Ergod. Th. & Dynam. Sys. (2006), **26**, 341–351
doi:10.1017/S0143385705000465
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Printed in the United Kingdom

Minimal non-ergodic directions on genus-2 translation surfaces

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(*Received* 14 *February* 2005 *and accepted in revised form* 7 *July* 2005)

Abstract. It is well known that on any Veech surface, the dynamics in any minimal direction is uniquely ergodic. In this paper it is shown that for any genus-2 translation surface which is not a Veech surface there are uncountably many minimal but not uniquely ergodic directions. The theorem can be applied to certain billiard tables.

1. *Introduction*

Suppose (X, ω) is a translation surface where the genus of X is at least 2. This means that X is a Riemann surface, and ω is a holomorphic 1-form on X. For each $\theta \in [0, 2\pi)$ there is a vector field defined on the complement of the zeros of ω such that $\arg \omega = \theta$ along this vector field. The corresponding flow lines are denoted ϕ_θ. For a countable set of θ there is a flow line of ϕ_θ joining a pair of zeros of ω. These flow lines are called *saddle connections*. For any θ such that there is no saddle connection in direction θ, the flow is minimal.

Veech [**13**] found examples of certain skew rotations over a circle which are minimal, but not uniquely ergodic. Namely, the orbits are dense but not uniformly distributed. Veech's examples can be interpreted [**12**] in terms of flows on (X, ω) where X has genus 2 and ω has a pair of simple zeros.

Take two copies of the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ and mark off a segment along the vertical axis from $(0, 0)$ to $(0, \alpha)$, where $0 < \alpha < 1$. Cut each torus along the segment and glue pairwise along the slits. The resulting surface (X_α, ω) is the connected sum of the pair of tori and is a branched double cover over the standard torus, branched over $(0, 0)$ and $(0, \alpha)$. These two endpoints of the slits become the zeros of order one of ω. There are a pair of

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circles on X_α such that the first return map of ϕ_θ to these circles gives a skew rotation over the circle. If α is irrational, then there are directions θ such that the flow ϕ_θ is minimal but not uniquely ergodic [**12**]. These reproduce the original Veech examples.

Additional results about these examples are known. Cheung [**2**] has shown that if α satisfies a Diophantine condition of the form that there exists $c > 0, s > 0$ such that $|\alpha - p/q| < c/q^s$ has no rational solutions p/q, then the Hausdorff dimension of the set of $\theta \in [0, 2\pi)$ such that ϕ_θ is not ergodic is exactly $\frac{1}{2}$. On the other hand Boshernitzan showed (in an appendix to the above paper) that there is a residual set of Liouville numbers α such that this set of θ has Hausdorff dimension 0. The dimension $\frac{1}{2}$ in the Cheung result is sharp, for it was shown [**10**] that for any (X, ω) (in any genus) this set of θ has Hausdorff dimension at most $\frac{1}{2}$.

Now in the slit torus case, if α is rational, then minimality implies unique ergodicity. This is part of a more general phenomenon called *Veech dichotomy*. There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on the moduli space of translation surfaces. A *Veech surface* is one whose stabilizer $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. These surfaces have the property that for any direction θ, either the flow ϕ_θ is periodic or it is minimal and uniquely ergodic [**14**].

This raises the question of whether every surface (X, ω) that is not a Veech surface has a minimal but non-uniquely ergodic direction. In [**11**] it was shown that for every component of every moduli space of (X, ω) (other than a finite number of exceptional ones), there exists $\delta > 0$ such that for almost every (X, ω) in that component (with respect to the natural Lebesgue measure class), the Hausdorff dimension of the set of θ such that ϕ_θ is minimal but not ergodic is δ. That theorem does not, however, answer the question for every surface.

In this paper we establish the following converse to Veech dichotomy in genus 2. Let $\mathcal{H}(1, 1)$ be the moduli space of translation surfaces in genus 2 with two simple zeros and $\mathcal{H}(2)$ the moduli space of translation surfaces with a single zero of order 2.

THEOREM 1.1. *For* any *surface* $(X, \omega) \in \mathcal{H}(1, 1)$ *or* $\mathcal{H}(2)$ *which is not a Veech surface, there are an uncountable number of directions* θ *such that the flow* ϕ_θ *is minimal but not uniquely ergodic.*

The theorem strengthens [**8**, Theorem 1.5]. There it was shown that for any surface of genus 2 which is not a Veech surface, there is a direction θ such that the flow ϕ_θ is not uniquely ergodic, and not all leaves are closed. We also remark that in genus 2, primitive Veech surfaces have been classified [**7**] and [**9**] (see also [**1**]). We also remark that the converse to Veech dichotomy is false in higher genus. B. Weiss (oral communication) using a construction of Hubert and Schmidt [**4**] has provided an example in genus 5 which is not a Veech surface and yet for which the Veech dichotomy holds. The surface is a branched cover over a Veech surface in $\mathcal{H}(2)$.

We also note that the theorem applies to show that certain billiard tables admit minimal non-uniquely ergodic directions. By a well-known procedure billiards in a polygon P with vertex angles rational multiples of π give rise to flows ϕ_θ on a surface (X, ω). If P is an L-shaped table then $(X, \omega) \in \mathcal{H}(2)$ and it is known precisely which L-shaped tables give rise to Veech surfaces [**5**]. Thus our theorem applies to show that any such table not on the list has a minimal non-uniquely ergodic direction.

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Another family of examples arise from billiards in a unit square with a barrier of length $\alpha < 1$ with one endpoint along a side at distance β from a corner. The barrier extends into the square perpendicular to the side. The resulting (X, ω) belongs to $\mathcal{H}(1, 1)$. If $\beta = 1/2$ then (X, ω) is the slit torus described earlier. In general, the surface is Veech if and only if both α and β are rational. Thus our theorem applies to all other possibilities.

2. *Splittings in $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$*

We begin by generalizing the construction of slit tori discussed in the introduction. Suppose T_1, T_2 are a pair of flat tori defined by lattices L_1, L_2. Let l_1, l_2 be simple segments on each, determining the same holonomy vector $w \in \mathbb{C}$. Cut each T_i along l_i and glue the resulting tori together along the cuts. The resulting surface is a connected sum of the pair of tori and belongs to the stratum $\mathcal{H}(1, 1)$. The endpoints of the slits become simple zeros. If (X, ω) is biholomorphically equivalent to this surface, we say (L_1, L_2, w) is a *splitting* of (X, ω).

Conversely, any surface in $\mathcal{H}(1, 1)$ can be constructed in this way for infinitely many possible triples (L_1, L_2, w) [**6**].

We can construct surfaces in $\mathcal{H}(2)$ in a similar fashion. Given a lattice L_1 we may cut along a simple segment with holonomy w as above, then identify opposite ends of the slit. This forms a torus with two boundary circles attached at a point. Glue in a cylinder, attaching a boundary component to each of the boundary circles. The holonomy of the boundary circles is w. Every surface in $\mathcal{H}(2)$ is found by such a construction [**6**]. We again refer to (L_1, L_2, w) as a splitting of (X, ω). In this case, we can think of the glued cylinder as a torus T_2 cut along a simple closed curve with holonomy w. We refer to this cylinder as a *degenerate* torus.

The result we use to construct non-uniquely ergodic directions is given in the following theorem [**11, 12**].

THEOREM 2.1. *Let (L_1^n, L_2^n, w^n) be a sequence of splittings of (X, ω) into tori T_1^n, T_2^n and assume the directions of the vectors w^n converge to some direction θ. Let $h_n > 0$ be the component of w^n in the direction perpendicular to θ and $a_n = \text{area}(T_1^n \Delta T_1^{n+1})$, the area of the regions exchanged between consecutive splittings. If*

(1) $\sum_{n=1}^{\infty} a_n < \infty$,

(2) *there exists $c > 0$ such that* $\text{area}(T_1^n) > c$, $\text{area}(T_2^n) > c$ *for all n, and*

(3) $\lim_{n\to\infty} h_n = 0$,

then θ is a non-ergodic direction.

The idea behind the proof of Theorem 1.1 is to construct uncountably many sequences of splittings satisfying the summability condition above with distinct limiting directions θ. Since there are only countably many non-minimal directions, there must be uncountably many limiting directions which are minimal but not ergodic.

In §3 we find conditions to generate new splittings out of old and we find a useful estimate for the change in area of the corresponding tori. Now if a surface (X, ω) is a Veech surface, then for any splitting (L_1, L_2, w) of (X, ω), the vector w is rational in each lattice L_i. In §4 we show that if a surface is not Veech, there is some splitting (L_1, L_2, w) such that the vector w is irrational in one of the lattices L_i. We use this splitting to begin the inductive process of finding sequences of splittings that satisfy the

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hypotheses of Theorem 2.1. The irrationality is used to make the change in area small and to ensure the inductive process can be continued. The proof of this in the case of $\mathcal{H}(2)$ is essentially the same although easier than in the case of $\mathcal{H}(1, 1)$. We will focus on the latter and point out the differences from the former as they occur.

3. *Building new splittings by twists*

We build new splittings out of old splittings by a Dehn twist operation. The construction works in exactly the same way for the two moduli spaces.

First let us adopt the notation that for vectors $v, v' \in \mathbb{R}^2$ given in rectangular coordinates by $v = \langle x_1, y_1 \rangle$ and $v' = \langle x_2, y_2 \rangle$ their cross product is

$$v \times v' := x_1 y_2 - x_2 y_1 \in \mathbb{R}.$$

Let (L_1, L_2, w) be a splitting of (X, ω). We may think of each slit torus T_i as a closed subsurface (with boundary) in X separated by a pair of saddle connections $\alpha_\pm$ given by

$$\partial T_1 = \alpha_+ - \alpha_-, \quad \text{hol}(\alpha_+) = w = \text{hol}(\alpha_-).$$

We would like to have an explicit construction of a class of simple closed curves having non-zero (geometric) intersection with $\alpha_+ \cup \alpha_-$. Let C_i be a cylinder contained in T_i that is disjoint from the line segment ℓ_i. (In the case where T_2 is degenerate we allow C_2 to consist of a single closed curve.) Furthermore, assume that the holonomy v_i of the core curve γ_i is not parallel to w. Each γ_i has a unique translate crossing the midpoint of ℓ_i. As closed curves joining the boundary of T_i in X to itself, these translates can be concatenated to form a simple closed curve γ whose holonomy is $v_1 + v_2$. Note that for γ to be well defined the curves γ_1 and γ_2 must have compatible orientations, i.e.

$$(v_1 \times w)(v_2 \times w) > 0. \tag{1}$$

Also, since $T_i \setminus C_i$ is a parallelogram with sides given by the vectors v_i and w, we have

$$|v_i \times w| \leq A_i = \text{area}(T_i) \quad \text{for } i = 1, 2. \tag{2}$$

Conversely, suppose $v_i \in L_i, i = 1, 2$ are primitive vectors for which (1) and (2) hold. The latter condition guarantees that v_i is the holonomy of a simple closed curve γ_i on T_i which can be realized by a saddle connection joining the initial point of the slit to itself, and another (provided T_i is not degenerate) joining the terminal point to itself. Neither of these intersect the interior of the slit. The pair of saddle connections bound a closed cylinder C_i in T_i of curves which do not cross the slit, where we allow C_2 to be a single closed curve if T_2 is degenerate. The complement of (the union of) these cylinders in X is an open annulus which, by (1), has a core curve γ with holonomy $v_1 + v_2$. Thus, any pair of primitive vectors $(v_1, v_2) \in L_1 \times L_2$ that satisfies (1) and (2) uniquely determines a unique simple closed curve $\gamma = \gamma(v_1, v_2)$.

For the purpose of geometric intuition, we shall give a normal form representation of the surface (X, ω) as an even-sided polygon in the plane with pairs of parallel sides identified. Each T_i decomposes into C_i and a parallelogram R_i, whose sides are given by the vectors v_i and w. Using the action of $\text{SL}_2(\mathbb{R})$ we may represent R_1 as a rectangle in the plane with

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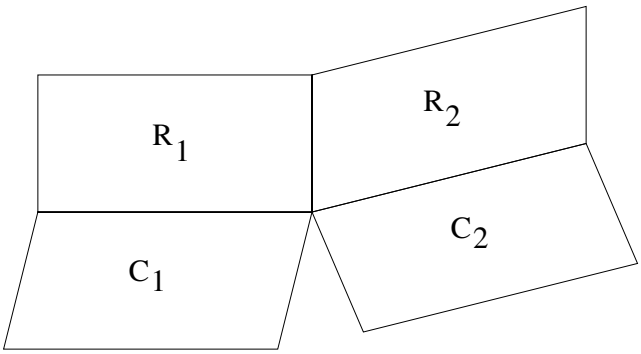


FIGURE 1. Normal form for (X, ω).

sides parallel to the coordinate axes. The parallelogram R_2 may be placed adjacent to the right edge of R_1 (i.e. the slit α_+), while a suitable choice of parallelograms representing the cylinders C_1 and C_2 may be placed adjacent to the lower edges of R_1 and R_2, respectively, so that there is no overlap. The boundary identifications are as follows. The right edge of R_2 is identified with the left edge of R_1 to form the slit α_- homologous to α_+. For each $i = 1, 2$, the upper edge of R_i is identified with the lower edge of C_i, while the left edge of C_i is identified with its right edge. In the case that (X, ω) has a double zero the cylinder forming the degenerate torus T_2 is represented as a parallelogram R_2 with upper and lower edges identified. (Recall that in this case C_2 is a closed curve and not a cylinder.) In both cases, the open annulus that forms the interior of $R_1 \cup R_2$ has γ as its core curve.

Now consider the curves $\beta_\pm^k$ obtained by twisting the slits $\alpha_\pm$ (relative to their endpoints) k times about γ in the positive sense, i.e. right-twist if $k > 0$, and left-twist if $k < 0$. In general, the geodesic representative of a twisted curve is a finite sequence of saddle connections. Let w^k be the common holonomy vector of $\beta_\pm^k$. Note that if $v_1 \times w > 0$, i.e. γ is *positively oriented* with respect to $\alpha_\pm$, then $w^k = w + k(v_1 + v_2)$.

LEMMA 3.1. *The twisted curves β_+^k and β_-^k are simultaneously realized by a (single) saddle connection if and only if w and w^k lie on the same side of v_1 and v_2. Thus, if γ is positively oriented with respect to $\alpha_\pm$, then the condition is*

$$v_1 \times w^k > 0 \quad \textit{and} \quad v_2 \times w^k > 0$$

while both inequalities are reversed in the case where γ is negatively oriented with respect to $\alpha_\pm$.

Proof. First, recall that the Dehn twist operation depends only on the orientation of the surface and not that of the curves $\alpha_\pm$ and γ. In particular, we may assume the orientation of γ is chosen positive with respect to $\alpha_\pm$. Since the hyperelliptic involution interchanges α_+ and α_- while fixing γ, β_+^k is realized by a saddle connection if and only if β_-^k is. If $v_1 \times v_2 \geq 0$ we show β_-^k is realized. Otherwise we consider β_+^k. Without loss of generality then assume $v_1 \times v_2 \geq 0$. Since the action of $\mathrm{SL}_2(\mathbb{R})$ preserves cross products, it is enough to verify that for *some* $g \in \mathrm{SL}_2(\mathbb{R})$, the vector gw^k is the holonomy of a curve

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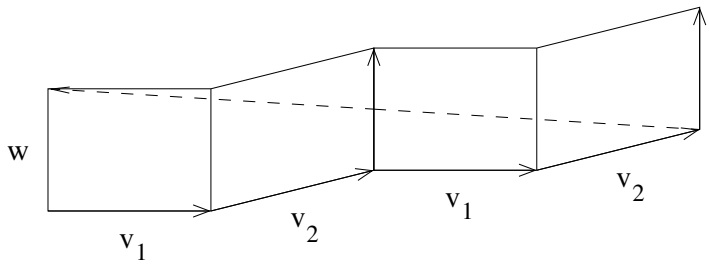


FIGURE 2. β_-^{-2} as a single saddle connection.

realized by a saddle connection on the flat surface $g \cdot (X, \omega)$. Consider first the case of a left twist $k < 0$. Choose g so that v_1 is horizontal and w is vertical, pointing upwards as in Figure 2. One sees easily that β_-^k is realized by a saddle connection if and only if $-k$ times the vertical component of v_2 is less than $|w|$, which is equivalent to $v_1 \times w^k > 0$. On the other hand, $k < 0$ implies $v_2 \times w^k = v_2 \times w + k v_2 \times v_1 > 0$.

The case of a right twist is similar. The condition $k > 0$ implies $v_1 \times w^k > 0$. By choosing g so that v_2 is horizontal and w is vertical we see that $v_2 \times w^k > 0$ is equivalent to the condition that β_-^k is realized by a saddle connection. □

Definition 3.2. Given a splitting (L_1, L_2, w) and a pair of primitive vectors $(v_1, v_2) \in L_1 \times L_2$ satisfying (1) and (2) we say v_2 is a *good partner* for v_1 with n twists if

$$|v_1 \times v_2| < \frac{1}{n} \max(|v_1 \times w|, |v_2 \times w|). \tag{3}$$

Note that if v_2 is a good partner for v_1 with n twists, then Lemma 3.1 implies there are n splittings (L_l^k, L_2^k, w^k) of (X, ω) with $w^k = w + k(v_1 + v_2)$, where $k = 1, \ldots, n$ or $k = -1, \ldots, -n$.

Next, we estimate the total area of the regions exchanged between the new and old splittings.

LEMMA 3.3. *Let (L_1', L_2', w') denote the new splitting of (X, ω) into tori T_1', T_2', determined by a particular value of k, $|k| \le n$. Then $T_1 \Delta T_1' = T_2 \Delta T_2'$ and*

$$\text{area}(T_1 \Delta T_1') \le |v_1 \times w| + |v_1 \times w'| \le 2|v_1 \times w| + n|v_1 \times v_2|. \tag{4}$$

Proof. The first statement is clear. For the second we note that the torus T_1' contains C_1 and T_2' contains C_2. Since $C_1 \subset T_1 \cap T_1'$, it follows that $T_1 \Delta T_1' \subset (T_1 \setminus C_1) \cup (T_1' \setminus C_1)$. By construction, $\text{area}(T_1 \setminus C_1) = |v_1 \times w|$ and $\text{area}(T_1' \setminus C_1) = |v_1 \times w'|$. This gives the desired inequality. □

4. *Irrational splittings*

To construct new splittings with small exchange of area, we need the notion of an irrational splitting.

Definition 4.1. We say the splitting (L_1, L_2, w) is *rational* in L_i if the vector w is a scalar multiple of some element in L_i; otherwise, it is *irrational* in L_i. The splitting is *irrational* if it is irrational in either L_1 or L_2.

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Note that if (X, ω) has a splitting that is rational in L_i then the flow in the direction of w is periodic in the torus T_i.

In Proposition 4.6 we will assume that the splitting is irrational in at least one of the two lattices. (In the case of (X, ω) having a double zero, we will assume that the splitting is irrational in L_1.) We will then find a new splitting so that the right-hand side of (4) is small. To continue the process of finding additional splittings we will need to know that the new splitting is still irrational. That fact will be accomplished by the next lemma.

LEMMA 4.2. *Let (L_1, L_2, w) be a splitting that is irrational in L_1 and suppose $v_1 \in L_1$ has a good partner $v_2 \in L_2$ with three twists. Then at least one of the three twists gives a splitting (L_1', L_2', w') that is irrational in L_1'.*

Proof. Let γ_0 be a simple segment in C_1 that concatenates with α_- to form a simple closed curve in T_1. Then the lattice L_1 is generated by $v_0 + w$ and v_1 where v_0 is the holonomy of γ_0. Observe that (L_1, L_2, w) is rational in L_1 if and only if w is a scalar multiple of a vector in the lattice L_0 generated by v_0 and v_1. Indeed, $w = c(av_0 + bv_1)$ for some $a, b \in \mathbb{Z}$ and $c > 0$ if and only if $(1+ac)w = c(a(v_0+w)+bv_1)$ where $1+ac \neq 0$ since $v_0 + w$ and v_1 are linearly independent. Now, since T_1 and T_1' share the same cylinder C_1, it follows that the splitting (L_1', L_2', w') is rational in L_1' if and only if $w' \in L_0$. Thus, if (L_1', L_2', w') with $w' = w+k(v_1+v_2)$ is rational in L_1' for $k = 1, 2, 3$ or $k = -1, -2, -3$, then the three vectors w' are parallel to elements in L_0. Let L_0' be the lattice generated by w and $v_1 + v_2$. Since L_0 and L_0' are lattices in $\mathbb{R}^2$ whose vectors share three non-parallel directions, they are isogenous (cf. proof of [**8**, Theorem 7.3]), i.e. $L_0' \subset \lambda L_0$ for some $\lambda > 0$. This implies that in fact they share all possible directions and so the splitting (L_1, L_2, w) is rational in L_1. □

The next proposition will help us find some initial splitting which is irrational so that the inductive process can begin.

PROPOSITION 4.3. *Let (L_1, L_2, w) be a splitting of (X, ω), rational in both L_1 and L_2. Suppose that there is a $v_1 \in L_1$ which has two good partners $v_2, v_2' \in L_2$ with at least two twists each and such that $v_2 - v_2'$ is not parallel to w. Then, if (X, ω) is a not Veech surface, at least one of the four twists gives a splitting (L_1', L_2', w') that is irrational in L_1'.*

Proof. Since the property of irrationality is invariant under the $\mathrm{SL}_2(\mathbb{R})$ action and scaling, we may normalize so that $v_1 = (1, 0)$ and $w = (0, 1)$. In terms of the normal form, the parallelogram R_1 is a unit square with lower right corner at the origin. Let v_0 be the holonomy of γ_0 as in the proof of the previous lemma. The rationality of (L_1, L_2, w) in L_1 and L_2 is equivalent to

$$v_0 \times w \in \mathbb{Q}, \quad \frac{v_2 \times w}{v_2' \times w} \in \mathbb{Q}. \tag{5}$$

Let $\theta := v_0 \times v_1$. Note that $-\theta$ is the height of the cylinder C_1; in particular, $\theta \neq 0$. The slope of the vector $w^k = w + k(v_1 + v_2)$ is given by

$$\sigma_k = \frac{-v_1 \times w^k}{w \times w^k} = \frac{1 + k(v_1 \times v_2)}{k(1 + v_2 \times w)}. \tag{6}$$

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The first return map to the base of R_1 under the flow in direction w^k is a rotation with rotation number ρ_k. Note that a point starting at the base of R_1 returns to a point at the top of R_1 with the same x-coordinate. The identification of the top of R_1 with the base of C_1 shifts the x-coordinate by $-v_0 \times w$. Provided that $\sigma_k \neq 0$, the total shift in the x-coordinate for the first return to the base of R_1 is given by

$$\rho_k = -\frac{\theta}{\sigma_k} - v_0 \times w. \tag{7}$$

Since v_2 is a good partner of v_1 with two twists, we have $\rho_k \in \mathbb{Q}$ for $\pm k = 1, 2$. Thus, together with the first part of (5) we have

$$\sigma_k \neq 0 \Rightarrow \frac{\theta(1 + v_2 \times w)}{1 + k(v_1 \times v_2)} \in \mathbb{Q}. \tag{8}$$

If $v_1 \times v_2 \neq 0$ and $\sigma_k \neq 0$ for both values of k then taking ratios in (8) we see that

$$v_1 \times v_2 \in \mathbb{Q}, \quad \theta(1 + v_2 \times w) \in \mathbb{Q}. \tag{9}$$

On the other hand, if either $v_1 \times v_2 = 0$ or $\sigma_k = 0$ then $v_1 \times v_2 \in \mathbb{Q}$, and using (8) for the other value of k we see that (9) still holds. The same argument can be applied to v_2' to yield

$$v_1 \times v_2' \in \mathbb{Q}, \quad \theta(1 + v_2' \times w) \in \mathbb{Q} \tag{10}$$

so that taking ratios again, we have

$$\frac{1 + v_2 \times w}{1 + v_2' \times w} \in \mathbb{Q}. \tag{11}$$

Since $v_2' - v_2$ is not parallel to w, the above, together with the second part of (5), implies both

$$v_2 \times w \quad \text{and} \quad v_2' \times w \in \mathbb{Q}. \tag{12}$$

Hence, $\theta \in \mathbb{Q}$, which together with (12) and the first parts of (5), (9), and (10) implies (X, ω) is a branched cover of the standard torus, branched over rational points. By [**3**], such a surface is Veech. □

The property that a surface (X, ω) admits a splitting (L_1, L_2, w) together with vectors v_1, v_2, v_2' satisfying the requirements in Proposition 4.3 defines an open subset in $\mathcal{H}(1, 1)$. (It is not hard to show that this property does not hold for *any* surface in $\mathcal{H}(2)$.) We show that this set is non-empty by constructing an explicit example. Let $v_1 = (1, 0)$, $w = (0, 1)$, $v_2 = (3, -1)$, $v_2' = (4, 1)$, $L_1 = \mathbb{Z} \times 2\mathbb{Z}$ and L_2 be the lattice generated by v_2 and v_2'. Then, $A_2 = 7$ and the fact that both v_2 and v_2' have slopes strictly between $\pm 1/2$ implies each has two twists.

Using results of McMullen we now obtain the following.

COROLLARY 4.4. *Suppose (X, ω) is not a Veech surface. Then it admits an irrational splitting.*

Proof. Case 1: (X, ω) is an eigenform for real multiplication. (We refer the reader to [**6**] for the definition of an *eigenform form for real multiplication*.) The corollary is given by [**8**, Theorem 7.5].

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Case 2: $(X, \omega) \in \mathcal{H}(2)$. Note that every cylinder determines a splitting. If every splitting is rational, then (X, ω) has the property that every cylinder belongs to a cylinder decomposition; by [**8**, Theorem 7.3] this means (X, ω) is an eigenform for real multiplication and in $\mathcal{H}(2)$ this implies that the surface is a Veech surface.

Case 3: $(X, \omega) \in \mathcal{H}(1, 1)$ is not an eigenform for real multiplication. By [**6**] the $\mathrm{SL}_2(\mathbb{R})$-orbit comes arbitrarily close (up to scale) to the example described after Proposition 4.3, which now implies (X, ω) admits an irrational splitting. □

Remark 4.5. Corollary 4.4 shows that the hypothesis in [**8**, Theorem 7.5] is unnecessary.

Theorem 1.1 is essentially a consequence of the preceding corollary and the next proposition, followed by an application of Theorem 2.1.

PROPOSITION 4.6. *If (L_1, L_2, w) is an irrational splitting of (X, ω) into tori T_1, T_2, then for any $\varepsilon > 0$ there exists a* new *irrational splitting (L_1', L_2', w') into tori T_1', T_2' such that the angle between w, w' is $< \varepsilon$ and* $\mathrm{area}(T_1 \Delta T_1') < \varepsilon$.

To prove Proposition 4.6 we adopt a suggestion of Yaroslav Vorobets and exploit a theorem of McMullen whose proof itself uses Ratner's theorem. This proof replaces our original more elementary but significantly longer proof.

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and regard G/Γ as the space of oriented lattices $\Lambda \subset \mathbb{R}^2$ of co-area 1. Let $N \subset G$ be the subgroup preserving horizontal vectors. Let $G_\Delta = \{(g, g) \mid g \in G\}$ and $N_\Delta = \{(g, g) \mid g \in N\}$.

THEOREM 4.7. [**6**, Theorem 2.6] *Let $z = (L_1, L_2) \in (G \times G)/(\Gamma \times \Gamma)$ be a pair of lattices, and let $Z = \overline{N_\Delta z}$. Then exactly one of the following holds.*

(1) *There are horizontal vectors $v_i \in L_i$ with $|v_1|/|v_2| \in \mathbb{Q}$. Then $Z = N_\Delta z$.*
(2) *There are horizontal vectors $v_i \in L_i$ with $|v_1|/|v_2| \notin \mathbb{Q}$. Then $Z = (N \times N)z$.*
(3) *One lattice, say L_2, contains a horizontal vector but the other does not. Then $Z = (G \times N)z$.*
(4) *Neither lattice contains a horizontal vector, but $L_1 \cap L_2$ is of finite index in both. Then $Z = G_\Delta z$.*
(5) *The lattices L_1 and L_2 are incommensurable, and neither contains a horizontal vector. Then $Z = (G \times G)z$.*

Proof of Proposition 4.6. Let (L_1, L_2, w) be a splitting of (X, ω) that is irrational in L_1. Let A_i be the co-area of L_i. It is enough to show that for any $\varepsilon > 0$ and sufficiently small $\varepsilon' < \varepsilon$ there is a pair of primitive vectors $(v_1, v_2) \in L_1 \times L_2$ satisfying:

(i) $|v_1 \times w| < \min(\varepsilon'/4, A_1)$ and $|v_2 \times w| \leq A_2$;
(ii) $|v_1 \times v_2| < \varepsilon'/6$ and $|v_1 \times v_2| < \frac{1}{3}\max(|v_1 \times w|, |v_2 \times w|)$.

Indeed, one of the three splittings obtained by twisting about $\gamma(v_1, v_2)$ will be irrational by Lemma 4.2. By Lemma 3.3

$$\mathrm{area}(T_1 \Delta T_1') \leq 2\varepsilon'/4 + 3\varepsilon'/6 < \varepsilon$$

and by choosing a sufficiently small ε', the direction of the vectors v_1, v_2, and hence that of $w + k(v_1 + v_2)$, can be made within ε of w.

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Let Z be the closure of the orbit of (L_1, L_2) under the action of N_Δ where $N \subset G$ is the subgroup that preserves vectors parallel to w.

Suppose first that (L_1, L_2, w) is irrational in both L_1 and L_2. Then we are in case 4 or 5 of Theorem 4.7. If case 5 holds, then the $G \times G$ action allows one to conclude that for any $\varepsilon' > 0$ there exist $(L_1', L_2') \in Z$ and primitive vectors $v_i' \in L_i'$ both perpendicular to w and having lengths

$$|v_i'| < \frac{1}{|w|} \min(\varepsilon', A_i).$$

If case 4 holds, then L_1 and L_2 are isogenous and so once one has found a vector $v_1' \in L_1'$ perpendicular to w by the G action on lattices, then automatically there is a vector $v_2' \in L_2'$ perpendicular to w as well. Then we can use the G_Δ action to make both vectors satisfy the above inequality as well.

It now follows that there exists $g_n \in N$ and $v_{n,i} \in L_i$ such that $g_n v_{n,i} \to v_i'$ as $n \to \infty$. Since $g \in G$ preserves cross products, and $g \in N$ fixes w, for large n we have vectors $v_{n,1}, v_{n,2}$ that satisfy:

- $|v_{n,1} \times w| = |g_n v_{n,1} \times g_n w| = |g_n v_{n,1} \times w| < \min(\varepsilon'/4, A_1)$;
- $|v_{n,2} \times w| = |g_n v_{n,2} \times g_n w| = |g_n v_{n,2} \times w| < A_2$;
- $|v_{n,1} \times v_{n,2}| = |g_n v_{n,1} \times g_n v_{n,2}| < \varepsilon'/6$;
- $|v_{n,1} \times v_{n,2}| = |g_n v_{n,1} \times g_n v_{n,2}| < \frac{1}{3} \max(|g_n v_{n,1} \times w|, |g_n v_{n,2} \times w|) = \frac{1}{3} \max(|v_{n,1} \times w|, |v_{n,2} \times w|)$.

This means that the vectors $v_{1,n}, v_{2,n}$ satisfy the desired conditions (i) and (ii).

Suppose next that (L_1, L_2, w) is rational in L_2. We are assuming that the splitting (L_1, L_2, w) is irrational in L_1. Now case 3 of Theorem 4.7 implies for any $\varepsilon' > 0$ there exist $(L_1', L_2') \in Z$ and primitive vectors $v_i' \in L_i'$ both perpendicular to w with $|v_1'| < (1/|w|) \min(\varepsilon', A_1)$ and $|v_2' \times w| = A_2$. Again, we approximate (L_1', L_2') by pairs of lattices in the N-orbit of (L_1, L_2) and note that $|v_2 \times w| = A_2$ for any pair (v_1, v_2) that approximates (v_1', v_2'). □

Proof of Theorem 1.1. Suppose (X, ω) is not a Veech surface. Since there are only countably many non-minimal directions, it suffices to construct an uncountable number of non-ergodic directions. By Corollary 4.4, there is some irrational splitting (L_1, L_2, w) of (X, ω) into tori T_1, T_2. Let A_i be the area of T_i. We inductively construct an infinite binary tree of splittings $(L_1^n(j), L_2^n(j), w^n(j))$ of (X, ω) so that there are 2^n splittings at level n, which we index by $j = 1, \ldots, 2^n$. We take the unique splitting of level zero to be (L_1, L_2, w). At the completion of the nth level, we define $\varepsilon_n > 0$ to be the minimum distance in angle between the directions of any two splittings that have been constructed up to this point. To construct the splittings of the next level, we apply Proposition 4.6 with $\varepsilon < \varepsilon_n/4$. Continuing *ad infinitum* we obtain an uncountable number of sequences (L_1^n, L_2^n, w^n) of splittings of (X, ω). Since $\varepsilon_n \leq (1/2)^n$ the directions of the $\{w^n\}$ for any infinite geodesic in the tree converge to a limiting direction θ. Let h_n be as in Theorem 2.1. Then

$$h_{n+1} = |w^{n+1}| \sin(\angle w^{n+1}\theta) \leq |w^{n+1}| \sin(2\angle w^n w^{n+1}) \leq \frac{2|w^n \times w^{n+1}|}{|w^n|}.$$

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We have $\lim |w^n| = \infty$. Now $|w^n \times (v_1 + v_2)| \leq A_1 + A_2$ since the left side is the area of an annulus contained in (X, ω). Since $w^{n+1} = w^n + k(v_1 + v_2)$ for some k, $|k| \leq 3$,

$$|w^n \times w^{n+1}| \leq 3(A_1 + A_2).$$

As a result we have $\lim_{n\to\infty} h_n = 0$. The minimum spacings ε_n at level n satisfy

$$\varepsilon_{n+1} \leq \varepsilon_n/2.$$

Since the change in areas is at most ε_n, the sum of the change of areas is at most $2\varepsilon_0$. If we choose $\varepsilon_0 < \frac{1}{2}\min(A_1, A_2)$ the remaining hypothesis of Theorem 2.1 is satisfied.

That the non-ergodic limiting directions that have been constructed are all distinct follows from the condition that the spacing between a splitting at level n and either of its descendants is at most $\varepsilon_n/4$. This completes the proof of Theorem 1.1. □

Acknowledgements. The authors are deeply indebted to Yaroslav Vorobets for correcting an error in the original proof of Corollary 4.4 and suggesting the more efficient proofs of Lemma 4.2 and especially of Proposition 4.6 which we have adopted over our original elementary, but significantly longer, arguments. This research is partially supported by NSF grant DMS0244472.

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