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Intersection K-theory for isolated conical singularities

by

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Abstract

Starting from the Karoubi multiplicative K-theory, we construct a Chern-Weil theory adapted to isolated conical singularities. The Chern character takes its values in the intersection cohomology of Goresky-MacPherson. We also propose an integer intersection K-theory for such singularities.

Key Words: Chern character, K-theory, intersection homology, conical singularities.

Mathematics Subject Classification 2010: 19D55, 55S99.

1. Introduction

Singularities are not homotopy invariants and so classical cohomology theories can be irrelevant in the singular setting. Many modifications of classical homology have been proposed to address this problem. Analysts often consider complexes of differential forms on the open regular part with a controlled behavior near the singular part, the control being L^2 or L^p or polar, [7], [16], [5], [4].

In 1980, a more geometric point of view was developed by Marc Goresky and Robert MacPherson who introduced the notion of intersection homology, in which rational Poincaré duality remains true for stratified varieties, [10]. Intersection cycles are those which are sufficiently transverse with the singular part. The defect of transversality is controlled by a sequence $\bar{p}$ of integers called the perversity.

The two points of view, analysis and geometry, give the same type of invariants. This results from several de Rham theorems that identify the dual of the real intersection homology with L^p-forms, [7], [16], or with polar forms, [5], [3]. The behavior of differential forms near the singularities is related to perversity.

In this paper we build K-theories adapted to spaces with isolated conical singularities.

According to the spirit of non commutative geometry, [8], the natural way to define a K-theory related to such controlled differential forms would be to consider the K-theory of a commutative algebra whose Hochschild homology consists precisely of these forms. But the asymptotically controlled forms are not an algebra, so they cannot be the Hochschild homology of an algebra. For instance

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it is proved in [6] that you must consider Hochschild homology with coefficients to obtain the intersection complex of polar forms, and the algebra structure is lost (see [9] for a "non commutative" study). So we replace the usual K-theories which have been considered in this situation by an adaption of multiplicative K-theory to intersection homology.

In the first part of the paper we define a real intersection K-theory using vector bundles which admit a connection whose curvature form satisfies some constraints near the singularities. These bundles belong to an adaptation of the multiplicative K-groups introduced by Max Karoubi in 1990.

Multiplicative K-groups are intermediate between algebraic K-groups and topological K-groups, [12]. These intermediate K-groups give a general framework in which it is possible to define secondary characteristic classes for bundles with additional structures when the base is a closed manifold, [11], [13].

Let us recall that a multiplicative K-group for a closed manifold is associated with a characteristic class P and a family of sub-complexes of the de Rham complex for this manifold. Such a group is the abelian group which is generated by the equivalence classes of triples (E,∇,ω) where E is a vector bundle equipped with a connection ∇ and ω is a suitable odd differential form. In terms of the Chern-Weil theory, suitable means that the closed even form $P(\nabla)-d\omega$ defines homology class belonging to the cohomology of the sub-complexes, [12] (also Section 2). In this paper we are only concerned with the Chern character.

We adapt the Karoubi multiplicative K-theory for manifolds with boundary, (Subsection 2.1) and we apply this theory to the manifold with boundary $\tilde{X}$ which is obtained by excising the singular points of the conical manifold X.

Let $\bar{p}$ be a perversity. Using suitable constraints on the boundary we define a subcomplex of the de Rham complex of $\tilde{X}$ which represents an intersection complex for X (Subsection 2.2). The associated Karoubi (multiplicative) Chern character takes its values in the real intersection cohomology and gives us an intersection Chern character "à la Chern-Weil".

But the Karoubi multiplicative groups are not $\mathbb{Z}$-modules of finite type (as can be seen from Theorem 2.1). The associated real theory is not finite-dimensional and cannot be isomorphic to the real intersection cohomology. We modify the Karoubi construction to get real K-groups which are finite-dimensional vector spaces (Proposition 3.1). Now, the corresponding Chern character $\overline{ch}$ is an isomorphism (Theorem 3.1). Let q denote the integer determined by the complementary perversity to $\bar{p}$, [10]. The previous real construction associated to the intersection complex with perversity $\bar{p}$ is denoted by $\mathcal{K}_{q,\mathbb{R}}(\widetilde{X})$.

In the second part of the paper we propose an integer lattice for the previous theory and define an integer "intersection K-theory" of finite type denoted by

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$IK_{\bar{p}}(X)$. As already noted, it cannot be the Karoubi multiplicative theory. We still consider the manifold with boundary $\tilde{X}$. Our intersection bundles (also called admissible bundles) are triples (E,F,t) where E and F are vector bundles on $\tilde{X}$, F is a sub-bundle of E, the restriction of F to the boundary is trivial and t is a given trivialization of this restriction. Moreover we assume that $ch_*(E)$ and $ch_*(F)$ satisfy suitable conditions depending on the perversity (Definition 4.1).

Then we define in a natural way the intersection Chern character ch' of an intersection bundle (E,F,t) with range in the rational intersection cohomology $IH_{\bar{p}}^{even}(X,\mathbb{Q})$, using the Chern characters of the bundles E and F/t, (Proposition 4.3). We prove the following theorem:

Theorem 4.1 *The intersection Chern character ch' induces an isomorphism*

$$ch' \otimes 1 : IK_{\bar{p}}(X) \otimes \mathbb{Q} \to IH_{\bar{p}}^{even}(X,\mathbb{Q}).$$

The intersection bundles have exactly the structure needed to be equipped with a connection (called admissible) and define multiplicative bundles in the sense of Karoubi. Our main theorem is the following.

Theorem 4.2 *A q-admissible bundle (E,F,t) always admits an admissible connection ∇ over E. For all admissible connections over E there exists a form $\omega \in \Omega^{odd}(\widetilde{X})$ such that $(E,\nabla,1,\omega)$ is a real multiplicative bundle. The class on $\mathcal{K}_{q,\mathbb{R}}(\widetilde{X})$ of this multiplicative bundle does not depend on the class of the admissible bundle on $IK_{\bar{p}}(X)$ and we have the relation*

$$ch' \otimes 1_{\mathbb{R}}([E,F,t] \otimes 1) = \overline{ch}([E,\nabla,1,\omega]).$$

Thus the real multiplicative theory $\mathcal{K}_{q,\mathbb{R}}$ defines an intersection Chern-Weil theory.

We conclude with some computations. In particular the local computation in intersection K-theory (i.e. the cone case) has exactly the same behavior as in intersection homology (Corollary 5.1).

A part of these results was mentioned in [14] (see also [15]).

2. Multiplicative K-theory and intersection cohomology

Max Karoubi, [12], built new K-theory groups called multiplicative K-theory groups for closed manifolds. These groups are parametrized by a family of sub-complexes of the de Rham complex. The cohomology of these sub-complexes is a receptacle for an adapted Chern character called the multiplicative Chern character.

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The family reduced to the whole complex gives topological K-theory. We will briefly extend these constructions for manifolds with a connected boundary. Then we build an "intersection" sub-complex such that the associated multiplicative Chern character takes its values in the intersection cohomology of the conical singular manifold obtained by clutching the boundary to a point.

2.1. Multiplicative K-theory of manifolds with boundary

Recall the construction of multiplicative K-theory, [12]. We shall extend this construction to manifolds with boundary, a situation which is more relevant to us.

Let us consider a compact manifold $\widetilde{X}$ with boundary L. A differentiable object on $\widetilde{X}$ means the restriction to $\widetilde{X}$ of a differentiable object on any open manifold with cylindrical ends which extends $\widetilde{X}$. To simplify, we only consider a single subcomplex $\mathcal{F}^\star(\widetilde{X})$ of the de Rham complex $\Omega^\star(\widetilde{X})$. We impose that $\mathcal{F}^0(\widetilde{X}) \neq 0$ and denote $\widetilde{\mathcal{F}}^*(\widetilde{X}) = \mathcal{F}^*(\widetilde{X}) + d\Omega^\star(\widetilde{X})$ where d is the differential of $\Omega^\star(\widetilde{X})$.

A multiplicative bundle over $\widetilde{X}$ is a triple (E,∇,ω) where $E \to \widetilde{X}$ is a bundle over $\widetilde{X}$, ∇ a connection over E, and $\omega \in \Omega^{odd}(\widetilde{X})$ such that $ch(\nabla) - d\omega \in \mathcal{F}^*(\widetilde{X})$. Let us consider two multiplicative bundles $\eta = (E,\nabla,\omega)$ and $\eta' = (E',\nabla',\omega')$. A morphism $\eta \longmapsto \eta'$ is given by an isomorphism of vector bundles $\alpha : E \xrightarrow{\approx} E'$ verifying the following condition :

$$\begin{aligned} \omega_i - \omega_i' &= \int_0^1 ch_i[(1-t)\nabla + t\alpha^*\nabla']dt \quad mod \quad \widetilde{\mathcal{F}}^i(\widetilde{X}) \\ &= \Theta(\nabla,\alpha^*\nabla') \qquad\qquad\qquad\quad mod \quad \widetilde{\mathcal{F}}^i(\widetilde{X}) \end{aligned}$$

where $\Theta(\nabla,\alpha^*\nabla')$ is the first transgression. The multiplicative bundles and their morphisms form a category. We denote by $[E,\nabla,\omega]$ the isomorphism class of (E,∇,ω). Let $\mathcal{K}(\widetilde{X})$ be the Grothendieck group of the abelian semigroup for the Whitney sum of isomorphism classes of multiplicative bundles,

$$[E,\nabla,\omega] + [E',\nabla',\omega'] = [E \oplus E', \nabla \oplus \nabla', \omega + \omega'].$$

Forgetting the odd form ω and the connection ∇, we define a homomorphism $u : \mathcal{K}(\widetilde{X}) \to K(\widetilde{X})$ with range in the classical K-theory group. The multiplicative Chern character $\overline{ch} : \mathcal{K}(\widetilde{X}) \to H^{even}(\mathcal{F}^*(\widetilde{X}))$ is defined by

$$\overline{ch}([E,\nabla,\omega]) = [ch(\nabla) - d\omega].$$

We define a morphism $v : H^{odd}(\Omega^*(\widetilde{X})/\mathcal{F}^*(\widetilde{X})) \to \mathcal{K}(\widetilde{X})$ by $v(cl(\omega)) = [T,\partial,\omega] - [T,\partial,0]$ where $cl(\omega)$ is the class of $\omega \in \Omega^{odd}(\widetilde{X})$ such that $d\omega \in \mathcal{F}^*(\widetilde{X})$ and ∂ is the de Rham connection of a trivial bundle T.

As in the multiplicative K-theory of M. Karoubi, [12], the multiplicative Chern character extends to a morphism of exact sequences.

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Theorem 2.1 *Let $\widetilde{X}$ be a compact manifold with a boundary and let $\mathcal{F}^\star(\widetilde{X})$ be a subcomplex of $\Omega^*(\widetilde{X})$. We have the following commutative diagram of $\mathbb{Z}$-modules whose rows are exact ($ev = even$):*

$$\begin{array}{ccccccccc} K^1(\widetilde{X}) & \stackrel{ch_*}{\to} & H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(\widetilde{X})}\right) & \stackrel{v}{\longrightarrow} & \mathcal{K}(\widetilde{X}) & \stackrel{u}{\longrightarrow} & K(\widetilde{X}) & \stackrel{ch_*}{\longrightarrow} & H^{ev}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(\widetilde{X})}\right) \\ \downarrow ch_* & & \downarrow = & & \downarrow \overline{ch_*} & & \downarrow ch_* & & \downarrow = \\ H^{odd}(\Omega^*(\widetilde{X})) & \to & H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(\widetilde{X})}\right) & \longrightarrow & H^{ev}(\mathcal{F}^*(\widetilde{X})) & \longrightarrow & H^{ev}(\Omega^*(\widetilde{X})) & \longrightarrow & H^{ev}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(\widetilde{X})}\right). \end{array}$$

The map ch_* is defined by the classical Chern character, we have $ch_j(\xi) \in H^{2j-1}(\widetilde{X};\mathbb{R})$ for the odd case and $ch_j(\xi) \in H^{2j}(\widetilde{X};\mathbb{R})$ for the even case.

2.2. Multiplicative Chern character and intersection cohomology

Let $\widetilde{X}$ be a N-dimensional compact manifold with boundary which is a connected closed manifold L. We denote by $X = \widetilde{X}/L$ the associated pseudomanifold with one isolated conical singularity and by $IH_p^*(X;\mathbb{K})$, where $\mathbb{K}$ is a field, the intersection cohomology of Goresky-MacPherson with coefficients in $\mathbb{K}$ associated with any perversity $\bar{p}$ such that $\bar{p}_N = p$, [10]. We have the following result where $q = N - p_N - 2$, [2] :

$$IH_p^i(X;\mathbb{K}) = \begin{cases} H^i(\widetilde{X};\mathbb{K}) & for \quad i \le q \\ Im(H^i(\widetilde{X},L;\mathbb{K}) \to H^i(\widetilde{X};\mathbb{K})) & for \quad i = q+1 \\ H^i(X;\mathbb{K}) & for \quad i \ge q+2\,. \end{cases} \tag{1}$$

Starting from the de Rham complex $\Omega^*(\widetilde{X})$ of $\widetilde{X}$, we will construct an intersection complex of X, i.e. a complex whose cohomology is the real intersection cohomology of X. Let $\Omega^*(X) = \ker[\Omega^*(\widetilde{X}) \stackrel{restriction}{\to} \Omega^*(L)]$ and consider the subcomplex $\mathcal{F}_{\bar{p}}^*(\widetilde{X}) \subset \Omega^*(\widetilde{X})$ such that

$$\mathcal{F}_{\bar{p}}^*(\widetilde{X}) : 0 \to \Omega^0(\widetilde{X}) \to \cdots \to \Omega^q(\widetilde{X}) \to \Omega^{q+1}(X) + d\Omega^q(\widetilde{X}) \to \Omega^{q+2}(X) \to \cdots \to \Omega^N(X)\,.$$

Proposition 2.1 *Let $\bar{p}$ be a perversity and denote $q = N - p_N - 2$. The subcomplex $\mathcal{F}_{\bar{p}}^*(\widetilde{X}) \subset \Omega^*(\widetilde{X})$ is an intersection complex for X associated to the perversity $\bar{p}$, i.e. we have an isomorphism $H^*(\mathcal{F}_{\bar{p}}^*(\widetilde{X})) \cong IH_p^*(X;\mathbb{R})$.*

Proof: From the definition of $\mathcal{F}_{\bar{p}}^*(\widetilde{X})$ we have

$$H^i(\mathcal{F}_{\bar{p}}^*(\widetilde{X})) = \begin{cases} H^i(\Omega^*(\widetilde{X})) & for \quad i \le q \\ H^i(\Omega^*(X)) & for \quad i \ge q+2\,. \end{cases}$$

Now consider the case $i = q+1$. By the natural map $\mu : H^{q+1}(\mathcal{F}_{\bar{p}}^*(\widetilde{X})) \to H^{q+1}(\Omega^*(\widetilde{X}))$ the class of the closed $(q+1)$-form $w + d\alpha \in \Omega^{q+1}(X) + d\Omega^q(\widetilde{X}) = \mathcal{F}_{\bar{p}}^{q+1}(\widetilde{X})$ is associated to the class $[w] \in H^{q+1}(\Omega^*(\widetilde{X}))$ which in fact belongs

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to $Im[H^{q+1}(\Omega^*(X)) \mapsto H^{q+1}(\Omega^*(\widetilde{X}))]$. The map μ is surjective, and $[w] = 0$ if $w = d\beta$ where $\beta \in \Omega^q(\widetilde{X})$, i.e. $[w] = 0$ in $H^{q+1}(\mathcal{F}^*_{\bar{p}}(\widetilde{X}))$. Hence μ is an isomorphism. □

To complete this section let us consider the intersection complex $\mathcal{F}^\star_{\bar{p}}(\widetilde{X})$. It results from Proposition 2.1 and from Theorem 2.1 that the associated multiplicative Chern character takes its values in the intersection cohomology. This gives a Chern-Weil definition of an "intersection Chern character".

Corollary 2.1 *As above let $X = \widetilde{X}/L$ be a N-dimensional compact pseudomanifold, with one isolated conical singularity. Denote by $\mathcal{K}_{\bar{p}}(\widetilde{X})$ the multiplicative K-theory group associated with the intersection subcomplex $\mathcal{F}^\star_{\bar{p}}(\widetilde{X})$. The multiplicative Chern character $\overline{ch}([E,\nabla,\omega]) = [ch(\nabla) - d\omega]$ takes its values in the intersection cohomology group $IH^{even}_{\bar{p}}(X,\mathbb{R})$ where the perversity $\bar{p}$ is such that $p_N = N - q - 2$. This multiplicative Chern character lifts up the classical Chern character*

$$\begin{array}{ccc} \mathcal{K}_{\bar{p}}(\widetilde{X}) & \stackrel{u}{\to} & K(\widetilde{X}) \\ \downarrow \overline{ch} & & \downarrow ch \\ IH^{even}_{\bar{p}}(X,\mathbb{R}) & \to & H^{even}(X,\mathbb{R})\,. \end{array}$$

3. Real intersection K-theory

Let us come back to Theorem 2.1. The first line is an exact sequence of $\mathbb{Z}$-modules where the group $H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(\widetilde{X})}\right)$ is a $\mathbb{R}$-vector space. So $\mathcal{K}_{\bar{p}}(\widetilde{X})$ is not a $\mathbb{Z}$-module of finite type unless the connecting homomorphism is 0. Then the $\mathbb{R}$-vector space $\mathcal{K}_{\bar{p}}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is not finite dimensional and cannot be isomorphic to $IH^{even}_{\bar{p}}(X,\mathbb{R})$, which is finite dimensional.

The classical Chern character $ch_* \otimes \mathbb{R}$ is an isomorphism. We plan to keep this important property within the intersection cohomology theory. We will modify the multiplicative K-theory of M. Karoubi and construct a real intersection K-theory with a Chern character that becomes an isomorphism. More precisely we will modify the construction of $\mathcal{K}(\widetilde{X})$ to get a finite dimensional $\mathbb{R}$-vector space structure such that the first exact sequence of Theorem 2.1 induces an exact sequence of $\mathbb{R}$-vector spaces.

This construction does not use that $\mathcal{F}^\star(\widetilde{X})$ is an intersection complex. We will add this hypothesis when we compare the real multiplicative theory with the intersection cohomology.

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3.1. Real multiplicative bundles

In order to obtain objects that admit a multiplication by real numbers, we will modify the definition of multiplicative bundles.

Consider the quadruples (E,∇,a,ω) where E is a vector bundle over $\widetilde{X}$, ∇ a connection over E, $a \in \mathbb{R}$ and ω an odd form of $\Omega^\star(\widetilde{X})$ such that

$$a\,ch_*(\nabla) = d\omega \ \ mod\ \mathcal{F}^\star(\widetilde{X}).$$

When $a \neq 0$, $(E,\nabla,\frac{\omega}{a})$ is a multiplicative bundle and when $a = 0$, ω defines an arbitrary cocycle of $\Omega^\star(\widetilde{X})/\mathcal{F}^\star(\widetilde{X})$. There is no relation between the connection ∇ and the form ω.

Two elements (E,∇,a,ω) and (E',∇',a',ω') are said to be equivalent if one of the two following conditions are satisfied (the notations are those of Subsection 2.1):

- (C_1) $a = a' \neq 0$ and the multiplicative bundles $(E,\nabla,\frac{\omega}{a})$ and $(E',\nabla',\frac{\omega'}{a'})$ are isomorphic;

- (C_2) $a = a' = 0$ and there exists a finite sequence of elements $a_j \in \mathbb{R}$ and a finite sequence of automorphisms α_j of a trivial bundle over $\widetilde{X}$ such that $\omega - \omega' = \sum_j a_j \Theta(\partial,\alpha_j^*\partial)$ modulo $\widetilde{\mathcal{F}}^\star(\widetilde{X})$.

This relation is an equivalence relation, that we denote by $\sim$ and $[E,\nabla,a,\omega]$ is the class of a quadruple (E,∇,a,ω). The (C_2) condition expresses that the classes of type $[E,\nabla,0,\omega]$ only depend on the form ω. We will keep the pair (E,∇) within the current notation for reasons of homogeneity. It is possible to choose $(E,\nabla) = (T,\partial)$ a trivial bundle with the de Rham connection.

Proposition 3.1 *Let us consider the $\mathbb{Z}$-module spanned by all classes of quadruples $[E,\nabla,a,\omega]$ and the sub-module spanned by elements of the following types*

- $(\mathcal{T}_1)$: $[E,\nabla,a,\omega] + [E,\nabla,a',\omega'] - [E,\nabla,a+a',\omega+\omega']$,

- $(\mathcal{T}_2)$: $[E,\nabla,a,\omega] + [E',\nabla',a,\omega'] - [E\oplus E',\nabla\oplus\nabla',a,\omega+\omega']$.

The quotient $\mathbb{Z}$-module is a $\mathbb{R}$-vector space when equipped with the following product ($b \in \mathbb{R}$)

$$b[E,\nabla,a,\omega] = [E,\nabla,ba,b\omega].$$

The zero element of this $\mathbb{R}$-vector space is the class of any quadruple $(E,\nabla,0,0)$.

Before proving the proposition let us define the real multiplicative K-theory.

Definition 3.1 The $\mathbb{R}$-vector space of Proposition 3.1 is called the real multiplicative K-theory group of $\widetilde{X}$ and is denoted by $\mathcal{K}_{\mathbb{R}}(\widetilde{X})$. The correspondence $[E,\nabla,\omega] \mapsto [E,\nabla,1,\omega]$ defines a natural morphism $\mathcal{K}(\widetilde{X}) \to \mathcal{K}_{\mathbb{R}}(\widetilde{X})$.

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Proof: Let us verify that the relations induced by elements of type $(\mathcal{T}_1)$ are compatible with the relation $\sim$. This is clear when the real coefficient is zero. If not, let us consider $(E,\nabla,a,\omega) \overset{\alpha}{\sim} (E_1,\nabla_1,a,\omega_1)$ and $(E,\nabla,a',\omega') \overset{\beta}{\sim} (E_1,\nabla_1,a',\omega'_1)$ where a,a' are non zero and $\alpha,\beta : E \cong E_1$ are the underlying isomorphisms. First we claim that it is possible to suppose $\alpha = \beta$. Introduce ω'_2 such that $(E,\nabla,a',\omega') \overset{\alpha}{\sim} (E_1,\nabla_1,a',\omega'_2)$. Then $(E_1,\nabla_1,a',\omega'_2)$ and $(E_1,\nabla_1,a',\omega'_1)$ are equivalent, so the multiplicative bundles $(E_1,\nabla_1,\frac{\omega'_2}{a'})$ and $(E_1,\nabla_1,\frac{\omega'_1}{a'})$ are equivalent and $\frac{\omega'_1}{a'} - \frac{\omega'_2}{a'}$ belongs to $\widetilde{\mathcal{F}}^\star(\widetilde{X})$. To verify this, pick a vector bundle E'_1 such that $E_1 \oplus E'_1 \cong T$ is trivial and consider the relation

$$0 = [E_1,\nabla_1,\frac{\omega'_1}{a'}] - [E_1,\nabla_1,\frac{\omega'_2}{a'}] - [E'_1,\nabla'_1,-\frac{\omega'_2}{a'}] + [E'_1,\nabla'_1,-\frac{\omega'_2}{a'}] = [T,\partial,w] - [T,\partial,0]$$

where $w = \frac{\omega'_1}{a'} - \frac{\omega'_2}{a'}$ modulo $\mathcal{F}^\star(\widetilde{X})$. According to the first exact sequence of Theorem 2.1, $w \in \widetilde{\mathcal{F}}^\star(\widetilde{X})$, hence our claim.
Now suppose $\alpha = \beta$. If $a + a'$ is nonzero the following relations

$$\begin{aligned} \frac{\omega}{a} - \frac{\omega_1}{a} &= \Theta(\nabla,\alpha^\star(\nabla_1)) \quad mod \quad \bar{\mathcal{F}}^\star(\widetilde{X}) \\ \frac{\omega'}{a'} - \frac{\omega'_2}{a'} &= \Theta(\nabla,\alpha^\star(\nabla_1)) \quad mod \quad \bar{\mathcal{F}}^\star(\widetilde{X}) \end{aligned}$$

involve

$$\frac{\omega+\omega'}{a+a'} - \frac{\omega_1+\omega'_2}{a+a'} = \frac{a}{a+a'}(\frac{\omega}{a} - \frac{\omega_1}{a}) + \frac{a'}{a+a'}(\frac{\omega'}{a'} - \frac{\omega'_2}{a'}) = \Theta(\nabla,\alpha^\star(\nabla_1)) \ mod \ \bar{\mathcal{F}}^\star(\widetilde{X}).$$

This proves the equivalence $(E,\nabla,a+a',\omega+\omega') \sim (E_1,\nabla_1,a+a',\omega_1+\omega'_2)$, as well the relations induced by the elements of type $(\mathcal{T}_2)$ are also compatible with $\sim$.

To finish let us verify that the product with $b \in \mathbb{R}$ is well defined. This is clear when $a = 0$. Let $(E,\nabla,a,\omega) \sim (E',\nabla',a,\omega')$ with $a \neq 0$ and $b \in \mathbb{R}^\star$. The equivalence $(E,\nabla,ba,b\omega) \sim (E',\nabla',ba',b\omega')$ (i.e. $b[E,\nabla,a,\omega] = b[E',\nabla',a,\omega']$) follows from the relation:

$$\frac{b\omega}{ba} - \frac{b\omega'}{ba} = \frac{\omega}{a} - \frac{\omega'}{a} = \Theta(\nabla,\alpha^\star(\nabla')) \ mod \ \bar{\mathcal{F}}^\star(\widetilde{X}).$$

Elements of type $(\mathcal{T}_1)$ or $(\mathcal{T}_2)$ are stable by real number multiplication so $\mathcal{K}_{\mathbb{R}}(\widetilde{X})$ is a $\mathbb{R}$-vector space. □

3.2. Exact sequence and real intersection K-theory for isolated conical singularities

Define the real multiplicative Chern character by

$$\overline{ch}([E,\nabla,a,\omega]) = ach(\nabla) - d\omega.$$

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With an additional homological property on $\mathcal{F}^*(\widetilde{X})$ we will see that $\overline{ch}$ is an isomorphism.

The rational cohomology $H^{even}(\widetilde{X};\mathbb{Q}) \subset H^{even}(\Omega^*(\widetilde{X}))$ defines a natural rational lattice in the cohomology groups $H^{even}(\Omega^*(\widetilde{X}))$ such that $H^{even}(\widetilde{X};\mathbb{Q}) \otimes \mathbb{R} \cong H^{even}(\Omega^*(\widetilde{X}))$.

In this section we will suppose that the subcomplex $\mathcal{F}^{even}(\widetilde{X}) \subset \Omega^{even}(\widetilde{X})$ verifies similar homological hypothesis, that is there exists a rational lattice $A_{\mathbb{Q}} \subset H^{even}(\mathcal{F}^*(\widetilde{X}))$ such that $A_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong H^{even}(\mathcal{F}^*(\widetilde{X}))$ and such that the map $w_{\mathbb{R}} : H^{even}(\mathcal{F}^*(\widetilde{X})) \to H^{even}(\Omega^*(\widetilde{X}))$ is induced by a $\mathbb{Q}$-linear map $w_{\mathbb{Q}} : A_{\mathbb{Q}} \to H^{even}(\widetilde{X};\mathbb{Q})$ between the lattices

$$\begin{array}{ccc} H^{even}(\mathcal{F}^\star(\widetilde{X})) & \overset{w_{\mathbb{R}}}{\to} & H^{even}(\Omega^*(\widetilde{X})) \\ \uparrow\cong & & \uparrow\cong \\ A_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} & \overset{w_{\mathbb{Q}}\otimes 1_{\mathbb{R}}}{\to} & H^{even}(\widetilde{X};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}. \end{array}$$

Let us denote by

- $u_{\mathbb{R}} : \mathcal{K}_{\mathbb{R}}(\widetilde{X}) \to K(\widetilde{X}) \otimes \mathbb{R}$ the forgetful homomorphism given by $u_{\mathbb{R}}([E,\nabla,a,\omega]) = [E]\otimes a$ where $[E]$ denotes the K-theory class of the bundle E;
- $v_{\mathbb{R}} : H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(X)}\right) \to \mathcal{K}_{\mathbb{R}}(\widetilde{X})$ such that $v_{\mathbb{R}}(cl(\omega)) = [T,\partial,1,\omega] - [T,\partial,1,0] = [T,\partial,0,\omega]$.

The real multiplicative K-theory satisfies (compare with Theorem 2.1) :

Theorem 3.1 *Suppose that the subcomplex $\mathcal{F}^*(\widetilde{X}) \subset \Omega^*(\widetilde{X})$ satisfies the rational homological property defined above. Then we have the following diagram of exact sequences of $\mathbb{R}$-vector spaces ($ev = even$)*

$$\begin{array}{ccccccccc} K^1(\widetilde{X})\otimes\mathbb{R} & \overset{ch_*\otimes 1}{\to} & H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(X)}\right) & \overset{v_{\mathbb{R}}}{\to} & \mathcal{K}_{\mathbb{R}}(\widetilde{X}) & \overset{u_{\mathbb{R}}}{\to} & K(\widetilde{X})\otimes\mathbb{R} & \overset{ch\otimes 1}{\to} & H^{ev}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(X)}\right) \\ \cong\downarrow ch_*\otimes 1 & & \downarrow= & & \downarrow \overline{ch_*} & & \cong\downarrow ch_*\otimes 1 & & \downarrow= \\ H^{odd}(\Omega^*(\widetilde{X})) & \to & H^{odd}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(X)}\right) & \to & H^{ev}(\mathcal{F}^*(\widetilde{X})) & \to & H^{ev}(\Omega^*(\widetilde{X})) & \to & H^{ev}\left(\frac{\Omega^*(\widetilde{X})}{\mathcal{F}^*(X)}\right). \end{array}$$

In particular

$$\overline{ch_*} : \mathcal{K}_{\mathbb{R}}(\widetilde{X}) \to H^{ev}(\mathcal{F}^*(\widetilde{X}))$$

is an isomorphism.

Before giving the proof of this theorem, let us come back to the intersection cohomology theory. If in Subsection 2.2 we start with $C^*(\widetilde{X};\mathbb{Q})$ instead of $\Omega^*(\widetilde{X})$, the result of Proposition 2.1 holds and we get a rational intersection

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complex $I_qC^*(\widetilde{X};\mathbb{Q}) \subset C^*(\widetilde{X};\mathbb{Q})$. The cohomology of $I_qC^*(\widetilde{X};\mathbb{Q})$ is the rational intersection cohomology and it follows from the commutative diagram

$$\begin{array}{ccc} IH^*_{\bar{p}}(X,\mathbb{R}) \cong H^*(\mathcal{F}^\star_q(\widetilde{X})) & \to & H^*(\Omega^*(\widetilde{X})) \cong H^*(\widetilde{X},\mathbb{R}) \\ \uparrow & & \uparrow \\ IH^*_{\bar{p}}(X,\mathbb{Q}) \cong H^*(I_qC^*(\widetilde{X};\mathbb{Q})) & \to & H^*(\widetilde{X};\mathbb{Q}) \cong H^*(\widetilde{X},\mathbb{Q}) \end{array}$$

that the intersection complex $\mathcal{F}^\star_q(\widetilde{X})$ satisfies the above rational homological property. Theorem 3.1 implies:

Corollary 3.1 *Let X be an N-dimensional pseudomanifold with one isolated conical singularity. Denote by $\mathcal{K}_{\bar{p},\mathbb{R}}(\widetilde{X})$ the real multiplicative K-theory associated to the particular subcomplex $\mathcal{F}^\star_{\bar{p}}(\widetilde{X})$ defined in Section 2.2. Then the real multiplicative Chern character*

$$\overline{ch}_* : \mathcal{K}_{\bar{p},\mathbb{R}}(\widetilde{X}) \to IH^{even}_{\bar{p}}(X,\mathbb{R})$$

defined by $\overline{ch}_*([E,\nabla,a,\omega]) = [a\,ch_*(\nabla) - d\omega]$, is an isomorphism.

For the proof of Theorem 3.1 we use the following computations.

Lemma 3.1 *(We recall that $\mathcal{F}_0 \neq 0$).*

1. *Let $E \oplus E' = T$ be a trivial bundle and ∇' a connection over E', then*

$$([E,\nabla,a,\omega] - [rk(E),\partial,a,0]) + ([E',\nabla',a,-\omega] - [rk(E'),\partial,a,0]) = 0$$

and thus

$$\begin{aligned} [E,\nabla,a,\omega] - [rk(E),\partial,a,0] &= -([E',\nabla',a,-\omega] - [rk(E'),\partial,a,0]) \\ &= [E',\nabla',-a,\omega] - [rk(E'),\partial,-a,0]. \end{aligned}$$

2. $[E,\nabla,a+b,\omega] = [E,\nabla,a,\frac{a\omega}{a+b}] + [E,\nabla,b,\frac{b\omega}{a+b}]$.

3. *For all q positive integer, the following relation holds*

$$[qE,q\nabla,a,qw] = [E,\nabla,qa,qw] = q[E,\nabla,a,w]\ .$$

4. *Let $r_i = rk(E_i)$, $r = \sum_{finite} r_i$ and $\sum_{finite} r_i a_i = ra$, then*

$$\sum_{finite} [E_i,\nabla_i,a_i,\omega_i] = [r,\partial,a,0] + \sum_{finite} [E_i,\nabla_i,a_i,\omega_i] - [r_i,\partial,a_i,0]\ .$$

5. *In the previous formula the a_i's can be assumed to be rationally free.*

First let us give the proof of Theorem 3.1.

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Proof: Let us verify that in the first sequence the composition of two homomorphisms is 0. In the commutative diagram the fourth and fifth vertical arrows are obviously isomorphisms, thus $(ch \otimes 1) \circ u_{\mathbb{R}} = 0 = u_{\mathbb{R}} \circ v_{\mathbb{R}}$. Notice that $v_{\mathbb{R}}$ is the composition of v with the natural homomorphism $\mathcal{K}(\widetilde{X}) \mapsto \mathcal{K}_{\mathbb{R}}(\widetilde{X})$ and so $v_{\mathbb{R}} \circ (ch_* \otimes 1) = 0$ results immediately from Theorem 2.1.

Now let us prove exactness at $K(\widetilde{X}) \otimes \mathbb{R}$. Let $x = \sum_{i \in I} [E_i] \otimes a_i \in K(\widetilde{X}) \otimes \mathbb{R}$ such that $x \in \ker(ch \otimes 1)$. The a_i's can be chosen rationally independent. The real vector space $V = H^*(\Omega^*(\widetilde{X})/\mathcal{F}^\star(\widetilde{X}))$ contains a rational vector subspace $A \subset V$ which is the image of the rational subspace $H^*(\widetilde{X}, \mathbb{Q}) \subset H^*(\Omega^*(\widetilde{X}))$ by the map $H^*(\Omega^*(\widetilde{X})) \to H^*(\Omega^*(\widetilde{X})/\mathcal{F}^\star(\widetilde{X}))$. Let $\sum a_i R_i = 0$ be a vanishing linear combination where $R_i = \overline{ch_*[E_i]} \in A$ (for $* > 0$). Let B_k be a basis (over $\mathbb{Q}$) of A. Decompose $R_j = \sum_k r_{j,k} B_k$ where, by hypothesis, the $r_{j,k}$'s are rational numbers. We deduce that $\sum_{i,k} a_i r_{i,k} B_k = 0$. The rational homological property of $\mathcal{F}^\star(\widetilde{X})$ implies that the B_k's are independent over $\mathbb{R}$, hence, for all k, the sum $\sum_{i,k} a_i r_{i,k} = 0$. The rational linear independence of the a_i's implies $r_{i,k} = 0$ for all i,k. Thus for all i, we have $R_i = \overline{ch_*[E_i]} = 0$ and the exactness of the Karoubi sequence (2.1) gives $x = u_{\mathbb{R}}(\sum_{i \in I} a_i [E_i, \nabla_i, 1, \omega_i])$.

Now let us prove exactness at $\mathcal{K}_{\mathbb{R}}(\widetilde{X})$. According to (4) and (5) of 3.1, we can write an element $x \in \mathcal{K}_{\mathbb{R}}(\widetilde{X})$ as

$$x = [r, \partial, a, 0] + \sum_{i \in I} [E_i, \nabla_i, a_i, w_i] = \sum_{i \in I} ([E_i, \nabla_i, a_i, w_i] - [r_i, \partial, a_i, 0])$$

where I is a finite set, $r_i = rk(E_i)$, $r = \sum_{i \in I} r_i$, $\sum_{i \in I} r_i a_i = ra$, and the a_i's are rationally independent. Suppose $x \in \ker u_{\mathbb{R}}$. Then the relation $u_{\mathbb{R}}(x) = \sum_{i \in I} ([E_i] - [r_i]) \otimes a_i + [r] \otimes a = 0$ holds. So there exist integers $k_i \neq 0$ such that the bundles $k_i E_i$'s are trivial over $\widetilde{X}$ and $r = 0$. For each integer i, there exists an isomorphism $\alpha_i : k_i E_i \cong T_i$ such that:

$$[k_i E_i, k_i \nabla_i, a_i, k_i w_i] - [k_i r_i, \partial, a_i, 0] = [T_i, \partial, a_i, \epsilon_i] - [T_i, \partial, a_i, 0] = [T_i, \partial, 0, \epsilon_i]$$

with $\epsilon_i = k_i \omega_i + a_i \displaystyle\int_0^1 ch_*(t\partial + (1-t)\alpha_i^\star \nabla_i) dt$. The relations

$$\begin{aligned} k_i [E_i, \nabla_i, a_i, w_i] - k_i [r_i, \partial, a_i, 0] &= a_i [T_i, \partial, 1, \epsilon_i] - a_i [T_i, \partial, 1, 0] \\ [E_i, \nabla_i, a_i, w_i] - [r_i, \partial, a_i, 0] &= \tfrac{a_i}{k_i} ([T_i, \partial, 1, \epsilon_i] - [T_i, \partial, 1, 0]), \end{aligned}$$

imply that

$$x = \sum_{i \in I} \frac{a_i}{k_i} v_{\mathbb{R}}(cl(\bar{\epsilon}_i)) = v_{\mathbb{R}} \left(\sum_{i \in I} \frac{a_i}{k_i} cl(\bar{\epsilon}_i) \right).$$

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Finally let $c = cl(\omega) \in H^{odd}(\Omega^\star(\widetilde{X})/\mathcal{F}^\star(\widetilde{X}))$ such that $v_{\mathbb{R}}(c) = 0$. According to condition (C_2), there exists a finite sequence of $a_j \in \mathbb{R}$ and a finite sequence of automorphisms α_j of a trivial bundle over $\widetilde{X}$ such that $\omega = \sum_j a_, \Theta(\partial, \alpha_j^* \partial)$ modulo $\widetilde{\mathcal{F}}^\star(\widetilde{X})$, and $c = (ch_* \otimes 1)\left(\sum_j \alpha_j \otimes a_j\right)$. □

It remains to prove Lemma 3.1.

Proof: 1. Let us remark that, modulo a coboundary, we have $ch(\nabla + \nabla') = rk(T) \in \mathbb{Z} = \mathcal{F}_0$. Thus $ch(\nabla') = -ch(\nabla)$ modulo ($\mathcal{F}+$ coboundaries) and $ch(\nabla) = dw$ implies that $ch(\nabla') = -dw$ modulo ($\mathcal{F}+$ coboundaries). Hence $[E', \nabla', a, -w]$ is well defined. Now

$$Q = ([E, \nabla, a, w] - [rk(E), \partial, a, 0]) + ([E', \nabla', a, -w] - [rk(E'), \partial, a, 0])$$

verifies

$$\begin{aligned} Q &= [E + E', \nabla + \nabla', a, 0] - [T, d, a, 0] \\ &= [T, d, a, 0] - [T, d, a, 0] \\ &= 0 \end{aligned}$$

and $[E', \nabla', a, -w] = -[E, \nabla, a, w] = [E, \nabla, -a, -w]$.
2. and 3. are immediate.
4. results from the following computation

$$\begin{aligned} \textstyle\sum_{finite} [r_i, d, a_i, 0] &= \textstyle\sum_{finite} a_i [r_i, d, 1, 0] = \sum_{finite} a_i r_i [1, d, 1, 0] \\ &= ar[1, d, 1, 0] = a[r, d, 1, 0] = [r, d, a, 0]\,. \end{aligned}$$

5. Let us prove that it is possible to reduce the number of a_i as long as they are not $\mathbb{Q}$-free. Consider a relation $\sum_{finite} q_i a_i = 0$ where the q_i's are all non zero integers. We divide by the product of the q_i's so the relation becomes $a_1/p_1 = -\sum_{i>1} a_i/p_i = S$. The sign of the a_i's can be chosen so that $p_i > 0$ for all i. The relation

$$[E_1, \nabla_1, a_1, w_1] \overset{3}{=} [p_1 E_1, p_1 \nabla_1, \frac{a_1}{p_1}, w_1] \overset{2}{=} \sum_{i>1} \left[p_1 E_1, p_1 \nabla_1, -\frac{a_i}{p_i}, -\frac{a_i w_1}{p_i S} \right]$$

implies

$$\begin{aligned} &\sum_{finite} [E_i, \nabla_i, a_i, w_i] - [r_i, d, a_i, 0] \\ &= \sum_{i>1} \left[p_1 E_1' + p_i E_i, p_1 \nabla_1' + p_i \nabla_i, \frac{a_i}{p_i}, w_i + \frac{a_i w_1}{p_i S} \right] - \sum_{i>1} [p_i r_i, \partial, \frac{a_i}{p_i}, 0]. \end{aligned}$$

In this last relation the number of a_i decreased. □

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4. Integer intersection K-theory of isolated conical singularities

We propose an integer intersection K-theory underlying the previous real intersection K-theory and build an associated intersection Chern character.

4.1. Admissible bundles and integer intersection K-theory

The integer multiplicative theory of M. Karoubi is not of finite type. We follow another way to define an integer theory.

Consider again an integer $0 \leq q \leq N$, a perversity $\bar{p}$ such that $p_N = N - q - 2$, a manifold $\widetilde{X}$ with boundary L and $X = \widetilde{X}/L$ the associated pseudomanifold with one conical singularity. The following constructions and results are valid for compact pairs of CW-complexes $(\widetilde{X}, L)$ such that the point $L/L \in \widetilde{X}/L$ admits a conical neighborhood with base L.

Definition 4.1 Let $q \geq 0$ be a fixed integer. A q-admissible bundle over $\widetilde{X}$ is a triple (E,F,t) such that :

1. $F \to \widetilde{X}$ is a subbundle of $E \to \widetilde{X}$ such that

$$\begin{array}{ll} ch_j[E] = ch_j[F] & if\ 2j \geq q+2\,, \\ ch_j[F] = 0 & if\ 0 < 2j \leq q+1\,. \end{array}$$

2. The restriction $F_{|L}$ of F over L is trivial and $t : F_{|L} \to C^{rk(F)} \times L$ is a trivialisation.

3. Moreover, if q is odd, we assume that $ch_{(q+1)/2}(E_{|L}) = 0$.

In particular $ch_j(E_{|L}) = 0$ when $2j \geq q+1$ and we say that E is rationally q-flat over the boundary.
The bundle deduced from $F \to \widetilde{X}$ by collapsing $F_{|L}$ in one fibre over L/L, using t, is denoted by $F/t \to X$.

The Whitney sum endows the set of q-admissible bundles with a semigroup structure

$$(E,F,t) \oplus (E',F',t') = (E \oplus E', F \oplus F', t \oplus t')\,.$$

We say that two q-admissible bundles (E,F,t) and (E',F',t') are isomorphic if there exists an isomorphism $\alpha : (E,F) \to (E',F')$ of pairs such that α restricted to $F_{|L}$ is exactly $t'^{-1} \circ t : F_{|L} \to F'_{|L}$. In particular the following diagram is

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commutative

$$\begin{array}{ccc} E & \stackrel{\alpha}{\to} & E' \\ \uparrow & & \uparrow \\ F_{|L} & \stackrel{\alpha_{|L}}{\to} & F'_{|L} \\ \downarrow t & & \downarrow t' \\ L\times\mathbb{C}^n & \stackrel{=}{\to} & L\times\mathbb{C}^n\ . \end{array}$$

This relation is compatible with Whitney sum. Let us denote the associated symmetrized group by $\widetilde{IK_{\bar{p}}}(X)$ and the isomorphism class of (E,F,t) by $[E,F,t]$.

Let us recall that an automorphism of the trivial bundle $t:\mathbb{C}^n\times L\mapsto\mathbb{C}^n\times L$ defines an element of $K^1(L)$ and all elements of $K^1(L)$ admit such a representation for n large enough.

The cofibration $L\subset\widetilde{X}\mapsto\widetilde{X}/L=X$ determines the following commutative diagram

$$\begin{array}{ccccccccc} K^1(\widetilde{X}) & \to & K^1(L) & \stackrel{\delta}{\to} & K^0(X) & \to & K^0(\widetilde{X}) \\ \downarrow ch_* & & \downarrow ch_* & & \downarrow ch_* & & \downarrow ch_* \\ H^{odd}(\widetilde{X},\mathbb{Q}) & \to & H^{odd}(L,\mathbb{Q}) & \stackrel{\delta}{\to} & H^{even}(X,\mathbb{Q}) & \to & H^{even}(\widetilde{X},\mathbb{Q}) \end{array}$$

where $\delta[t]=[n/t]$ and thus $\delta ch_*[t]=ch[n/t]$ (n denotes the trivial bundle of rank n and $[n]$ its K-theory class).

Introduce S_q the subgroup of $\widetilde{IK_{\bar{p}}}(X)$ spanned by the formal differences $[n,m,t]-[n,m',t']$ with $m,m'\le n$ and t, t' such that

$$\delta ch_j t=\delta ch_j t'=0\quad when\ \ 2j\ge q+2$$

(recall that $ch_j:K^1(\widetilde{X})\to H^{2j-1}(\widetilde{X};\mathbb{Q})$).

Notation 4.1 *Let* $IK_{\bar{p}}(X)=\dfrac{\widetilde{IK_{\bar{p}}}(X)}{S_q}$ *be the quotient group. We still denote by* $[E,F,t]$ *the elements of this quotient group.*

The class $[E,F,t]$ of an admissible bundle (E,F,t) has the following characterization.

Proposition 4.1 *(Stable equivalence). Two q-admissible bundles* (E,F,t) *and* (E',F',t') *are in the same class in* $IK_{\bar{p}}(X)$ *if and only if there exist positive integers* n,p_1,p_2 *with* $p_1,p_2\le n$ *and two automorphisms* $t_i:[p_i]\to[p_i]$, $i=1,2$, *with* $\delta ch_j t_i=0$ *when* $2j\ge q+2$, *such that*

$$(E\oplus n,F\oplus p_1,t\oplus t_1)\cong(E'\oplus n,F'\oplus p_2,t'\oplus t_2).$$

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The proof of this proposition follows from the classical one according to the following properties.

Lemma 4.1 *1. For all q-admissible bundles (E,F,t) there is a q-admissible bundle (E',F',t') such that $(E\oplus E',F\oplus F',t\oplus t')\cong(n,m,id)$.*

2. All elements of $\widetilde{IK}_{\bar{p}}(X)$ can be written $[E,F,t]-[n,m,id]$.

Proof: 1. Let $F_0\to X$ be a bundle such that $F/t\oplus F_0$ is trivial. Consider the pull back F' of F_0 over $\widetilde{X}$ with t' a trivialisation of $F'_{|L}$. The trivialisation t' is defined modulo an isomorphism between the fiber over the singular point and $\mathbb{C}^{rk(F')}$. Changing this isomorphism does not alter the class because the group $U_{rk(F')}(\mathbb{C})$ is connected. Hence $F/t\oplus F'/t'=m$ is trivial. Now let us choose F_1 such that $F\oplus F_1=E$ and let F'_1 be such that $F_1\oplus F'_1=m_1$. Then

$$\begin{aligned}(E'=F'_1\oplus F',F',t')\oplus(E,F,t)&=(E'=F'_1\oplus F',F',t')\oplus(F_1\oplus F,F,t)\\&=(m+m_1,m,id).\end{aligned}$$

2. The second assertion is proved using the first one by the same argument as in the classical K-theory. □

Now let us construct admissible bundles. We define a homomorphism $IK_{\bar{p}}(X)\to K(\widetilde{X})$ by associating to an admissible bundle (E,F,t) the class $[E]\in K(\widetilde{X})$ of the bundle E. By hypothesis E is rationally q-flat over the boundary (see Definition 4.1). Conversely let $E\to\widetilde{X}$ be a bundle rationally q-flat over the boundary. We will prove that a stable multiple of this bundle provides us with the first term of an admissible bundle.

Proposition 4.2 *Consider a bundle $E\to\widetilde{X}$ rationally q-flat over the boundary. Then there exist integers p,n and bundles F_1 and F_2 over $\widetilde{X}$ such that*

- *$pE\oplus n=F_1\oplus F_2$,*
- *$(F_2)_{|L}$ is trivial,*
- *$(pE\oplus n,F_2,t)$ is q-admissible for any trivialisation t of $(F_2)_{|L}$.*

Proof: Denote by $[E]_{\mathbb{Q}}$ the rational class of the bundle E. The elements of $K(\widetilde{X})$ can be written $[E]-[n]$. Thus the elements of $K(\widetilde{X})\otimes\mathbb{Q}$ can be written $1/p([E]_{\mathbb{Q}}-[n]_{\mathbb{Q}})=([E]-[n])\otimes 1/p$. Hence :

- for all classes $C\in K(\widetilde{X})\otimes\mathbb{Q}$, there are integers p,n and a bundle E such that $pC=[E]_{\mathbb{Q}}-[n]_{\mathbb{Q}}$.

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- $[E]_{\mathbb{Q}} = [F]_{\mathbb{Q}}$ implies that $[E]-[F]$ is a torsion element and there is an integer k such that $k([E]-[F]) = 0$. Thus there is an integer m such that $kE + m = kF + m$. In particular if $[E]_{\mathbb{Q}} = [rk(E)]_{\mathbb{Q}}$ there are two integers k,m such that $kE + m = k(rk(E)) + m$ is trivial.

Define $G_1 = ch_*^{-1}(\oplus_{2j \leq q+1} H^{2j}(\widetilde{X},\mathbb{Q}))$ and $G_2 = ch_*^{-1}(\oplus_{2j \geq k+2} H^{2j}(\widetilde{X},\mathbb{Q}))$ then $K(\widetilde{X}) \otimes \mathbb{Q} = G_1 \oplus G_2$. Given a bundle $E \to \widetilde{X}$ rationally q-flat over the boundary we have $[E]_{\mathbb{Q}} = C_1 + C_2$, where $C_i \in G_i$, with the relations :

- $(ch_{|L})_{2j}(C_1)_{|L} = 0$ for $2j = q+1$ (empty condition if q is even);

- $(ch_{|L})_{2j}(C_2)_{|L} = 0$ for $2j \geq q+2$.

Now let us introduce :

- an integer p_1 such that $p_1 C_1 = [E_1]_{\mathbb{Q}} - [n_1]_{\mathbb{Q}}$;

- an integer p_2 such that $p_2 C_2 = [E_2]_{\mathbb{Q}} - [n_2]_{\mathbb{Q}}$.

The bundle $(E_2)_{|L}$ is rationally trivial and there are integers q,m such that $q(E_2)_{|L} + m$ is trivial. Set $E_2' = qE_2 + m$ and $p' = p_1 p_2 q$, $F_1' = p_2 q E_1$, $F_2' = p_1 q E_2 + p_1 m$ and $n' = p_1 q n_2 + p_2 q n_1 + p_1 m$. Then

$$\begin{aligned} p'[E]_{\mathbb{Q}} = p'(C_1 + C_2) &= [p_2 q E_1 \oplus p_1 E_2']_{\mathbb{Q}} - [p_2 q n_1 + p_1 q n_2 + p_1 m]_{\mathbb{Q}} \\ &= [F_1' \oplus F_2']_{\mathbb{Q}} - [n']_{\mathbb{Q}} \,. \end{aligned}$$

□

4.2. Intersection Chern character

We will define in a natural way the intersection Chern character of an admissible bundle (E,F,t) with range in the rational intersection cohomology. This Chern character is rationally an isomorphism (Theorem 4.1). Admissible bundles are just what you need to define multiplicative bundles (Theorem 4.2) and the associated real multiplicative Chern character (see Corollary 3.1) computes the rational intersection Chern character. Recall that N is the dimension of the singular manifold $X = \widetilde{X}/L$.

Proposition 4.3 *Let $\bar{p}$ be a perversity and $\bar{q}$ be the complementary perversity and let $0 < q = q_N \leq N-2$. Then the intersection Chern character*

$$ch' : IK_{\bar{p}}(X) \to IH_{\bar{p}}^{even}(X,\mathbb{Q})$$

with range in the rational intersection cohomology of X is defined by

$$ch'[E,F,t] = \oplus_j ch'_j[E,F,t]$$

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where

$$ch'_j[E,F,t] := \begin{cases} ch_j[E] \in H^{2j}(\widetilde{X},\mathbb{Q}) & \forall\ 2j \le q+1 \\ ch_j[F/t] \in H^{2j}(X,\mathbb{Q}) & \forall\ 2j \ge q+2\,. \end{cases}$$

Proof: The intersection Chern character is well defined because ch' vanishes on S_q. This follows from the fact that the differences $[n,m,t]-[n,m',t']$, such that $ch_j([n/t]) = ch_j([n/t']) = 0$ for $2j \ge q+2$, span S_q, and their rank is $n-n=0$.

To prove that $ch'[E,F,t] \in IH_{\bar{p}}^{even}(X,\mathbb{Q})$, the only point we have to check is that for q odd, $ch_{\frac{q+1}{2}}[E] \in Im(H^{q+1}(X,\mathbb{Q}) \mapsto H^{q+1}(\widetilde{X},\mathbb{Q}))$. This results from the following commutative diagram using $ch_{\frac{q+1}{2}}[E_{|L}] = 0$:

$$\begin{array}{ccccc} K(X) = K(\widetilde{X}/L) & \to & K(\widetilde{X}) & \to & K(L) \\ \downarrow ch_{\frac{q+1}{2}} & & \downarrow ch_{\frac{q+1}{2}} & & \downarrow ch_{\frac{q+1}{2}} \\ H^{q+1}(X,\mathbb{Q}) & \to & H^{q+1}(\widetilde{X},\mathbb{Q}) & \to & H^{q+1}(L,\mathbb{Q})\,. \end{array}$$

□

The admissible bundles give a convenient integer intersection K-theory because of the following result.

Theorem 4.1 *The intersection Chern character ch' induces an isomorphism*

$$ch' \otimes 1 : IK_{\bar{p}}(X) \otimes \mathbb{Q} \mapsto IH_{\bar{p}}^{even}(X,\mathbb{Q}).$$

Proof: Denote $q = N - p_N - 2$. Let us show the surjectivity.
Suppose $0 < 2j \le q$. Let $c \in IH_{\bar{p}}^{2j}(X,\mathbb{Q}) = H^{2j}(\widetilde{X},\mathbb{Q})$. For degree $j=0$ the rank gives the result. Because $ch \otimes 1 : K(\widetilde{X}) \otimes \mathbb{Q} \mapsto H^{even}(\widetilde{X},\mathbb{Q})$ is an isomorphism, it follows that $c = ch_*(\frac{1}{n}([E]_{\mathbb{Q}} - [k]_{\mathbb{Q}})) = \frac{1}{n} ch_j[E]$ where $\frac{1}{n}([E]_{\mathbb{Q}} - [k]_{\mathbb{Q}}) \in K(\widetilde{X}) \otimes \mathbb{Q}$ with $ch_i[E] = 0$ when $i \neq 0, j$. Now there are two positives integers p,m and a subbundle F_2 such that $(F_2)_{|L}$ is trivial and for all trivialisation t of $(F_2)_{|L}$, we have $[pE \oplus m, F_2, t] \in IK_q(X)$ (Proposition 4.2). Let us compute $ch'_i[pE \oplus m, F_2, t]$.

For $2i \le q+1$, we have $ch'_i[pE \oplus m, F_2, t] = ch_i[pE \oplus m] = pnc$ when $i = j$ and 0 otherwise.

For $2i > q+1$, $ch'_i[pE \oplus m, F_2, t] = ch_i[F_2/t]$ and we know that $ch_i[F_2] = pch_i[E] = 0$. Consider the commutative diagram where the vertical arrows are isomorphisms,

$$\begin{array}{ccccc} K^1(L) \otimes \mathbb{Q} & \stackrel{b}{\to} & K(\widetilde{X}/L) \otimes \mathbb{Q} & \to & K(\widetilde{X}) \otimes \mathbb{Q} \\ \downarrow ch_* \otimes 1 & & \downarrow ch_* \otimes 1 & & \downarrow ch_* \otimes 1 \\ H^{odd}(L,\mathbb{Q}) & \to & H^{even}(\widetilde{X}/L,\mathbb{Q}) & \to & H^{even}(\widetilde{X},\mathbb{Q})\,. \end{array}$$

The class $[F_2/t]_{\mathbb{Q}}$ belongs to the image of $K^1(L) \otimes \mathbb{Q}$ by the boundary operator b of the first exact sequence. So there exists a trivialisation t_0 such that $[F_2/t_0]_{\mathbb{Q}} = 0$.

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Indeed $t't^{-1} \in K^1(L)$ acting via the boundary operator on $[F/t]_{\mathbb{Q}}$ gives $[F/t']_{\mathbb{Q}}$. Consider $\tau/k \in K^1(L) \otimes \mathbb{Q}$ such that $b(\tau) = k[F_2/t]_{\mathbb{Q}}$ and define t_0 by the relation $\tau^{-1} = t_0 t^{-1}$. Hence we have $ch_i[pE \oplus m, F_2, t_0] = 0$.
Thus all classes $c \in IH_{\bar{p}}^{2j}(X,\mathbb{Q})$ where $0 < 2j \leq q$ belong to the range of ch'.

When $2j = q+1$ (q odd) the same arguments as the previous ones imply that $c \in IH_{\bar{p}}^{q+1}(X,\mathbb{Q})$ belongs to the range of ch'.

Suppose now $2j \geq q+2$. Then $c \in IH_{\bar{p}}^{2j}(X,\mathbb{Q}) = H^{2j}(X,\mathbb{Q})$ and there exists $\frac{1}{n}([F_2]_{\mathbb{Q}} - [k]_{\mathbb{Q}}) \in K(X) \otimes \mathbb{Q}$ such that $ch(\frac{1}{n}([F_2]_{\mathbb{Q}} - [k]_{\mathbb{Q}})) = ch_j(\frac{1}{n}([F_2]_{\mathbb{Q}} - [k]_{\mathbb{Q}})) = c$. Let us denote by $F \mapsto \widetilde{X}$ the pull back of $F_2 \mapsto X$. The bundle $F_{|L}$ admits a trivialisation t such that $[F/t] = [F_2]$. The triple (F,F,t) is admissible because $F_{|L}$ is trivial and $ch_i[F] = 0$ for all $0 < i \leq q+1$. The surjectivity results from the relation

$$ch'([F,F,t]_{\mathbb{Q}} - [k,k,id]_{\mathbb{Q}}) = ch_j([F/t]_{\mathbb{Q}} - [k]_{\mathbb{Q}}) = nc.$$

Now let us show that ch' is injective. Let $z = \frac{1}{r}([E,F,t]_{\mathbb{Q}} - [n,m,id]_{\mathbb{Q}}) \in \widetilde{IK}(X) \otimes \mathbb{Q}$. Suppose that

$$0 = ch'_j(\bar{z}) = \begin{cases} ch_j[E] \otimes \frac{1}{r} = 0 & for \quad 2j \leq q+1 \\ ch_j[F/t] \otimes \frac{1}{r} = 0 & for \quad 2j \geq q+2\,, \end{cases}$$

and let F_1 be a bundle such that $F \oplus F_1 = E$. The Chern character verifies $ch_j[E] = 0 = ch_j[F_1]$ when $0 < 2j \leq q+1$ and $ch_j[F/t] = 0$ when $2j \geq q+2$. Therefore $ch_j[F] = 0$ when $j > 0$. There exist integers p,s,p',s',n,n' such that $[p'F_1' \oplus s'] = [n']$ and $[pF \oplus s] = [n]$, and

$$\begin{aligned} &pp'[E,F,t] \oplus [ps' + p's, p's, id] \\ &= [p'n + pn' + ps' + p's, p'n + p's, pp't \oplus id] = [m, n_1, t_1]. \end{aligned}$$

As $\delta ch_j t_1 = \delta ch_j t = ch_j[F/t] = 0$ for all $2j \geq q+2$, we obtain $[E,F,t] \in S_q$, and thus ch' is injective. □

Now let us show that the intersection Chern character is determined by the real multiplicative character of a suitable connection.

A connection ∇ on the first argument E of an admissible bundle (E,F,t) is called admissible if:

- ∇ induces a connection ∇_0 over F,
- the restriction of ∇_0 over a collar neightborhood U of the boundary can be identified with the pull back connection $t^*\partial$ where ∂ is the de Rham connection on the trivial bundle $U \times \mathbb{C}^{rk(F)}$.

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Theorem 4.2 *A q-admissible bundle (E,F,t) always admits an admissible connection ∇ over E. For all admissible connections over E there exists a form $\omega \in \Omega^{odd}(\widetilde{X})$ such that the triple $(E,\nabla,1,\omega)$ is a real multiplicative bundle. The class on $\mathcal{K}_{q,\mathbb{R}}(\widetilde{X})$ of this multiplicative bundle does not depend on the class of (E,F,t) on $IK_{\bar{p}}(X)$ and we have the relation*

$$ch' \otimes 1_{\mathbb{R}}([E,F,t] \otimes 1) = \overline{ch}([E,\nabla,1,\omega]).$$

Proof: An admissible connection is obtained in the following way: choose a connection ∇_0 over F extending the connection $t^*\partial$ and consider any connection ∇_1 over a sub-bundle F_1 such that $F \oplus F_1 = E$. Then $\nabla = \nabla_0 + \nabla_1$ is an admissible connection over $E = F \oplus F_1$.

Let ∇ be an admissible connection over E. We will construct the form ω_* such that $ch_j(\nabla) = d\omega_j$ modulo $\mathcal{F}_q^*(\widetilde{X})$.

- For $2j \leq q$ we define $\omega_j = 0$.

- For $2j \geq q+2$ the relation $[ch_j(\nabla)] = [ch_j(\nabla_F)] \in H^{even}(\Omega^*(\widetilde{X}))$ implies that there is a $(2j-1)$-form ω_j such that $ch_j(\nabla) - ch_j(\nabla_F) = d\omega_j$.

- For $2j = q+1$ (q odd), we have $ch_j(\nabla)_{|L} = d((\omega_j)_{|L})$ because $ch_j(E_{|L} \otimes 1) = 0$ and the $(q+1)$-form $ch_j(\nabla) - d\omega_j$ restricted to L is 0 and thus belongs to $\mathcal{F}_q^{q+1}(\widetilde{X})$.

Thus $(E,\nabla,1,\omega)$ is a real multiplicative bundle such that $\overline{ch}_j(E,\nabla,1,\omega) = (ch_j(\nabla) - d\omega_j) \in \mathcal{F}_q^*(\widetilde{X})$ which is the relation of the theorem. As $\overline{ch}$ and $ch' \otimes 1$ are isomorphisms (Theorems 3.1 and 4.1), the class $[E,\nabla,1,\omega] \in \mathcal{K}_{q,\mathbb{R}}(\widetilde{X})$ is well defined. □

5. Some computations of intersection K-theory groups

The notations are those of the previous section.

5.1. Intersection K-theory for extremal perversities

The previous constructions are associated to a perversity $\bar{p}$ but only depend on the integer $q = N - p_N - 2$ that can be considered as the complementary perversity $\bar{q}$ (i.e. $q = q_N$) in the sense of Goresky-MacPherson, [10]. Notice that the previous K-theory constructions are valid for $0 \leq q \leq N = dim(X)$. Let us remark that $q = 0$ corresponds to the total perversity $\bar{p} = tot$ but $q = N = \dim(X)$ is not a Goresky-MacPherson perversity. Only $0 \leq q \leq N-2$ is allowable for a Chern

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character with values in the intersection cohomology of Goresky-MacPherson. For $q = N$ we note $\bar{p} = \overline{-2}$ the " complementary perversity". It is not a Goresky-MacPherson perversity.

Proposition 5.1 *Let X be a pseudomanifold with an isolated conical singularity. If the group $K(\widetilde{X})$ (or $H^{even}(\widetilde{X},\mathbb{Z})$) is torsion-free, we have the isomorphisms*

$$IK_{tot}(X) \cong K(X)\ ;\ IK_{\overline{-2}}(X) \cong K(\widetilde{X}).$$

Without the torsion-free assumption we can only assume the inclusions

$$K(X) \subset IK_{tot}(X)\ ;\ K(\widetilde{X}) \subset IK_{\overline{-2}}(X).$$

Proof: Let $p : \widetilde{X} \mapsto \widetilde{X}/L = X$ be the projection. The pull back $p^*E \to \widetilde{X}$ of a locally trivial bundle $E \to X$ restricted to the boundary L is trivial. Given an isomorphism $u : E_s \mapsto \mathbb{C}^{rk(E)}$ of the fibre of E over the singularity $s = L/L$ with a numerical space, we define a trivialisation

$$t_u : p^*E/L = E_s \times L \xrightarrow{u\times 1} \mathbb{C}^{rk(E)} \times L.$$

The class $[p^*(E),p^*(E),t_u] \in IK_{tot}(X)$ does not depend on the isomorphism u chosen because $GL_{rk(E)}(\mathbb{C})$ is connected. We define a homomorphism $\varphi : K(X) \longmapsto IK_{tot}(X)$ by $\varphi[E] = [p^*(E),p^*(E),t_u]$. Conversely $\psi : IK_{tot}(X) \longmapsto K(X)$ is defined by $\psi[E_1,F,t] = [F/t] + [rk(E_1) - rk(F)]$. The relation $\psi \circ \varphi = id_{K(X)}$ is evident and $\varphi : K(X) \longmapsto IK_{tot}(X)$ is a monomorphism. Moreover the relations $\varphi(\psi[E_1,F,t]) = ([F + [rk(E_1) - rk(F)],F,t)$ and $ch_*([E_1]) = ch_*([F + [rk(E_1) - rk(F)])$ (since $q = 0$) imply the first isomorphism because ch_* is a monomorphism when $K(\widetilde{X})$ is torsion-free, [1]. Now we compute $IK_{\overline{-2}}(X)$. The following homomorphisms

$$\begin{array}{ccccc} K(\widetilde{X}) & \overset{u}{\to} & IK_{\overline{-2})}(X) & \overset{v}{\to} & K(\widetilde{X}) \\ E & \mapsto & [E,0,id] & & \\ & & [E,F,t] & \mapsto & [E] \end{array}$$

satisfy $v \circ u = id_{K(\widetilde{X})}$ and $u \circ v([E,F,t]) = [E,0,id]$ ($ch([F]) = rk(F)$ and $[E,F,t] = [E,0,id]$ because of the definition of $S_{dim\widetilde{X}}$, see Section 4). This proves the second isomorphism. □

5.2. *Relation between intersection K-theory and K-theory*

We will extend to all perversities the comparison with the K-theory. Following [1] let us denote by $K_p(\widetilde{X}) \subset K(\widetilde{X})$ the kernel of $\oplus_{2j<p} ch_j : K(\widetilde{X}) \to H^{even}(\widetilde{X};\mathbb{Q})$

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and by ${}^pK(\widetilde{X}) \subset K(\widetilde{X})$ the kernel of $\oplus_{2j>p} ch_j : K(\widetilde{X}) \to H^{even}(\widetilde{X};\mathbb{Q})$.
We let also $K^1_p(L) \subset K^1(L)$ and ${}^pK^1(L) \subset K^1(L)$ as being, respectively, spanned by the elements $[n,t] \in K^1(L)$ such that $ch_j[n,t] = 0$ for $2j < p$, respectively for $2j > p$ (ch_j takes its values in $H^{2j-1}(\widetilde{X};\mathbb{Q})$).

Proposition 5.2 *Let $\widetilde{X}$ be a manifold with a boundary L, and let $X = \widetilde{X}/L$ be the associated pseudomanifold with a conical singularity. Suppose that $H^\star(L,\mathbb{Z})$ and $H^\star(\widetilde{X},\mathbb{Z})$ are free. Then the following sequence is exact*

$$\begin{array}{ccccccc} \frac{K^1(L)}{q+1K^1(L)} & \stackrel{\varphi}{\to} & IK_{\bar{p}}(X) & \stackrel{\psi}{\to} & K_{q+2}(\widetilde{X}) \oplus {}^{q+1}K(\widetilde{X}) & \stackrel{\alpha}{\to} & \frac{K(L)}{{}^qK(L)} \\ \text{class}[n,t] & \mapsto & [n,n,t]-[n,n,id] & & \eta \oplus \eta' & \mapsto & \text{class}(\eta+\eta')_{|L} \\ & & [E,F,t] & \mapsto & [F]-[rk(F)] \oplus [E]-[F]+[rk(F)]\,. & & \end{array}$$

Proof: Firstly let us verify that the homomorphisms φ, ψ are well defined. Let (E,F,t) be a q-admissible bundle. For $0 < 2j \leq q+1$ the relation $ch_j([F]-[rk(F)]) = 0$ holds and of course $ch_0([F]-[rk(F)]) = 0$. It results that $[F]-[rk(F)] \in K_{q+2}(\widetilde{X})$. For $2j \geq q+2$, we have $ch_j([E]-[F]+[rk(F)]) = 0$ and $[E]-[F]+[rk(F)]$ belongs to ${}^{q+1}K(\widetilde{X})$. Finally, if $(E,F,t) \in S_q$, the two components of ψ are zero and ψ is well defined.
With respect to how φ is defined, the group ${}^{q+1}K^1(L)$ is embedded in S_q, and φ is well defined.
To prove $\alpha \circ \psi = 0$ consider (E,F,t) admissible. Then

$$\begin{aligned} \alpha \circ \psi([E,F,t]) &= \alpha([F]-rk(F) \oplus [E]-[F]+rk(F)) \\ &= \text{class}([F_{|L}]-rk(F)+[E_{|L}]-[F_{|L}]+rk(F)) \\ &= \text{class}([E_{|L}]). \end{aligned}$$

For all integer j, such that $2j \geq q+1$, $ch_j[E_{|L}] = 0$. Thus $[E_{|L}] \in {}^qK(L)$. Hence the class $cl[E_{|L}] \in \frac{K_q(L)}{{}^qK(L)}$ is zero and $Im\psi \subset \ker\alpha$.
Now consider $(\eta \oplus \eta') \in \ker\alpha$. Thus $(\eta+\eta')_{|L} \in {}^qK(L)$, so for all j, such that $2j \geq q+1$, $ch_j(\eta+\eta'_{|L}) = 0$. As $\eta \oplus \eta' \in K_{q+2}(\widetilde{X}) \oplus {}^{q+1}K(\widetilde{X})$ it results that

$$\begin{cases} ch_j[\eta \oplus \eta'] = ch_j[\eta] & \forall 2j \geq q+2 \\ ch_j[\eta] = 0 & \forall 2j \leq q+1\,. \end{cases}$$

Moreover we know that $ch_j[\eta \oplus \eta']_{|L} = 0 = ch_j[\eta_{|L}]$ when $2j \geq q+1$ and $ch_j[\eta_{|L}] = 0$ when $2j \leq q+1$. Thus, for all $j \in \mathbb{N}$ the class $ch_j[\eta_{|L}]$ is 0. Let $\eta = [E_1]-[n_1]$ (so $rk(E_1) = n_1$) and $\eta' = [E_2]-[n_2]$. As $H^\star(L,\mathbb{Z})$ is free, $(ch_*)_{|L}$ is injective by [1]. Therefore there exists an integer n, such that $(E_1 \oplus n)_{|L}$ is trivial. Let $E = E_1 \oplus E_2 \oplus n$, $F = E_1 \oplus n$ and choose a trivialisation t of $(E_1 \oplus n)_{|L}$. Then (E,F,t) is a q-admissible triple. A simple computation gives:

$$\psi([E,F,t]-[n,n,id]-[n_1+n_2,n_1,id]) = \eta \oplus \eta'.$$

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The relation $\varphi \circ \psi = 0$ is a consequence of the following equalities

$$\psi \circ \varphi[n,t] = \psi([n,n,t]-[n,n,id]) = (n-n)\oplus(n-n+n)-(n-n)\oplus(n-n+n) = 0.$$

Now let $[E,F,t]-[n,m,id] \in \ker\psi$. We get $([F]-[rk(F)])\oplus([E]-[F]+[rk(F)]) = 0 \oplus n \in K_{q+2}(\widetilde{X}) \oplus {}^{q+1}K(\widetilde{X})$. As $H^{even}(\widetilde{X},\mathbb{Z})$ is free, E, F are stably trivial and there exist p and a trivialisation $(E\oplus p, F\oplus p) \xrightarrow{u} (rk(E)\oplus p, rk(F)\oplus p)$ such that $[E\oplus p, F\oplus p, t\oplus id] = [rk(E)\oplus p, rk(F)\oplus p, (t\oplus id)\circ u^{-1}]$. This implies the equalities

$$\begin{array}{rl} & \varphi(rk(E)\oplus p,(t\oplus id)\circ u^{-1}) \\ = & [E\oplus p,F\oplus p,t\oplus id]-[rk(E)\oplus p,rk(E)\oplus p,id] \\ = & [E,F,t]-[rk(E),rk(E),id] \end{array}$$

and the relation $\mathrm{Im}\varphi = \ker\psi$. □

Finally, consider $X = cL = \dfrac{[0,1[\times L}{\{0\}\times L}$ the cone over a connected compact differentiable manifold L, so $\widetilde{X} = [0,1[\times L$. For all integers q, we have ${}^qK(\widetilde{X}) = {}^qK(L)$ and $K_q(\widetilde{X}) = K_q(L)$ and the exact sequence of Proposition 5.2 becomes

$$\frac{K^1(L)}{{}^{q+1}K^1(L)} \xrightarrow{\varphi} IK_{\bar{p}}(cL) \xrightarrow{\psi} K_{q+2}(L)\oplus {}^{q+1}K(L) \xrightarrow{\alpha} \frac{K(L)}{{}^qK(L)}.$$

Since ${}^qK(L) \subset {}^{q+1}K(L)$, it is clear that $\ker\alpha = {}^qK(L)$. Because $U(n)$ is connected it is possible to push t along $[0,1]$ and we see that $[n,n,t] = [n,n,id]$. Thus $\varphi = 0$ and $IK_{\bar{p}}(cL) = {}^qK(L)$.

To summarize, we proved, for intersection K-theory, the following local computation:

Corollary 5.1 *Let L be a C^∞ connected compact manifold of dimension $N-1$ such that $H^\star(L,\mathbb{Z})$ is free. Then the following results hold:*

1. $IK_{tot}(cL) = \mathbb{Z}$.
2. $IK_{\bar{p}}(cL) = {}^qK(L)$ *for* $0 < q = N - p_N - 2$.
3. $IK_{\overline{-2}}(cL) = K(L)$.

Final remark. When $L = S^{N-1}$, we have no topological singularity, because $cL = B^N$ is the open N-ball. For any Goresky-MacPherson perversity it results from Corollary 5.1 that $IK_{\bar{p}}(cS^{N-1}) = \mathbb{Z}$. Without topological singularities the intersection K-theory is identical to the classical K-theory.

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