fur, FK, BFM, BM**]). These are the sets of the form

$$\Gamma = \{g_{n_1} + g_{n_2} + \cdots + g_{n_k} \mid k \in \mathbb{N}, n_1, n_2, \ldots, n_k \text{ are distinct}\},$$

where $\{g_n\}_{n=1}^{\infty}$ is an arbitrary sequence of elements of G. In addition, for the case of $\mathbb{Z}^d$-actions, the IP-sets considered in [**BKM1**] and [**BKM2**] are symmetric: $\Gamma = -\Gamma$.

In a recent paper [**delJ**], del Junco obtained most of the results from [**BKM1**] and [**BKM2**] by using methods of commutative harmonic analysis. The use of the spectral theorem allows to view the problem as a linear one. (Del Junco's method, however, is not sufficient for the treatment of groups of the form $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{n_i}$.)

In this paper, the theorem of Krengel is generalized to weakly mixing actions of arbitrary countable Abelian groups. The result we obtain essentially states the following.

THEOREM. (General Abelian action) *Let G be a countable Abelian group. Then the set of weakly wandering vectors for a weakly mixing G-action on a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$.*

Our method, like that of [**delJ**], makes use of the spectral theorem. The main novelty in our proof is the use of the spectral decomposition for the '2-fold action' $g \mapsto U_{2g}$. This allows us to prove the wandering result in the required generality. As will be seen from the proof, wandering can be forced to occur along IP-sets and, for $\mathbb{Z}^d$-actions, along symmetric IP-sets. Therefore, our theorem generalizes results from [**BKM1**], [**BKM2**] and [**delJ**].

Another interesting direction for investigation is weak wandering for *mildly mixing* group actions. This is a stronger notion of mixing—an action is called mildly mixing if it has no *rigid vectors*. Let us say that a sequence $\{g_i\}_{i=1}^{\infty}$ in a countable group G goes to infinity ($g_i \to \infty$) if for any finite set $F \subset G$, the set $\{i \mid g_i \in F\}$ is also finite. A non-zero vector f is called rigid if there exists a sequence $\{g_i\}_{i=1}^{\infty}$ in G such that $g_i \to \infty$ and $U_{g_i} f \to f$. For this stronger notion of mixing one can impose a stronger condition on the set of wandering, namely, every IP-set contains an IP-subset along which there is orthogonality. This result was proved in [**BKM2**].

Going back to $\mathbb{Z}$ there is one more possibility for generalizations. Let p be a non-constant integer-valued polynomial ($p(\mathbb{Z}) \subset \mathbb{Z}$). Then for a weakly mixing U one can

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force wandering along an infinite subset of the set $\{p(n) \mid n \in \mathbb{Z}\}$. The proof is readily obtained by combining Lemma 3.14 from [**Fur**] with Theorem 2 of [**delJ**] (or Lemma 3 of [**BKM2**]). Furthermore, one can crossbreed polynomial and IP-aspects of the result. One way to do this is to get orthogonality along $p(n)$, where the values of n form an IP-set—see [**Beg**] for the results for weakly and mildly mixing $\mathbb{Z}$-actions. As the referee has noted there is another possible crossbreed—one can demand orthogonality along an IP-set generated by values of $p(n)$. This result (its weak mixing and mild mixing versions) follows from the above results in [**Fur**] and [**delJ**] in a straightforward way. We are not sure which version of the polynomial result, if any, is a 'natural' one.

For completeness, we mention here a more distant line of investigation. In [**Kre**], Krengel pioneered an additional direction of research, related in spirit but pursued by completely different methods. Given an invertible measure-preserving transformation of a probability space, one can consider its natural action on finite partitions of the space. Krengel called a partition *weakly independent* if it has infinitely many jointly independent iterates under the action. He proved that weakly independent partitions are dense in the set of all partitions. In [**JRW**], Krengel's result was generalized for weakly mixing free actions of Abelian groups subject to a requirement that the 2-fold action is weakly mixing as well. The methods used in that paper involve Rokhlin towers and extension of shift-invariant measures from Ω^A to Ω^G, Ω—a finite set, $A \subset G$. In the language of stochastic processes, this extension amounts to building of a stationary stochastic process from its marginal.

In the forthcoming work [**BJ**], the following result for any amenable action is established.

THEOREM. (Partitions) *Suppose a discrete amenable group G acts freely on a probability space $(X, \mathcal{B}, \mu)$, and the action $\{T_g\}_{g\in G}$ is weakly mixing. Then for any $\varepsilon > 0$ and finite partition P of X there is a partition Q, ε-close to P, and an infinite sequence $\{g_i\}_{i=1}^{\infty}$ in G such that the partitions $T_{g_i}P$ are jointly independent.*

Similarly to the results above on wandering of vectors, in [**JRW**] and [**BJ**] the independence of partitions is obtained along IP-sets in G.

Preliminaries

Throughout the paper G stands for a countably infinite discrete Abelian group with additively written operation. We also fix a Følner sequence $\{F_n\}_{n=1}^{\infty}$ for G, that is, a sequence of finite subsets F_n of G with the property that for any $g \in G$

$$\frac{|(F_n + g) \cap F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 1;$$

the existence of such a sequence is implied by the amenability of G (see, e.g., [**Gre, Pat**]). For example, for $G = \mathbb{Z}^d$ a possible choice is $F_n = \{-n \le x \le n\}^d$ or $F_n = \{0 \le x \le n\}^d$.

The upper density of a set $A \subset G$ (relative to $\{F_n\}_{n=1}^{\infty}$) is defined by

$$\bar{d}(A) = \limsup \frac{|A \cap F_n|}{|F_n|}.$$

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If this ratio tends to a limit as $n \to \infty$, then the upper density is simply called *density* and is denoted by $d(A)$. We say that $A \subset G$ is *of zero density* if $d(A) = 0$.

Let $\{U_g\}_{g\in G}$ be a unitary action of G on a Hilbert space $\mathcal{H}$ (all Hilbert spaces will be considered complex). We will often find it convenient to work with the following definition of weak mixing. The action is *weakly mixing* if for any $f, h \in \mathcal{H}$ and any $\varepsilon > 0$ the set $\{g \in G : |\langle U_g f, h\rangle| > \varepsilon\}$ is of zero density in G or, equivalently, if $(1/|F_N|)\sum_{g\in F_N} |\langle U_g f, h\rangle| \xrightarrow[N\to\infty]{} 0$. (A notion of *strong mixing*, or simply *mixing*, corresponds to the case when the sets $\{g \in G : |\langle U_g f, h\rangle| > \varepsilon\}$ are finite.) It should be noted that the property of weak mixing is independent of the choice of the Følner sequence. An equivalent definition of weak mixing (for Abelian group actions) is that the operators U_g, $g \in G$, have no common non-zero eigenvector in $\mathcal{H}$. (See [**Dye**] for different equivalent characterizations of weak mixing actions of any amenable topological semigroup.) An action $\{U_g\}_{g\in G}$ is called *compact* if all the orbits $\{U_g f \mid g \in G\}$ are totally bounded in $\mathcal{H}$. (Equivalently, that means that $\mathcal{H}$ is spanned by the eigenvectors of the action, i.e. the action has discrete spectrum.)

The notion of *IP-set*, which plays the decisive role in [**FW1**] as well as in applications of ergodic theory to combinatorics (see [**Fur, FK, FW2, BFM, BM**]), is defined in the following way. Let $\{g_j\}_{j=1}^{\infty}$ be a sequence in G and let $\mathcal{F}$ denote the set of all finite non-empty subsets of $\mathbb{N}$. For $\alpha \in \mathcal{F}$ denote $g_\alpha := \sum_{j\in\alpha} g_j$. Given $\{g_j\}_{j=1}^{\infty}$, the set $\{g_\alpha \mid \alpha \in \mathcal{F}\}$ is denoted by $\mathrm{FS}(\{g_j\}_{j=1}^{\infty})$ ('finite sums') and is called an IP-set (one of the possible meanings of 'IP' is 'Infinite-dimensional Parallelepiped', see [**Fur**, p. 53]). We note that repetitions are allowed in the sequence $\{g_j\}_{j=1}^{\infty}$ (even in infinite quantities), thus the notion of IP-set is a generalization of the notion of semigroup.

Again let $\{U_g\}_{g\in G}$ be a unitary action of G on a Hilbert space $\mathcal{H}$. A non-zero vector $f \in \mathcal{H}$ is called *weakly wandering* if its orbit $\{U_g f \mid g \in G\}$ contains an infinite subset of pairwise orthogonal vectors.

Wandering along IP-sets in weakly mixing actions

Here we present the exact statement of the wandering result and prove it. Special cases of the theorem were previously proved in [**BKM1, BKM2, delJ**]. Our proof has much in common with these proofs, but it has several new features.

THEOREM. *Let $\{U_g\}_{g\in G}$ be a weakly mixing unitary action of a countable Abelian group G on a Hilbert space $\mathcal{H}$ and let A be a subset of G with $\bar{d}(A) > 0$. Then for any $f \in \mathcal{H}$ and $\varepsilon > 0$ there exist a vector $\tilde{f}$ with $\|\tilde{f} - f\| \le \varepsilon$ and a sequence $\{g_j\}_{j=1}^{\infty}$ of elements of A such that*

$$U_{g_\alpha}\tilde{f} \perp U_{g_\beta}\tilde{f}$$

for any $\alpha, \beta \in \mathcal{F}$, $\alpha \neq \beta$.

If a stronger condition on A, $\bar{d}(A) = 1$, holds, then one can enforce, in addition, $\mathrm{FS}(\{g_j\}_{j=1}^{\infty}) \subset A$.

Remark. If a unitary action has an eigenvector, then its weakly wandering vectors are clearly not dense. Therefore, the above theorem in fact provides a characterization of weak mixing.

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Before we pass to the proof of the theorem some preliminary work has to be done. We will need the following lemma.

LEMMA 1. *Suppose that $(X, \mathcal{B}, \mu)$ is a probability space, $\{\varphi_n\}_{n\in I}$ (I is at most countable) is a family of measurable complex-valued functions on X satisfying $|\varphi_n(\cdot)| \equiv 1$ for all $n \in I$, and F is a bounded real-valued measurable function on X. Assume that $L := \mathrm{Span}(\{\varphi_n\}_{n\in I})$ (in $L^2(X)$) is closed under complex conjugation, i.e. $\varphi \in L \Longrightarrow \bar{\varphi} \in L$. Let $\rho \geq 0$, $0 \leq \sigma < 1$ and suppose that the following conditions hold:*

(i) $\sum_{n\in I} |\langle F, \varphi_n\rangle| \leq \rho$;

(ii) $\sum_{\substack{n\in I \\ n\neq m}} |\langle \varphi_m, \varphi_n\rangle| \leq \sigma$ *for any* $m \in I$.

Then there exists $\tilde{F} \in L^\infty(X)$ with the following properties:

(a) $\|\tilde{F} - F\|_\infty \leq \rho/(1-\sigma)$;

(b) $\tilde{F}$ *is a real-valued function;*

(c) $\tilde{F} \perp L$.

Proof. We will search for $\tilde{F}$ in the form

$$\tilde{F} = F - \sum_{n\in I} a_n \varphi_n \tag{1}$$

with

$$\sum_{n\in I} |a_n| \leq \frac{\rho}{1-\sigma}. \tag{2}$$

Clearly this will imply (a). To get (c) we need

$$\langle \tilde{F}, \varphi_m\rangle = 0 \quad \text{for all } m \in I, \tag{3}$$

or, substituting (1),

$$\langle F, \varphi_m\rangle = \sum_{n\in I} a_n \langle \varphi_n, \varphi_m\rangle. \tag{4}$$

Now consider two vectors in $l_1(I)$: $\bar{a} = \{a_n\}_{n\in I}$ to be determined, and $\bar{b} = \{b_n\}_{n\in I} = \{\langle F, \varphi_n\rangle\}_{n\in I}$ (by (i), $\|b\|_1 \leq \rho$), and the (generally, infinite) Hermitian matrix $T = \{t_{m,n}\}_{m,n\in I} = \{\langle \varphi_n, \varphi_m\rangle\}_{m,n\in I}$. This matrix defines a linear transformation from $l_1(I)$ to itself ($(T\bar{a})_m = \sum_{n\in I} t_{m,n} a_n$), σ-close to the identity:

$$\|T - \mathrm{Id}\|_{\mathrm{End}(l_1)} = \sup_{n\in I} \sum_{m\in I\setminus\{n\}} |\langle \varphi_n, \varphi_m\rangle| \leq \sigma$$

by (ii). Therefore, (4) can be rewritten in the form $\bar{b} = T\bar{a}$, and we also obtain that, since $\sigma < 1$, T is invertible and (the operator norms are estimated in $\mathrm{End}(l_1)$)

$$\|T^{-1}\| = \|(\mathrm{Id} - (\mathrm{Id} - T))^{-1}\| \leq \left\| \sum_{k=0}^{\infty} (\mathrm{Id} - T)^k \right\| \leq \sum_{k=0}^{\infty} \sigma^k = \frac{1}{1-\sigma}.$$

Thus, we may write

$$\bar{a} = T^{-1}\bar{b}, \quad \|a\|_1 \leq \|T^{-1}\|_{\mathrm{End}(l_1)} \|b\|_1 \leq \frac{\rho}{1-\sigma}.$$

We have found $\bar{a} = \{a_n\}_{n\in I}$ such that (2) is satisfied, so for $\tilde{F}$ defined by (1), the condition (a) holds. By (3), (c) also holds for this $\tilde{F}$.

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Finally, (b) is implied by (c) and the 'self-conjugacy' of F and L. Indeed, by (1), $F - \tilde{F} \in L$, where $\tilde{F} \perp L$ by (c). Therefore, $F - \tilde{F}$ is the orthogonal projection of F onto $L : \pi_L F = F - \tilde{F}$, which implies

$$\overline{F - \tilde{F}} = \pi_{\bar{L}} \bar{F} = \pi_L F = F - \tilde{F}.$$

Therefore, $F - \tilde{F}$ is real and so is $\tilde{F} = F - (F - \tilde{F})$, which proves (b). □

Now let G be an Abelian group; we will write G additively. The following notation will be used: if Γ is a subset of G, then $\Gamma - \Gamma := \{g_1 - g_2 \mid g_1, g_2 \in \Gamma\}$.

If H is a subgroup and Δ, Δ' are subsets of G, we say that Δ, Δ' are *equivalent modulo* H and write $\Delta \equiv \Delta' \pmod{H}$ if the images of Δ and Δ' in the quotient group G/H coincide. In other words, $\Delta \equiv \Delta' \pmod{H}$ if and only if for each $\delta \in \Delta$ there exists a $\delta' \in \Delta'$ such that $\delta - \delta' \in H$, and *vice versa*. (To avoid confusion, we remark that no one-to-one correspondence between Δ and Δ' is assumed.)

The following lemma plays the key role in the sequel.

LEMMA 2. *Suppose $\{U_g\}_{g \in G}$ is a unitary action of an Abelian group G on a Hilbert space $\mathcal{H}$. Denote by $\bar{G}$ the subgroup of G whose action on $\mathcal{H}$ is scalar: $\bar{G} = \{g \in G \mid U_g = \lambda(g)\mathrm{Id}_{\mathcal{H}},\ \lambda(g) \in \mathbb{C}\}$. Let $\mathcal{C}$ be a class of subsets of G such that $\mathcal{C}$ satisfies the following requirement: for any $h, F \in \mathcal{H}$ and any $\rho > 0$, $\sigma > 0$ there exists a set $\Gamma = \Gamma(h, F, \rho, \sigma) \in \mathcal{C}$ such that for $\Delta := \Gamma - \Gamma$ the following holds:*

$$\Delta \cap \bar{G} = \{0\}, \tag{o}$$

and for some $\Delta_0 \subset G$, $\Delta_0 \equiv \Delta \setminus \bar{G} (\mathrm{mod}\ \bar{G})$ one has:

(i) $\sum_{\delta \in \Delta_0} |\langle U_\delta h, F\rangle| \le \rho$;

(ii) *for every $\delta \in \Delta_0$, $\sum_{\substack{\delta' \in \Delta_0 \\ \delta' \neq \delta}} |\langle U_{\delta'} h, U_\delta h\rangle| \le \sigma$.*

Then for any $f \in \mathcal{H}$ and any $\varepsilon > 0$ there exist $\tilde{f} \in \mathcal{H}$ with $\|\tilde{f} - f\| \le \varepsilon$ and a set $\Gamma \in \mathcal{C}$ such that $U_{\gamma_1} \tilde{f} \perp U_{\gamma_2} \tilde{f}$ for any $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

Remarks. (1) Clearly, in choosing Δ_0 satisfying (i) and (ii) it is in one's interest to choose 'the minimal possible' Δ_0, i.e. to take just one representative from each coset in $G/\bar{G}$ that is different from $\bar{G}$ itself and intersects Δ non-trivially.

(2) Certainly, one can assume without loss of generality that $\rho = \sigma$, but we prefer to keep use of both variables to make it easier for the reader to follow the proof.

(3) As one will see from the proof, the conclusion of the lemma remains true if its requirements are weakened by fixing any value of $\sigma = \sigma_0 < 1$ and demanding the existence of $\Gamma = \Gamma(h, F, \rho) \in \mathcal{C}$.

Proof. By a result from the spectral theory of Abelian group actions (see, e.g., [**Zim**]), the given G-action $\{U_g\}$ is unitarily isomorphic to a G-action on a Hilbert space $\mathcal{H} = L^2(X)$ (for a probability space $(X, \mathcal{B}, \mu)$) given by

$$U_g h = \psi_g \cdot h \quad \text{for any } g \in G, h \in \mathcal{H},$$

where $\psi_g \in L^\infty(X)$, $|\psi_g(\cdot)| \equiv 1$, for all $g \in G$. From now on we consider this new action and denote it in the same way as before, by U_g. (The method of [**BKM1**] and [**BKM2**]

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reduces proofs of the results about wandering to a nonlinear fixed point problem. The use of the spectral theorem allows us to view the problem as a linear one.)

We are looking for $\tilde{f} \in L^\infty(X) \subset L^2(X)$, $\|\tilde{f} - f\|_2 \le \varepsilon$, and a set $\Gamma \in \mathcal{C}$ such that $U_{\gamma_1}\tilde{f} \perp U_{\gamma_2}\tilde{f}$ for all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \ne \gamma_2$, which is equivalent to

$$U_\delta \tilde{f} \perp \tilde{f} \quad \text{for all } \delta \in \Delta = \Gamma - \Gamma, \delta \ne 0. \tag{$*$}$$

Γ will be found first and $\tilde{f}$ after that. $\Gamma \in \mathcal{C}$ will be chosen so that (o), (i) and (ii) hold for ρ and σ small enough, whose values are to be determined below. The condition (o) gives $\Delta \setminus \{0\} = \Delta \setminus \bar{G}$, thus for each $\delta \in \Delta$, $\delta \ne 0$ there is $\delta' \in \Delta_0$ with $\delta - \delta' \in \bar{G}$, which means $U_\delta = \lambda(\delta - \delta')U_{\delta'}$, $\lambda(\delta - \delta') \in \mathbb{C}$. Therefore, it is enough to check the condition $(*)$ for the case $\delta \in \Delta_0$ only. For $\delta \in G$ and $\tilde{f} \in \mathcal{H} = L^2(X)$ we have

$$\langle U_\delta \tilde{f}, \tilde{f}\rangle = \int_X U_\delta \tilde{f} \cdot \overline{\tilde{f}}\, d\mu = \int_X \psi_\delta \tilde{f} \cdot \overline{\tilde{f}}\, d\mu = \int_X \psi_\delta |\tilde{f}|^2\, d\mu = \int_X U_\delta \mathbf{1} \cdot |\tilde{f}|^2\, d\mu,$$

where $\mathbf{1} \in L^2(X)$ is the function that identically equals 1 on X.

Thus, the requirement $(*)$ can be rewritten as

$$\int_X U_\delta \mathbf{1} \cdot |\tilde{f}|^2\, d\mu = 0 \quad \text{for all } \delta \in \Delta_0. \tag{$**$}$$

The function $\tilde{f} \in L^\infty(X)$ satisfying $\|\tilde{f} - f\|_2 \le \varepsilon$ and $(**)$ can be found in two steps. First, we pick $f' \in L^2(X)$ such that $\|f' - f\|_2 \le \varepsilon/2$, $c = \operatorname{ess\,inf}|f'| > 0$ and $C = \|f'\|_\infty = \operatorname{ess\,sup}|f'| < \infty$, and put $F = |f'|^2$.

Next, we seek for $\tilde{f} \in L^\infty(X)$, $\|\tilde{f} - f'\|_2 \le \varepsilon/2$, such that $(**)$ holds. To find $\tilde{f}$, we look for $\tilde{F} = |\tilde{f}|^2$—a real-valued function $\tilde{F} \in L^\infty(X)$ such that $U_\delta \mathbf{1} \perp \tilde{F}$ for all $\delta \in \Delta_0$, and $\|\tilde{F} - F\|_\infty \le \tau$, where τ is any positive constant that satisfies (a) $\tau \le c^2$, and (b) $\tau \le (c^2/2C)\varepsilon$. If such $\tilde{F}$ is found, then we are done: since $\tilde{F} \ge 0$ by (a), one can define $\tilde{f} = f'\sqrt{\tilde{F}/F}$, which gives $|\tilde{f}|^2 = \tilde{F}$ and

$$\begin{aligned}\|\tilde{f} - f'\|_2 \le \|\tilde{f} - f'\|_\infty &\le \|f'\|_\infty \left\|\sqrt{\frac{\tilde{F}}{F}} - \mathbf{1}\right\|_\infty = \|f'\|_\infty \left\|\sqrt{\mathbf{1} + \frac{\tilde{F} - F}{F}} - \mathbf{1}\right\|_\infty \\ &\le \|f'\|_\infty \left\|\frac{\tilde{F} - F}{F}\right\|_\infty \le \|f'\|_\infty \|F^{-1}\|_\infty \|\tilde{F} - F\|_\infty \le Cc^{-2}\tau \le \frac{\varepsilon}{2}\end{aligned}$$

by (b).

It is now left to explain how to find $\tilde{F}$ with the two properties listed above. The assumptions of the lemma hold in particular for the functions $h = \mathbf{1}$, $F = |f'|^2$ and for $\rho = \tau/2$, $\sigma = \frac{1}{2}$. There exists a set $\Gamma \in \mathcal{C}$ such that $\Delta = \Gamma - \Gamma$ satisfies (o), and some $\Delta_0 \subset G$, $\Delta_0 \equiv \Delta \setminus \bar{G} \pmod{\bar{G}}$ has the properties (i) and (ii) for the functions and constants above. So we may apply Lemma 1 to F, ρ, σ and to the system

$$\{\varphi_\delta\}_{\delta \in \Delta_0} := \{U_\delta \mathbf{1}\}_{\delta \in \Delta_0}.$$

Indeed, (i) and (ii) provide us with the corresponding requirements of Lemma 1, and since the set $\Delta \setminus \{0\} = (\Gamma - \Gamma) \setminus \{0\}$ is symmetric in G relative to 0 and $\overline{\varphi_\delta} = \varphi_{-\delta}$ for any $\delta \in G$, the subspace $L = \operatorname{Span}(\{\varphi_\delta\}_{\delta \in \Delta_0}) = \operatorname{Span}(\{\varphi_\delta\}_{\delta \in \Delta \setminus \{0\}})$ (the identity is due to (o)) is closed under complex conjugation. Thus, by Lemma 1 there exists a real-valued function $\tilde{F} \in L^\infty(X)$, $\|\tilde{F} - F\|_\infty \le \rho/(1 - \sigma) = \tau$ such that $\tilde{F} \perp U_\delta \mathbf{1}$ for all $\delta \in \Delta_0$. So, $\tilde{F}$ is found and the lemma is proved. □

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As a matter of fact, in the proof of the theorem we will need the following generalization of Lemma 2, which deals with several actions of G simultaneously.

LEMMA 3. *Let $\mathcal{H}_0, \dots, \mathcal{H}_N$ be Hilbert spaces, and for each $i = 0, \dots, N$ let $\{U_g^{(i)}\}_{g\in G}$ be a unitary action of an Abelian group G on $\mathcal{H}_i$. For $i = 0, \dots, N$ denote by $\bar{G}_i$ the subgroup of G whose action on $\mathcal{H}_i$ is scalar. Let $\mathcal{C}$ be a class of subsets of G such that $\mathcal{C}$ satisfies the following requirement: for any set of vectors $h_i, F_i \in \mathcal{H}_i$, $i = 0, \dots, N$, and any $\rho > 0$, $\sigma > 0$ there exists a set $\Gamma \in \mathcal{C}$ such that for $\Delta = \Gamma - \Gamma$*

$$\Delta \cap \bar{G}_i = \{0\}, \quad i = 0, \dots, N, \tag{o'}$$

and for some $\Delta_i \subset G$, $\Delta_i \equiv \Delta \setminus \bar{G}_i \pmod{\bar{G}_i}$, $i = 0, \dots, N$ one has:

(i) $\sum_{\delta\in\Delta_i} |\langle U_\delta^{(i)} h_i, F_i\rangle| \le \rho$, $i = 0, \dots, N$;

(ii) *for every $\delta \in \Delta_i$, $\sum_{\substack{\delta'\in\Delta_i \\ \delta'\neq\delta}} |\langle U_{\delta'}^{(i)} h_i, U_\delta^{(i)} h_i\rangle| \le \sigma$, $i = 0, \dots, N$.*

Then for any set of vectors $f_i \in \mathcal{H}_i$, $i = 0, \dots, N$, and any $\varepsilon > 0$ there exist a set $\Gamma \in \mathcal{C}$ and vectors $\tilde{f}_i \in \mathcal{H}_i$ with $\|\tilde{f}_i - f_i\| \le \varepsilon$, $i = 0, \dots, N$, such that $U_{\gamma_1}^{(i)} \tilde{f}_i \perp U_{\gamma_2}^{(i)} \tilde{f}_i$ simultaneously for all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, and all $i = 0, \dots, N$.

Proof. The proof of this lemma is completely analogous to the proof of its special case, Lemma 2. Namely, we pass to realizations of $\mathcal{H}_0, \dots, \mathcal{H}_N$ as functional spaces, where $U_g^{(0)}, \dots, U_g^{(N)}$ are given by operators of multiplication. Then we slightly (in the Hilbert space norm) perturb $f_0, \dots, f_N$ to get $0 < c < |f_0(\cdot)|, \dots, |f_N(\cdot)| < C < \infty$, and put $F_i = |f_i|^2$, $i = 0, \dots, N$, $\tau = \min\{c^2, \varepsilon c^2/2C\}$ and $\rho = \tau/2$, $\sigma = \frac{1}{2}$. After that, due to the assumptions of the lemma, we find $\Gamma \in \mathcal{C}$ such that $\Delta = \Gamma - \Gamma$ satisfies (o') and some $\Delta_0, \dots, \Delta_N \subset G$, $\Delta_i \equiv \Delta \setminus \bar{G}_i \pmod{\bar{G}_i}$, $i = 0, \dots, N$, satisfy (i) and (ii) for $h_i \equiv \mathbf{1}$, $i = 0, \dots, N$. Thus, we can apply Lemma 1 several times to find $\tilde{F}_0, \dots, \tilde{F}_N$ such that $\|\tilde{F}_i - F_i\|_\infty \le \tau$ and $\tilde{F}_i \perp U_\delta^{(i)} \mathbf{1}$ for all $i = 0, \dots, N$ and all $\delta \in \Delta_i$. Then the functions $\tilde{f}_i = f_i \sqrt{\tilde{F}_i / F_i}$ will satisfy the conclusion of the lemma. □

Proof of theorem. We will proceed with the help of Lemma 3, after splitting $\mathcal{H}$ into the orthogonal sum of subspaces such that the G-action $g \mapsto U_{2\cdot g}$ is weakly mixing on one of them and scalar on the others. To do that, we need a well-known fact, which follows easily from the characterization of weak mixing in terms of (the absence of) eigenvectors. The orthocomplement of the subspace spanned by all the eigenvectors of the action is an action-invariant space on which the action has no eigenvectors, i.e. is weakly mixing. (We remark that this is the maximal invariant subspace on which the action is weakly mixing, as one can show using the spectral theorem.)

THEOREM. (Decomposition) *For any unitary action $\{W_g\}_{g\in G}$ of an Abelian group G on a Hilbert space $\mathcal{H}$ there exists an orthogonal decomposition $\mathcal{H} = \mathcal{H}_{\text{comp}} \oplus \mathcal{H}_{\text{wm}}$, where $\mathcal{H}_{\text{comp}}$ and $\mathcal{H}_{\text{wm}}$ are W_g-invariant, $W_g|_{\mathcal{H}_{\text{wm}}}$ is weakly mixing, and $W_g|_{\mathcal{H}_{\text{comp}}}$ is compact.*

Further on, $\mathcal{H}_{\text{comp}} = \bigoplus_{i\in I} \mathcal{H}_i$, where for any $i \in I$ and $g \in G$, $W_g|_{\mathcal{H}_i} = \lambda_i(g)\text{Id}_{\mathcal{H}_i}$ with $\lambda_i(g) \in \mathbb{C}$.

We use the decomposition of $\mathcal{H}$ with respect to the unitary action $W_g = U_{2g}$ of G on $\mathcal{H}$. Note that the spaces $\mathcal{H}_i$, $i \in I$, being eigenspaces of the action U_{2g}, are invariant under any

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action that commutes with U_{2g}, in particular, under U_g. Therefore, the orthocomplement $\mathcal{H}_{\mathrm{wm}}$ of the sum of all these spaces $\mathcal{H}_{\mathrm{comp}} = \bigoplus_{i\in I} \mathcal{H}_i$, is also U_g-invariant, and one can consider the restriction of the action U_g on $\mathcal{H}_{\mathrm{wm}}$ as well.

Since we are looking for a perturbation of f, we may assume without loss of generality that f can be represented as a *finite* sum $f = f_0 + \sum_{i=1}^{N} f_i$ with $f_i \in \mathcal{H}_i$ for $i = 0, \dots, N$, where U_{2g} is weakly mixing on $\mathcal{H}_0 := \mathcal{H}_{\mathrm{wm}}$, while $U_{2g}|_{\mathcal{H}_i} = \lambda_i(g)\mathrm{Id}_{\mathcal{H}_i}$ for all $g \in G$ and $i = 1, \dots, N$.

Now consider the action $\{U_g\}_{g\in G}$ on each of these $N+1$ subspaces. Let $U_g^{(i)} = U_g|_{\mathcal{H}_i}$, $g \in G, i = 0, \dots, N$. For each $i = 0, \dots, N$, let $\bar{G}_i$ be the subgroup of G that acts scalarly on $\mathcal{H}_i$, i.e. $\bar{G}_i = \{g \in G \mid U_g^{(i)} = \lambda(g)\mathrm{Id}_{\mathcal{H}_i}\}$. (By the definition of the subspaces $\mathcal{H}_i$, each of the subgroups $\bar{G}_1, \dots, \bar{G}_N$ contains the subgroup $2G = \{2g \mid g \in G\}$.)

Denote by $\mathcal{C}$ the class of IP-sets $\Gamma = \mathrm{FS}(\{g_j\}_{j=1}^{\infty})$ with $g_j \in A$, $j = 1, 2, \dots$, satisfying

$$\mathcal{F} \to \Gamma,\ \alpha \mapsto g_\alpha,\ \text{is injective.} \tag{5}$$

The main technical part of our proof is the verification of the requirements of Lemma 3 for this class $\mathcal{C}$. Namely, we will show that for any set of $2(N+1)$ vectors $F_i, h_i \in \mathcal{H}_i$, $i = 0, \dots, N$, and any $\rho > 0$, $\sigma > 0$ there exists $\Gamma = \mathrm{FS}(\{g_j\}_{j=1}^{\infty})$ with $g_j \in A$, $j = 1, 2, \dots$, satisfying (5), such that the conditions (o′), (i) and (ii) of the lemma hold for $\Delta = \Gamma - \Gamma$ and some $\Delta_i \subset G$, $\Delta_i \equiv \Delta \setminus \bar{G}_i \pmod{\bar{G}_i}$, $i = 0, \dots, N$. (Here we need (5) to proceed from the requirement $\gamma_1 \neq \gamma_2$ in the conclusion of Lemma 3 to the condition $\alpha \neq \beta$ in the conclusion of the theorem.) That will end the proof.

Fix any $\rho > 0, \sigma > 0$. Suppose the desired set $\Gamma = \mathrm{FS}(\{g_j\}_{j=1}^{\infty})$ has already been built. Then

$$\Delta = \Gamma - \Gamma = \left\{ \sum_{j=1}^{l} c_j g_j \,\middle|\, l \in \mathbb{N}, c_j = 0, \pm 1 \text{ for } 1 \le j \le l \right\}.$$

For such Δ, put $\Delta_0 = \Delta \setminus \{0\}$ and

$$\Delta_1 = \cdots = \Delta_N = \left\{ \sum_{j=1}^{l} c_j g_j \,\middle|\, l \in \mathbb{N}, c_j = 0, 1 \text{ for } 1 \le j \le l \right\} \setminus \{0\},$$

then $\Delta_i \equiv \Delta \setminus \{0\} \equiv \Delta \setminus \bar{G}_i \pmod{\bar{G}_i}$ for every $i = 0, \dots, N$ as demanded (the first congruence for $i = 1, \dots, N$ is due to the fact that $\bar{G}_1, \dots, \bar{G}_N$ contain $2G$, and the second one for any i is due to the condition (o′) to be satisfied by Δ). Let

$$R = \left\{ \sum_{j=1}^{l} c_j g_j \,\middle|\, l \in \mathbb{N},\ c_j = 0, \pm 1 \text{ for } 1 \le j \le l, \sum_{j=1}^{l} c_j^2 \neq 0 \right\},$$

and

$$S = \left\{ \sum_{j=1}^{l} c_j g_j \,\middle|\, l \in \mathbb{N},\ c_j = 0, \pm 1, \pm 2 \text{ for } 1 \le j \le l, \sum_{j=1}^{l} c_j^2 \neq 0 \right\}.$$

Then $\Delta_i \subset R$ for all i, and $(\Delta_i - \Delta_i)\setminus\{0\} \subset R$ for $i = 1, \dots, N$, while $(\Delta_0 - \Delta_0)\setminus\{0\} \subset S$. Thus, condition (i) of Lemma 3 is implied by

$$\sum_{\alpha\in R} |\langle U_\alpha^{(i)} h_i, F_i\rangle| \le \rho, \quad i = 0, \dots, N, \tag{6}$$

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and, since $\langle U_{\delta'}h, U_\delta h\rangle = \langle U_{\delta'-\delta}h, h\rangle$ and $((\delta'-\delta=\delta''-\delta)\Rightarrow(\delta'=\delta''))$, condition (ii) is implied by

$$\sum_{\alpha\in S}|\langle U_\alpha^{(0)}h_0, h_0\rangle|\le\sigma \quad\text{and}\quad \sum_{\alpha\in R}|\langle U_\alpha^{(i)}h_i, h_i\rangle|\le\sigma,\quad i=1,\dots,N. \tag{7}$$

Let us present R as the union of

$$R_l=\left\{\sum_{j=1}^{l}c_jg_j\;\middle|\;c_j=0,\pm1 \text{ for } 1\le j\le l \text{ and } c_l\ne0\right\},\quad l=1,2,\dots,$$

and S as the union of

$$S_l=\left\{\sum_{j=1}^{l}c_jg_j\;\middle|\;c_j=0,\pm1,\pm2 \text{ for } 1\le j\le l \text{ and } c_l\ne0\right\},\quad l=1,2,\dots.$$

Then (6) and (7) are implied by

$$\sum_{\alpha\in R_l}|\langle U_\alpha^{(i)}h_i, F_i\rangle|\le\frac{\rho}{2^l},\quad i=0,\dots,N,\ l=1,2,\dots, \tag{8}$$

and

$$\sum_{\alpha\in S_l}|\langle U_\alpha^{(0)}h_0, h_0\rangle|\le\frac{\sigma}{2^l},\quad \sum_{\alpha\in R_l}|\langle U_\alpha^{(i)}h_i, h_i\rangle|\le\frac{\sigma}{2^l},\quad i=1,\dots,N,\ l=1,2,\dots. \tag{9}$$

To satisfy the condition (o′) of Lemma 3, we add the following requirement:

$$R_l\cap\bar G_i=\emptyset \quad\text{for all } i=0,\dots,N \text{ and } l=1,2,\dots. \tag{10}$$

We will build the sequence $\{g_j\}_{j=1}^{\infty}$ of the elements of A inductively, in such a way that the conditions (8), (9) and (10) are satisfied (in the meantime, we do not worry about (5)). For $l=1$ these conditions are implied by

$$\begin{gathered}|\langle U_{\lambda g_1}^{(i)}h_i, F_i\rangle|\le\rho/4 \quad\text{for } \lambda=\pm1, i=0,\dots,N,\\ |\langle U_{\lambda g_1}^{(i)}h_i, h_i\rangle|\le\sigma/8 \quad\text{for } \lambda=\pm1, i=0,\dots,N,\\ |\langle U_{\mu g_1}^{(0)}h_0, h_0\rangle|\le\sigma/8 \quad\text{for } \mu=\pm2,\\ \text{and, finally, } g_1\notin\bigcup_{0\le i\le N}\bar G_i.\end{gathered} \tag{11}$$

Since the action $g\mapsto U_g$ of G is weakly mixing on $\mathcal{H}$, all actions $g\mapsto U_g^{(i)}$ and $g\mapsto U_{-g}^{(i)}$, $i=0,\dots,N$, are weakly mixing on corresponding $\mathcal{H}_i$. By the definition of $\mathcal{H}_0$, the actions $g\mapsto U_{2g}^{(0)}$ and $g\mapsto U_{-2g}^{(0)}$ are also weakly mixing on $\mathcal{H}_0$. In addition, each subgroup $\bar G_i$, $i=0,\dots,N$, has infinite index in G (otherwise the action $U_g^{(i)}$ on $\mathcal{H}_i$ would be compact), thus it is a subset of zero density in G (that follows from the finite additivity of the functional $\bar d$ on any family of disjoint shifts of a given subset of G, and from its boundedness by 1). Therefore, $\bigcup_{i=0}^{N}\bar G_i$ is also a subset of zero density in G. It follows that the set of elements $g_1\in G$ satisfying (11) is the intersection of a finite number of sets

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of density 1, and so has density 1 as well. Therefore, this set has non-empty intersection with the set A of positive upper density, and we can pick g_1 from this intersection.

Now let $l \geq 2$, and assume that $g_1, \ldots, g_{l-1} \in A$ satisfying (8), (9) and (10) have already been chosen. Every $\alpha \in R_l$ has the form $\alpha = \pm g_l + \alpha'$ with $\alpha' \in \tilde{R}_{l-1} := \bigcup_{j=1}^{l-1} R_j \cup \{0\}$, and every $\alpha \in S_l$ has the form $\alpha = \pm g_l + \alpha'$ or $\alpha = \pm 2g_l + \alpha'$ with $\alpha' \in \tilde{S}_{l-1} := \bigcup_{j=1}^{l-1} S_j \cup \{0\}$. Thus, for a fixed $l \geq 2$, the conditions (8), (9) and (10) are implied by

$$\sum_{\substack{\alpha' \in \tilde{R}_{l-1} \\ \lambda = \pm 1}} |\langle U^{(i)}_{\lambda g_l + \alpha'} h_i, F_i \rangle| \leq \frac{\rho}{2^l}, \qquad \sum_{\substack{\alpha' \in \tilde{R}_{l-1} \\ \lambda = \pm 1}} |\langle U^{(i)}_{\lambda g_l + \alpha'} h_i, h_i \rangle| \leq \frac{\sigma}{2^{l+1}}, \quad i = 0, \ldots, N,$$

$$\sum_{\substack{\alpha' \in \tilde{S}_{l-1} \\ \mu = \pm 2}} |\langle U^{(0)}_{\mu g_l + \alpha'} h_0, h_0 \rangle| \leq \frac{\sigma}{2^{l+1}},$$

$$\text{and} \quad g_l + \alpha' \notin \bigcup_{0 \leq i \leq N} \bar{G}_i, \quad \alpha' \in \tilde{R}_{l-1}. \tag{12}$$

Since the sets $\tilde{R}_l$ and $\tilde{S}_l$, $l = 1, 2, \ldots$, are finite, the first three conditions in (12) are implied by (all the actions are unitary)

$$|\langle U^{(i)}_{\lambda g_l + \alpha'} h_i, F_i \rangle| = |\langle U^{(i)}_{\lambda g_l} h_i, U^{(i)}_{-\alpha'} F_i \rangle|$$
$$\leq \frac{\rho}{2^{l+1} |\tilde{R}_{l-1}|} \quad \text{for any } \lambda = \pm 1, 0 \leq i \leq N \text{ and } \alpha' \in \tilde{R}_{l-1}, \tag{13}$$

and two other similar conditions, while the last condition in (12) can be rewritten in the form

$$g_l \notin \Big(\bigcup_{\substack{0 \leq i \leq N \\ \alpha' \in \tilde{R}_{l-1}}} (\bar{G}_i - \alpha') \Big). \tag{14}$$

Again, the set of elements g_l of G that satisfy all these requirements is the intersection of finitely many subsets of G of density 1. Therefore, it has non-empty intersection with A, and that allows one to pick g_l.

Up to now, we have shown how to build a sequence $\{g_j\}_{j=1}^{\infty}$ of the elements of A that satisfies (8), (9) and (10). It is left to ensure (5). For that end, we only need to demand that $g_1 \neq 0$, and $g_l \notin \mathrm{FS}(\{\pm g_1, \ldots, \pm g_{l-1}\})$ for any $l > 1$. Clearly, this can be achieved as well.

For the proof of the last statement of the theorem, note that to get $\mathrm{FS}(\{g_j\}_{j=1}^{\infty}) \subset A$ we need to satisfy the additional requirement $g_l \in \bigcap_{\alpha \subset \{1, \ldots, l-1\}} (A - g_\alpha)$ (where $g_\emptyset := 0$). The assumption $\bar{d}(A) = 1$ implies that this intersection is again of upper density 1. Therefore, we can pick g_l from the finite intersection of sets, where all of the sets but one have density 1 and the remaining set has upper density 1. □

Remark. In fact, in the case of a weakly mixing $\mathbb{Z}$- or $\mathbb{Z}^d$-action (or, more generally, any action $g \mapsto U_g$ that is weakly mixing together with the action $g \mapsto U_{2g}$) one may impose, as is done in [**BKM1**], an additional constraint on the IP-set we are searching for—the condition of symmetry $\Gamma = -\Gamma$. In terms of sequences, one can demand in the

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theorem that, say, $-g_{2k} = g_{2k-1} \in A$ for any $k \in \mathbb{N}$. The necessary changes in the proof are clear. (The condition on the sequence $\{g_k\}$ is included then in the definition of the class $\mathcal{C}$ involving (5). The set $\Delta = \Gamma - \Gamma$ will contain linear combinations of the elements of the sequence $\{g_k\}$ with the coefficients not exceeding 2 in their absolute value. The decomposition lemma is applied to the actions $g \mapsto U_{3g}$, $g \mapsto U_{4g}$ to split $\mathcal{H}$ into subspaces such that on each of them each of the actions $g \mapsto U_{3g}$ and $g \mapsto U_{4g}$ is either weakly mixing or scalar. The sets Δ_i are built according to the case: $c_j = 0, 1, 2$ if $g \mapsto U_{3g}$ is scalar and $c_j = 0, 1, 2, 3$ if $g \mapsto U_{4g}$ is scalar.)

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