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ARISTOTLE AND EUCLID'S POSTULATES

In memoriam Ian Mueller

Book 1 of Euclid's *Elements* opens with a set of unproved assumptions: definitions (ὅροι), postulates, and 'common notions' (κοιναὶ ἔννοιαι).[1] The common notions are general rules validating deductions that involve the relations of equality and congruence. The attested postulates are five in number, even if a part of the manuscript tradition adds a sixth, almost surely spurious ('two straight lines do not contain a space'), that in some manuscripts features as the ninth, and last, common notion. The postulates are called αἰτήματα both in the manuscripts of the *Elements* and in the ancient exegetic tradition. It is not said, however, that this denomination is original, as it coincides with the *nomen rei actae* associated with the verbal form introducing the postulates themselves. Since antiquity it has been recognized that the postulates naturally split into two groups of quite different character: the first three are rules licensing basic constructions, the fourth and the fifth are assertions stating properties of particular geometric objects (see Procl. *In Euc*. 182–4 and 188–93). I transcribe postulates 1–3 in the text established by Heiberg (*Euclidis Elementa* 1.4.14–5.2):

ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν
καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν
καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι

At the beginning of the *Analytica posteriora*, Aristotle repeatedly presents a tripartite division of the principles of any deductive science: 'one is what is proved, the conclusion (this is what applies to some genus *per se*), one are the axioms (the axioms are from which), the third is the underlying genus, the properties and the *per se* attributes of which the proof makes clear' (*An. post.* A 7, 75a39–b2, and cf. for instance A 10, 76b11–16, where the ἀξιώματα are qualified as κοινά). Two other divisions, it is not clear whether supplementing this or being simply grounded on different ordering criteria, are introduced in the *Analytica posteriora*. One is at *An. post.* A 10, 76b23–34: Aristotle distinguishes between ὑποθέσεις and αἰτήματα, the postulates being 'something not in accordance with the opinion of the learner'. In the subsequent paragraph, ὑποθέσεις are instead opposed to ὅροι (76b35–77a4): ὑποθέσεις have a propositional structure, ὅροι do not, and simply need to be grasped. The same distinction surfaced for the first time as a sub-specification of the classification at A 2, 72a14–22: ἀξιώματα are

[1] I shall refer to the following articles by the name of the author and the date of publication: H.G. Zeuthen, 'Die geometrische Construction als "Existenzbeweis" in der antiken Geometrie', *Mathematische Annalen* 47 (1896), 222–8; H.D.P. Lee, 'Geometrical method and Aristotle's account of first principles', *CQ* 29 (1935), 113–24; A. Gomez-Lobo, 'Aristotle's hypotheses and the Euclidean postulates', *RMeta* 30 (1977), 430–9; I. Mueller, 'On the notion of a mathematical starting point in Plato, Aristotle, and Euclid', in A.C. Bowen (ed.), *Science and Philosophy in Classical Greece* (New York and London, 1991), 59–97. The references to the Aristotelian commentators are to the page(s) of the corresponding volumes in the *Commentaria in Aristotelem Graeca* series.

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separated from θέσεις, on the basis of whether or not they need to be grasped by anyone who is to learn anything (that is, the difference is between general principles and assumptions typical of some discipline); the θέσεις are further specified as ὑποθέσεις – that have a propositional structure and state existence – or ὁρισμοί.

Given the obvious resonances of these denominations with the established mathematical lexicon, interpreters since Proclus have tried to map the tripartite division(s) of principles proposed by Aristotle on to the one attested in the *Elements*. Proclus has no qualms in proposing an exact correspondence (*In Euc*. 76.4–77.2 and 178.1–8). He applies the Aristotelian designations ὑποθέσεις, αἰτήματα, ἀξιώματα to the ἀρχαί of the Euclidean treatise; expressly mentioning ὁ δαιμόνιος Ἀριστοτέλης as his source, he justifies his mapping by showing that the three kinds of principles are in one-to-one correspondence with the three possible combinations of the two polar oppositions 'self-evident'/'unknown' and 'conceded by the learner'/'not conceded by the learner'.[2] Since Aristotle more or less explicitly recognizes both definitions and 'axioms' as principles, an interpretative crux has been debated among modern interpreters, namely, whether or not the Euclidean postulates match any of the residual kinds introduced by Aristotle. An affirmative answer, grounded on Heath's popularization of a very influential paper by H.G. Zeuthen (1896), was given in modern times by H.D.P. Lee:[3] Aristotle's ὑποθέσεις exactly correspond to the first three postulates, since the latter do nothing more than establish the existence of the objects involved. Lee's view became an integral part of the folklore about pre-Euclidean mathematics, since the correspondence between existence and construction presents itself as 'obvious' to modern eyes. However, well-argued criticisms have recently been made against this thesis, either showing that we have no grounds for supposing that the constructive postulates were read, by Aristotle or by any ancient mathematician, as assumptions of existence (see Mueller [1991], 77–8), or arguing that the Aristotelian ὑποθέσεις most fittingly correspond to the local stipulations set out in the ἔκθεσις of a mathematical proposition.[4]

Once the link between Aristotle's ὑποθέσεις and Euclid's postulates is severed, the extant evidence, or rather lack of evidence, from Aristotelian texts can easily be transformed into a strong argument *ex silentio* against the possibility that rules analogous to the first three postulates were recognized as principles before the *Elements*: all interpreters agree that not one of the many Aristotelian passages dealing with mathematics or with mathematical practice contains any references to geometrical operations that could be likened to the postulates.

Yet there is an Aristotelian text that has been completely neglected in all previous treatments of the subject, namely *Top.* A 1, 101a13–17 (referred to hereafter as T); since I shall be repeatedly quoting it, I also give it in translation:

ἐκ τῶν οἰκείων μὲν τῇ ἐπιστήμῃ λημμάτων οὐκ ἀληθῶν δὲ τὸν συλλογισμὸν ποιεῖται· τῷ γὰρ ἢ τὰ ἡμικύκλια περιγράφειν μὴ ὡς δεῖ ἢ γραμμάς τινας ἄγειν μὴ ὡς ἂν ἀχθείησαν τὸν παραλογισμὸν ποιεῖται.

[2] The case of a self-evident yet not conceded principle of course does not apply.

[3] See Lee (1935), 114–17. As a matter of fact, these four pages are not much more than a cento of quotations from two of Heath's books.

[4] See Gomez-Lobo (1977), whose interpretation, however, must be corrected on a crucial point: the verb 'to be' which often introduces the ἔκθεσις has a 'presential', not a copulative value: this makes it unnecessary to assign to the same verb a 'predicative value with ellipsis of the predicate' (p. 433) in Aristotelian passages such as *An. post.* A 2, 72a18–24.

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He effects his syllogism from assumptions which are, on the one hand, appropriate to the science in question but, on the other, are not true – for he effects his paralogism either by describing the semicircles not as one should or by drawing certain lines not as they should be drawn.

The subject of Aristotle's sentences is the ψευδογραφῶν, the 'pseudographer' or proposer of false proofs in geometry; the context is a short description of such paralogisms. The two λήμματα mentioned by Aristotle bear an obvious resemblance, respectively, to the third and first postulates; my aim is to show that they can actually be taken as evidence for the status of pre-Euclidean mathematics. I shall first discuss some terminological points directly pertaining to the Aristotelian formulation of the 'assumptions'.

The first point is the presence of the verb περιγράφειν. In the mathematical lexicon, this is the standard term for 'circumscribing' one figure about another, but we are not entitled to assume that such a specific meaning was intended by Aristotle, or even that it was the current one in his times. In fact, appealing to the third postulate is not the only way circles can be constructed starting from a suitable straight line. In the technical literature, circles can also be 'circumscribed' about a straight line taken as a diameter, as in Euclid, *Optica* 24 (*aliter* 28, recension **A**), very much as segments are 'circumscribed' to a chord in *Optica* 48 (both recensions) and 49 (recension **A**). In these occurrences the prefix περι- is repeated as a preposition (περιγράφειν περί). The same kind of constructions prominently feature in Simplicius' digression on the quadrature of lunes: a semicircle is consistently said to be circumscribed about a straight line as its diameter (only in the version whose source is Alexander: see *In Ph.* 56.2.8.22.23 and 57.2); a segment is circumscribed about a chord (only in the version ascribed to Eudemus: see *In Ph.* 62.21, 67.30, 68.1). In Archytas' solution to the problem how to find two mean proportional, a circle is 'described around' (a form of γράφειν περί) a straight line taken as a diameter, and a point περιγράψει a semicircle (*Archimedis Opera Omnia* 3.84.15 and 29).[5] In sum, this evidence – for the most part pre-Euclidean – suggests that the verb περιγράφειν was a standard one to denote the operation of 'describing' (semi)circles starting from one of their diameters. Only the canonization of the third postulate, which refers to the *different* operation of 'tracing' a circle starting from its centre and its radius, caused the prefix περι- to be dropped.[6]

These considerations suggest also an answer to a natural question: why on earth should a geometer take 'to describe *semi*circles' (and not circles) as an assumption? The hypothesis that this is because Aristotle is cryptically alluding to some specific result – as Alexander implicitly suggests in his commentary by offering two examples of a fallacious proof arising from wrongly describing semicircles (*In Top*. 23.22–25.9) – is simply unmethodical. Very simply, once the datum is the diameter, tracing a semicircle is equivalent to tracing the entire circle, and *El*. 11.def.14 and the constructions of the regular polyhedra in *El*. 13 show that a fundamental constructive definition such as that of a sphere was effected by rotating a semicircle. What is more, every construction after *El*. 1.3 can be performed by tracing semicircles, because 1.1 can as well.[7] In short, 'describing semicircles' is not such a marginal geometric construction as it might seem

[5] This evidence is of limited value since the whole proof was rewritten by Eutocius.

[6] In the *Elements*, a semicircle is said to be 'described on' (forms of γράφειν ἐπί) a diameter, as for instance in propositions 6.13, 13.13–16. The canonization process was not immediate: in his duplication of a cube, Philon περιέγραψε a semicircle 'making use' of a point and a line segment as centre and radius respectively (110.9–10 Marsden).

[7] The peculiar form of the third postulate is strictly functional to its application in *El*. 1.2–3 – the two constructions that give to the basic operation of cutting off a segment from a straight line the

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at first sight; it starts from a set of data different from that of the third postulate and is therefore alternative to it. One need only think of the variety of *neusis* techniques to realize that a construction that seems contrived or redundant to us need not have been so to the ancient geometers.

The second assumption mentioned by Aristotle amounts to 'drawing' (ἄγειν) 'lines' (γραμμάς). As for the substantive, Aristotle elsewhere employs it freely for 'straight line',[8] and this must be the intended meaning in our text **T** too.[9] The verb ἄγειν is the one featuring in the enunciation of the first postulate, whereas in the applications forms of ἐπιζευγνύναι invariably appear, probably to make more explicit the fact that what is done is that two points are 'joined'. In the mathematical tradition, ἄγειν acquired a more specialized meaning, namely, 'drawing' straight lines whose function is specified, usually by means of constraints imposed on their direction or on their relations with other geometric objects (parallelism, perpendicularity, tangency, sectioning …). A straight line that is 'drawn' often passes διά 'through' one or more points, without 'joining' them. This preposition can also feature as a prefix: the result is διάγειν 'to draw across', whose meaning is highly specialized for the situation in which the straight line 'drawn across' passes through one point but otherwise is arbitrary. There are, then, remarkable terminological affinities between the second assumption mentioned by Aristotle and the first postulate, but the terms involved are of widespread use and denote basic geometrical objects or operations: the coincidences could simply point to a shared lexicon, and do not corroborate the thesis that Aristotle is actually quoting some 'principle'.

As a matter of fact, the most important elements suggesting that the two assumptions actually refer to mathematical principles come from the way Aristotle presents them: 1) he calls them λήμματα; 2) he puts an explicit modal emphasis on what the pseudographer 'should' have done: cf. the verbal form δεῖ in the first assumption and the subordinate with ἄν + optative in the second.

Concerning the first of these points, the word λῆμμα is quite uncommon in the Aristotelian corpus: it is attested in the above passage but elsewhere only in *Top*. Θ 1, 156a21, *Soph. el.* 33, 183a15, *Eth. Nic.* Δ 1 1122a10, *Pol*. B 10, 1272a41, E 8, 1309a21, Z 4, 1318b16, Z 7, 1321a41, *Rh*. A 12, 1372b2. The only significant occurrences, however, are the first two in this list, where the term is a strict synonym of 'premise' of a syllogism; a similar meaning must be supposed for the occurrence in **T** too. In contrast to this poor record, the verb λαμβάνειν is extremely frequent in Aristotle in the sense of 'assuming' some assertion as a premise or as a principle, as opposed to 'proving' it: for instance, in the first 34 lines (Bekker) of *An. post.* A 10 the verb occurs with this meaning nine times. I feel entitled to conclude that the λήμματα from which the pseudographer makes his paralogisms have a well-defined mathematical content and admit a linguistic expression on the basis of which one can decide whether they are actually used in a proof or not. Recall also that Archimedes, who provides us with the best appoximation to pre-Euclidean evidence concerning lexical and syntactical features of a proof, uses the word λαμβανόμενα for those assumptions with an explicit existential import which he lists at the beginning of *Sph. cyl*. 1.

epistemic status of a proposition (i.e. a statement supported by a proof) and that can be used in place of the postulate (i.e. an unproven statement) itself.

[8] See e.g. the much-debated passage on the definition of 'ratio' at *Top*. Θ 3, 158b30–5.

[9] This designation was apparently a rigid one before Aristotle, and the only one to be found in Plato.

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As for the second point above, there are two possibilities: either the λήμματα were formulated as prescriptions, or the modal shade is simply a reflection of the prescriptive context of the Aristotelian passage: the pseudographer simply does not do things in the way that he 'should'. I think that the first possibility is worth exploring, and I shall do so in the rest of this note.

Notice first the position of the negative particle μή: it precedes and colours the whole of the modal clauses ὡς δεῖ and ὡς ἂν ἀχθείησαν; therefore, such clauses cannot be translated 'as one should not', but 'not as one should'. This suggests that the prescriptive mode is already operating in the λήμματα.

Three questions naturally arise at this point: first, what assumption or premise could correspond to 'drawing a straight line as it should be drawn'; second, how could Aristotle call such an assumption a λῆμμα if it were not made explicit (actually, it is enough that it *could* be made explicit; cf. *An. post.* A 10, 76b16–22); and third, if the assumption was or could be made explicit, why did Aristotle not range it among the principles? Now, the kind of premise to which the λήμματα belong is specified just before the passage discussed above: the pseudographer does not syllogize from true and primary things, nor from acceptable ones (this second branch is of no use in mathematics). But how could an assumption corresponding to 'drawing straight lines not as they should be drawn' be qualified as 'not true' (or indeed 'true', for that matter)? I think that the only way it can is if the clause οὐκ ἀληθῶν δέ in **T** refers to the fact that the assumption is taken in the wrong way by the pseudographer, not to the fact that it is *false*. In this case, the modal connotation that Aristotle emphasizes might well correspond to a stylistic trait well established in mathematical practice: constructions, be they postulates or less basic operations performed to complete a given configuration in view of the proof, are characterized by a verbal form in the imperative. Accordingly, I submit that the specific λήμματα Aristotle alludes to are not in assertoric form (in the same way as the ὅροι are not), and therefore they cannot be true or false. The pseudographer does not simply draw a false diagram; he wrongly applies some constructive assumptions that have inescapable consequences on the development of the subsequent proof, even though they may well be faultless from the logical point of view.

Support for this interpretation can be found in the way that the deductive activity of the pseudographer is described. When his activity is viewed in abstract terms at *Top.* 101a10, the pseudographer συλλογίζεται. When his actual practice is at issue, however, the expression changes to τὸν συλλογισμὸν/τὸν παραλογισμὸν ποιεῖται 'he effects his paralogism'/'syllogism', which I take as pointing to the fact that the λήμματα are necessary for, but only preliminary to, setting up the strictly deductive portion of a proof (the ἀπόδειξις in ancient mathematical jargon), of which they are not an integral part (cf. Mueller [1991], 77).

If the (auxiliary) construction is not an integral part of the deductive core of a proof, the 'assumptions' on which it is grounded, either implicitly or explicitly, cannot be counted among the principles of a deductive science. Yet, the *Topica* passage attests Aristotle's recognition that there are such 'constructive assumptions', and it is of course not really important whether these λήμματα were explicitly stated as such at the beginning of some pre-Euclidean exposition of '*Elements*' or whether they simply featured as fixed formulaic expressions characterizing certain operations on geometrical objects that were only enucleated and given the dignity of ἀρχαί or the status of basic constructive steps in the *Elements*. For these reasons, the tripartite division of principles found in the *Elements* and the one(s) proposed by Aristotle do not match, and in fact *could not* match. It is no surprise, then, that he does not mention the 'postulates' which we

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know from the mathematical tradition and that he uses this term to characterize some principles on the basis of a dialectical criterion such as the degree of assent to them manifested by some learner. Nor can we assume that Aristotle, had he found the mathematical postulates stated as independent assumptions in some technical text, would have felt compelled to 'replace' them with some other kind of principles, if he considered them irrelevant to the ἀπόδειξις: his classification of principles is simply grounded on different parameters.

As a matter of fact, I suspect that not even the seemingly unproblematic Aristotelian references to the rule 'if equals be subtracted from equals the remainders are equal'[10] can safely be taken to allude to axioms that presumably featured as principles in pre-Euclidean '*Elements*' (the rule is attested in the *Elements* as the third common notion). Take the mathematical passage at *An. pr*. A 24, 41b15–22.[11] Aristotle points out that the proof he reports is invalid, and is actually a circular one, unless one makes explicit three general rules of increasing generality, validating particular assertions that are either instantiations of these rules or their applications to specific objects: the general rules are that 'in general the <angles> of the semicircles are equal', that the two angles of a circular segment are equal in every instance, and finally, the 'equals-from-equals' rule. Now, if we accept that the three rules are put on the same logical footing by Aristotle, and since he apparently considers the proof he expounds to be both representative of the mathematical style and fallacious, I cannot help thinking that none of the rules was made explicit as a principle in any mathematical writing, a hypothesis that is by no means in contradiction with the plain fact that Aristotle did recognize the importance of the 'equals-from-equals' rule as a starting point of so many mathematical deductions.

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[10] At *An. pr*. A 24, 41b21–2, *An. post.* A 10, 76a41 and 76b20–1, A 11, 77a30–1, *Metaph*. K 4, 1061b20.

[11] Notice in it the three occurrences of forms of (προσ)λαμβάνειν: two of them are referred to as assumptions, one as an instantiation of an assumption.

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