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A New Weak Galerkin Finite Element Scheme for the Brinkman Model

Qilong Zhai[1], Ran Zhang[1,*] and Lin Mu[2]

[1] *School of Mathematics, Jilin University, Changchun 130012, P.R. China.*
[2] *Department of Mathematics, Michigan State University, East Lansing MI48824, United States.*

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Abstract. The Brinkman model describes flow of fluid in complex porous media with a high-contrast permeability coefficient such that the flow is dominated by Darcy in some regions and by Stokes in others. A weak Galerkin (WG) finite element method for solving the Brinkman equations in two or three dimensional spaces by using polynomials is developed and analyzed. The WG method is designed by using the generalized functions and their weak derivatives which are defined as generalized distributions. The variational form we considered in this paper is based on two gradient operators which is different from the usual gradient-divergence operators for Brinkman equations. The WG method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. Optimal-order error estimates are established for the corresponding WG finite element solutions in various norms. Some computational results are presented to demonstrate the robustness, reliability, accuracy, and flexibility of the WG method for the Brinkman equations.

AMS subject classifications: 65N30, 65N15, 65N12, 74N20

Key words: Weak Galerkin finite element methods, weak gradient, Brinkman equation, polyhedral meshes.

1 Introduction

Incompressible viscous flows around and through porous arise in many fields, such as underground water hydrology, petroleum industry, automotive industry, biomedical engineering, and heat pipes modeling [11, 21, 22]. In these applications, the simple Darcy model is just suit for modeling slow flow problems [10, 16], thus can not describe cavity

*Corresponding author. *Email addresses:* diql15@mails.jlu.edu.cn (Q. Zhai), zhangran@jlu.edu.cn (R. Zhang), linmu@math.msu.edu (L. Mu)

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problems well. Furthermore, even though the Stokes-Darcy interface model is suit for the flow of a viscous fluid in porous and cavity media, it is not practical for the accuracy information about the number and locations of the interfaces between vugs and the porous matrix are not accessible. In 1949, H. C. Brinkman proposed a so called Brinkman model [3] for this transport phenomena in porous media. The unified equations in the Brinkman model represent flow of fluid in complex porous media with a permeability coefficient highly varying such that the flow is dominated by Stokes in some regions and by Darcy in others. Thus, comparing with the popular Stokes-Darcy interface model, the Brinkman model can describe both a Stokes and a Darcy flow without employing complex interface conditions.

Mathematically speaking the Brinkman model is a parameter dependent combination of the Darcy and Stokes models. The challenging aspects are mainly coming from the high variability in the PDE coefficients, that may take extremely large or small values, negatively affects the conditioning of the discrete problem which poses a substantial challenge for developing efficient and stable algorithm to suit for both the Stokes and Darcy simultaneously. In literature, a great deal of effort has been made in meeting this challenge by modifying either existing Stokes elements or Darcy elements to obtain new Brinkman stable elements. In [12], the numerical experiments show that for certain stable Stokes elements, such as Taylor-Hood, P_2-P_0, mini elements, will lead to non-convergent discretizations as the Brinkman equations become Darcy-dominated and similarly, for certain stable Darcy elements, such as Raviart-Thomas elements will lead to non-convergence as the Brinkman becomes Stokes-dominated. A large number of approaches have been proposed in the literature to address the numerical stability of various discretized methods. Among these work, in [4], [5] and [23], the authors introduce jumps penalization on the normal component of the velocity field or on the pressure field to stabilize the Crouzeix-Raviart-P_0 finite elements or P_1-P_0 finite elements, respectively. In [6], an augmented Lagrangian approach and a least squares stabilization are explored in order to use inf-sup compatible Taylor-Hood elements also in the Darcy case, while in [12] and [9] high order non-conforming elements are investigated.

In the present paper, we analyze the development of stable numerical methods for the Brinkman equations by using weak Galerkin finite element methods. It shows that the WG presents a natural and straightforward framework for constructing stable numerical algorithms for the Brinkman equations. Weak Galerkin (WG) refers to finite element techniques for partial differential equations in which differential operators are approximated by weak forms as distributions. Weak Galerkin methods were first introduced in [17] for the second order elliptic problem and was further developed in [13, 15, 18, 19, 24, 25] with other applications. The central idea of the WG method is to interpret the differential operators (e.g., gradient, Laplacian, Hessian, curl, divergence etc.) as distributions over a space of generalized functions and employ some proper stabilizations to enforce weak continuities for approximating functions. It has shown that the WG methods are efficient and robust by allowing the use of discontinuous approximating functions. The flexibility of discontinuous functions gives WG methods many advantages, such as high

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order of accuracy, high parallelizability, localizability, and easy handling of complicated geometries.

In a simple form, the Brinkman model seeks unknown functions u and p satisfying

$$-\mu\Delta\mathbf{u}+\nabla p+\mu\kappa^{-1}\mathbf{u}=\mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0, \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u}=\mathbf{g}, \quad \text{on } \partial\Omega, \tag{1.3}$$

where μ is the fluid viscosity and κ denotes the permeability tensor of the porous media which occupies a polygonal or polyhedral domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$. $\mathbf{u}$ and p represent the velocity and the pressure of the fluid, and $\mathbf{f}$ is a momentum source term.

Assume that there exist two positive constant numbers $\alpha_1,\alpha_2>0$ such that

$$\alpha_1\boldsymbol{\xi}^t\boldsymbol{\xi}\leq\boldsymbol{\xi}^t\kappa^{-1}\boldsymbol{\xi}\leq\alpha_2\boldsymbol{\xi}^t\boldsymbol{\xi}, \quad \boldsymbol{\xi}\in\mathbb{R}^d. \tag{1.4}$$

Here $\boldsymbol{\xi}$ is understood as a column vector and $\boldsymbol{\xi}^t$ is the transpose of $\boldsymbol{\xi}$. We consider the case where α_1 is of unit size and α_2 is possibly of large size.

Throughout this paper, we would follow the standard definitions for Lebesgue and Sobolev spaces: $L^2(\Omega)$, $H^1(\Omega)$, $[L^2(\Omega)]^d$,

$$[H_0^1(\Omega)]^d=\{\mathbf{v}\in[H^1(\Omega)]^d:\mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}$$

and

$$L_0^2(\Omega):=\left\{q\in L^2(\Omega):\int_\Omega q dx=0\right\}$$

are the natural spaces for the weak form of the Stokes problem [2, 8]. Denote $(\cdot,\cdot)$ for inner products in the corresponding spaces.

For ease of presentation, we assume that μ is absorbed into other terms, and consider the homogeneous Dirichlet (no-slip) boundary condition: $\mathbf{g}=\mathbf{0}$. Then the standard variational formulation for the Brinkman problem (1.1)-(1.3) is to find $\mathbf{u}\in[H_0^1(\Omega)]^d$ and $p\in L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u},\nabla\mathbf{v})+(\kappa^{-1}\mathbf{u},\mathbf{v})-(\nabla\cdot\mathbf{v},p)=(\mathbf{f},\mathbf{v}), \tag{1.5}$$

$$(\nabla\cdot\mathbf{u},q)=0, \tag{1.6}$$

for all $\mathbf{v}\in[H_0^1(\Omega)]^d$ and $q\in L_0^2(\Omega)$. Here $\nabla\mathbf{u}$ denotes the velocity gradient tensor $(\nabla\mathbf{u})_{ij}=\partial_j\mathbf{u}_i$. It is well known that under our assumptions on the domain and the data, problem (1.5)-(1.6) has a unique solution $(\mathbf{u};p)\in[H_0^1(\Omega)]^d\times L_0^2(\Omega)$.

For any $q\in L^2(\Omega)$, the distributional derivative ∇q is known to be a bounded linear functional in $[H_0^1(\Omega)]^d$ such that its action on any $\mathbf{v}\in[H_0^1(\Omega)]^d$ is given by

$$\langle\nabla q,\mathbf{v}\rangle=-(\nabla\cdot\mathbf{v},q), \quad \forall\,\mathbf{v}\in[H_0^1(\Omega)]^d.$$

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Then the weak form (1.5)-(1.6) can be rewritten as follows: find $(\mathbf{u};p)\in[H_0^1(\Omega)]^d\times L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u},\nabla\mathbf{v})+(\kappa^{-1}\mathbf{u},\mathbf{v})+\langle\nabla p,\mathbf{v}\rangle=(\mathbf{f},\mathbf{v}), \tag{1.7}$$

$$\langle\nabla q,\mathbf{u}\rangle=0, \tag{1.8}$$

for all $\mathbf{v}\in[H_0^1(\Omega)]^d$ and $q\in L_0^2(\Omega)$.

In [14], the authors considered WG methods for Brinkman equations (1.5)-(1.6). To the best of our knowledge, the numerical analysis of methods based on the variational form (1.7)-(1.8) for the Brinkman equations has never been done before. It is known that as permeability tensor κ is getting quite small the Brinkman equations becomes Darcy-dominated. The variational formulation (1.7)-(1.8) is suit for the mixed formulation of Darcy which would presents a better approximation for this case. In this paper, we propose a WG method based on this weak form of the primary problem, and numerically, the new method seems more efficient than the WG method based on (1.5)-(1.6) for small permeability tensor.

The rest of this paper is organized as follows. Section 2 is devoted to the definitions of weak functions and weak derivatives. The WG finite element schemes for variational form of the Brinkman equation (1.7)-(1.8) are presented in Section 3. This section also contains some local L^2 projection operators and some related approximation properties. In Section 4, we prove the solvability and drive an error equation for the WG finite element approximation. In Section 5 we present some technical estimates for quantities related to the local L^2 projections into various finite element spaces. Optimal-order error estimates for the WG finite element approximations are derived in Section 6 in an H^1-equivalent norm for both the velocity and pressure, and L^2 norm for the velocity. In Section 7, we present a Schur complement scheme by eliminating all the variables on the element, yielding a system of linear equations with significantly reduced number of unknowns defined on the element boundary. In Section 8, some numerical experiments are conducted to demonstrate the reliability, flexibility and accuracy of the weak Galerkin method for the Brinkman equations.

2 Weak differential operators and discrete weak gradient

Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$, $d=2,3$. We use the standard definition for the Sobolev space $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for any $s\geq 0$. For example, for any integer $s\geq 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$|v|_{s,D}=\left(\sum_{|\alpha|=s}\int_D|\partial^\alpha v|^2\mathrm{d}D\right)^{\frac{1}{2}}$$

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with the usual notation

$$\alpha=(\alpha_1,\cdots,\alpha_d),\quad |\alpha|=\alpha_1+\cdots+\alpha_d,\quad \partial^\alpha=\prod_{j=1}^{d}\partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,D}$ is given by

$$\|v\|_{m,D}=\left(\sum_{j=0}^{m}|v|_{j,D}^2\right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D=\Omega$, we shall drop the subscript D in the norm and in the inner product notation.

The space $H(\mathrm{div};D)$ is defined as the set of vector-valued functions on D which, together with their divergence, are square integrable; i.e.,

$$H(\mathrm{div};D)=\{\mathbf{v}:\mathbf{v}\in[L^2(D)]^d,\nabla\cdot\mathbf{v}\in L^2(D)\}.$$

The norm in $H(\mathrm{div};D)$ is defined by

$$\|\mathbf{v}\|_{H(\mathrm{div};D)}=(\|\mathbf{v}\|_D^2+\|\nabla\cdot\mathbf{v}\|_D^2)^{\frac{1}{2}}.$$

Let T be a polygonal or polyhedral domain with boundary ∂T.

A weak function on the region T refers to a scalar function $p=\{p_0,p_b\}$ such that $p_0\in L^2(T)$ and $p_b\in L^2(\partial T)$. Let

$$\mathcal{W}(T)=\{p=\{p_0,p_b\}:p_0\in L^2(T),p_b\in L^2(\partial T)\}. \tag{2.1}$$

Definition 2.1. ([17]) For any $p\in\mathcal{W}(T)$, the weak gradient of p, denoted by $\nabla_w p$, is defined as a linear functional in the dual space of $[H^1(T)]^d$ whose action on each $\mathbf{q}\in[H^1(T)]^d$ is given

$$\langle\nabla_w p,\mathbf{q}\rangle_T:=-(p_0,\nabla\cdot\mathbf{q})_T+\langle p_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}, \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal direction along ∂T, $(\cdot,\cdot)_T$ stands for the L^2-inner product in $L^2(T)$ and $\langle\cdot,\cdot\rangle_{\partial T}$ is the L^2 inner product on ∂T.

A discrete version of the weak gradient operator ∇_w is an approximation, denoted by $\nabla_{w,r,T}$ in the space of polynomials of degree no more than r such that

$$(\nabla_{w,r,T}p,\mathbf{q})_T=-(p_0,\nabla\cdot\mathbf{q})_T+\langle p_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T},\quad\forall\mathbf{q}\in[P_r(T)]^d. \tag{2.3}$$

For any integer $k\geq 1$, denote by $W_k(T)$ the discrete function space as follows

$$W_k(T)=\{q=\{q_0,q_b\}:\{q_0,q_b\}|_T\in P_{k-1}(T)\times P_{k-1}(e),e\subset\partial T\}.$$

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For any $p=\{p_0,p_b\}\in W_k(T)$, from the integration by parts, we have

$$(p_0,\nabla\cdot\mathbf{q})_T=-(\nabla p_0,\mathbf{q})_T+\langle p_0,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}.$$

Substituting the above identity into (2.3) yields

$$(\nabla_{w,r,T}p,\mathbf{q})_T-(\nabla p_0,\mathbf{q})_T=\langle p_b-p_0,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \tag{2.4}$$

for all $\mathbf{q}\in[P_r(T)]^d$.

A weak vector-valued function on the region T refers to a vector-valued function $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}$ such that $\mathbf{v}_0\in[L^2(T)]^d$ and $\mathbf{v}_b\in[L^2(\partial T)]^d$. Let

$$\mathcal{V}(T)=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\mathbf{v}_0\in[L^2(T)]^d,\mathbf{v}_b\in[L^2(\partial T)]^d\}. \tag{2.5}$$

Similarly, for any $\mathbf{v}\in\mathcal{V}(T)$, the weak gradient of $\mathbf{v}$, denoted by $\hat{\nabla}_w\mathbf{v}$, is defined as a linear functional in the dual space of $[H^1(T)]^{d\times d}$. It is given by

$$(\hat{\nabla}_w\mathbf{v},\tau)_T:=-(\mathbf{v}_0,\nabla\cdot\tau)_T+\langle\mathbf{v}_b,\tau\cdot\mathbf{n}\rangle_{\partial T}. \tag{2.6}$$

Denote the discrete weak vector-valued function space as follows

$$V_k(T)=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\{\mathbf{v}_0,\mathbf{v}_b\}|_T\in[P_k(T)]^d\times[P_k(e)]^d,e\subset\partial T\}.$$

A discrete version of the weak gradient operator $\hat{\nabla}_w$ is an approximation, denoted by $\hat{\nabla}_{w,r,T}$ in the space of polynomials of degree r such that

$$(\hat{\nabla}_{w,r,T}\mathbf{v},\tau)_T=-(\mathbf{v}_0,\nabla\cdot\tau)_T+\langle\mathbf{v}_b,\tau\cdot\mathbf{n}\rangle_{\partial T},\quad\forall\tau\in[P_r(T)]^{d\times d}. \tag{2.7}$$

From the integration by parts, for any $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_k(T)$, we have

$$(\mathbf{v}_0,\nabla\tau)_T=-(\nabla\mathbf{v}_0,\tau)_T+\langle\mathbf{v}_0,\tau\cdot\mathbf{n}\rangle_{\partial T}.$$

Substituting the above identity into (2.7) yields

$$(\hat{\nabla}_{w,r,T}\mathbf{v},\tau)_T-(\nabla\mathbf{v}_0,\tau)_T=\langle\mathbf{v}_b-\mathbf{v}_0,\tau\cdot\mathbf{n}\rangle_{\partial T} \tag{2.8}$$

for all $\tau\in[P_r(T)]^{d\times d}$.

3 A weak Galerkin formulation

The goal of this section is to introduce a weak Galerkin finite element algorithm.

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3.1 Notations

Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [18]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0=\mathcal{E}_h\setminus\partial\Omega$ be the set of all interior edges or flat faces. Denote by h_T the diameter of $T\in\mathcal{T}_h$ and $h=\max_{T\in\mathcal{T}_h}h_T$ the meshsize for the partition $\mathcal{T}_h$.

For $k\geq 1$, we define two weak Galerkin finite element spaces W_h and V_h by patching $W_k(T)$ and $V_k(T)$ over all the elements $T\in\mathcal{T}_h$ as follows

$$W_h=\{\{q_0,q_b\}:\{q_0,q_b\}\big|_T\in W_k(T),\ \forall T\in\mathcal{T}_h\}$$

and

$$V_h=\{\{\mathbf{v}_0,\mathbf{v}_b\}:\{\mathbf{v}_0,\mathbf{v}_b\}\big|_T\in V_k(T),\ \forall T\in\mathcal{T}_h\}.$$

It should be noticed that $\mathbf{v}_b$ and q_b are single-valued on each edge.

Denote by W_h^0 and V_h^0 the subspace of W_h and V_h consisting of discrete weak functions with proper conditions that

$$W_h^0=\Big\{p=\{p_0,p_b\}\in W_h:\int_\Omega p_0=0\Big\},$$

and

$$V_h^0=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h:\mathbf{v}_b=0\ \text{ on }\partial\Omega\},$$

respectively.

Denote the discrete weak gradient operators by $\nabla_{w,k-1}$ and $\hat{\nabla}_{w,k-1}$ on the finite element space W_h and V_h which can be computed by using (2.3) and (2.7) on each element T, respectively. For simplicity of notation, we shall drop the subscript $k-1$ in above two notations and use ∇_w and $\hat{\nabla}_w$ to denote $\nabla_{w,k-1}$ and $\hat{\nabla}_{w,k-1}$.

For each element $T\in\mathcal{T}_h$, denote by Q_0 and $\hat{Q}_0$ the L^2 projection operator onto $P_{k-1}(T)$ and $[P_k(T)]^d$, respectively. For each edge or face $e\in\mathcal{E}_h$, denote by Q_b and $\hat{Q}_b$ the L^2 projection onto $P_{k-1}(e)$ and $[P_k(e)]^d$, respectively. We shall combine Q_0 with Q_b by writing $Q_h=\{Q_0,Q_b\}$ and $\hat{Q}_0$ with $\hat{Q}_b$ by writing $\hat{Q}_h=\{\hat{Q}_0,\hat{Q}_b\}$. Then we can define two projections onto the finite element space W_h and V_h such that on each element T,

$$Q_hq=\{Q_0q,Q_bq\},\quad \hat{Q}_h\mathbf{v}=\{\hat{Q}_0\mathbf{v},\hat{Q}_b\mathbf{v}\}.$$

In addition, denote by $\hat{\mathbb{Q}}_h$ the local L^2 projection onto $[P_{k-1}(T)]^{d\times d}$. The projection operators defined above have some useful properties as stated in the following lemma.

Lemma 3.1. *The projection operators Q_h, $\hat{Q}_h$, and $\hat{\mathbf{Q}}_h$ satisfy the following properties*

$$\hat{\nabla}_w(\hat{Q}_h\boldsymbol{v})=\hat{\mathbf{Q}}_h(\nabla\boldsymbol{v}),\quad \forall\boldsymbol{v}\in[H^1(\Omega)]^d. \tag{3.1}$$

$$(\nabla_w(Q_hq),\boldsymbol{w})=(\hat{Q}_0(\nabla q),\boldsymbol{w})-\sum_{T\in\mathcal{T}_h}\langle q-Q_bq,\boldsymbol{w}\cdot\boldsymbol{n}\rangle_{\partial T},$$
$$\forall q\in H^1(\Omega),\forall\boldsymbol{w}\in[P_k(T)]^d. \tag{3.2}$$

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Proof. From the definition of projection operators and the integration by parts, we have

$$\begin{aligned}(\hat{\nabla}_w(\hat{Q}_h\mathbf{v}),\tau)_T &= -(\hat{Q}_0\mathbf{v},\nabla\cdot\tau)_T+\langle\hat{Q}_b\mathbf{v},\tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(\mathbf{v},\nabla\cdot\tau)_T+\langle\mathbf{v},\tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\mathbf{v},\tau)_T\\ &= (\hat{\mathbf{Q}}_h(\nabla\mathbf{v}),\tau)_T\end{aligned}$$

for any $\tau\in[P_{k-1}(T)]^{d\times d}$, which verifies the identity (3.1).

Similarly, applying the definition of ∇_w and the integration by parts, it reaches

$$\begin{aligned}(\nabla_w(Q_hq),\mathbf{w})_T &= -(Q_0q,\nabla\cdot\mathbf{w})_T+\langle Q_bq,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(q,\nabla\cdot\mathbf{w})_T+\langle q,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial T}-\langle q-Q_bq,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla q,\mathbf{w})_T-\langle q-Q_bq,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\hat{Q}_0(\nabla q),\mathbf{w})_T-\langle q-Q_bq,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial T}\end{aligned}$$

for any $\mathbf{w}\in[P_k(T)]^d$. Summing over all $T\in\mathcal{T}_h$ arrives at (3.2). Thus we complete the proof of the lemma. □

3.2 Algorithm

On each element $T\in\mathcal{T}_h$, we introduce four bilinear forms given as follows:

$$s_{1,T}(\mathbf{v},\mathbf{w}) = h_T^{-1}\langle\mathbf{v}_0-\mathbf{v}_b,\mathbf{w}_0-\mathbf{w}_b\rangle_{\partial T}, \tag{3.3}$$

$$a_T(\mathbf{v},\mathbf{w}) = (\hat{\nabla}_w\mathbf{v},\hat{\nabla}_w\mathbf{w})_T+s_{1,T}(\mathbf{v},\mathbf{w})+(\kappa^{-1}\mathbf{v}_0,\mathbf{w}_0)_T, \tag{3.4}$$

$$b_T(\mathbf{v},q) = (\mathbf{v}_0,\nabla_w q)_T, \tag{3.5}$$

$$s_{2,T}(p,q) = h_T\langle p_0-p_b,q_0-q_b\rangle_{\partial T} \tag{3.6}$$

for $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}$, $\mathbf{w}=\{\mathbf{w}_0,\mathbf{w}_b\}\in V_k(T)$ and $p,q\in W_k(T)$.

The sum over all $T\in\mathcal{T}_h$ yields four globally-defined bilinear forms:

$$s_1(\mathbf{v},\mathbf{w}) = \sum_{T\in\mathcal{T}_h}s_{1,T}(\mathbf{v},\mathbf{w}),\quad \mathbf{v},\mathbf{w}\in V_h, \tag{3.7}$$

$$a(\mathbf{v},\mathbf{w}) = \sum_{T\in\mathcal{T}_h}a_T(\mathbf{v},\mathbf{w}),\quad \mathbf{v},\mathbf{w}\in V_h, \tag{3.8}$$

$$b(\mathbf{v},q) = \sum_{T\in\mathcal{T}_h}b_T(\mathbf{v},q),\quad \mathbf{v}\in V_h,q\in W_h, \tag{3.9}$$

$$s_2(p,q) = \sum_{T\in\mathcal{T}_h}s_{2,T}(p,q),\quad p,q\in W_h. \tag{3.10}$$

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From these bilinear forms we can conduct some norms. For any $\mathbf{v}\in V_h$, $q\in W_h^0$, define

$$|||\mathbf{v}|||^2=a(\mathbf{v},\mathbf{v})=\|\kappa^{-\frac{1}{2}}\mathbf{v}_0\|^2+\|\hat{\nabla}_w\mathbf{v}\|^2+\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2, \tag{3.11}$$

$$\|\mathbf{v}\|_{1,h}^2=\|\hat{\nabla}_w\mathbf{v}\|^2+\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2, \tag{3.12}$$

$$|||q|||_1^2=\|\kappa^{\frac{1}{2}}\nabla_w q\|^2+\sum_{T\in\mathcal{T}_h}h_T^{-1}\|q_0-q_b\|_{\partial T}^2, \tag{3.13}$$

$$|q|_{0,h}^2=s_2(q,q)=\sum_{T\in\mathcal{T}_h}h_T\|q_0-q_b\|_{\partial T}^2. \tag{3.14}$$

It is easy to confirm that $|||\cdot|||$ and $\|\cdot\|_{1,h}$ are both equivalent to H^1 norms.

We are now in a position to describe a weak Galerkin finite element method for the Brinkman equations (1.1)-(1.3).

Weak Galerkin Algorithm 1. A numerical approximation for (1.1)-(1.3) can be obtained by seeking $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h^0$ and $p_h\in W_h^0$ such that

$$a(\mathbf{u}_h,\mathbf{v})+b(\mathbf{v},p_h)=(\mathbf{f},\mathbf{v}_0), \tag{3.15}$$

$$b(\mathbf{u}_h,q)-s_2(p_h,q)=0, \tag{3.16}$$

for all $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0$ and $q\in W_h^0$.

4 Solvability and error equations

Although the WG finite element scheme (3.15)-(3.16) is a saddle-point problem, it is difficult to get its solvability and stability directly by the theory of Babuška and Brezzi [1] for the uncertainty of the permeability tensor κ. But we can still have the solvability from the flexibility of the discontinuous weak finite element spaces.

Lemma 4.1. *The weak Galerkin finite element scheme* (3.15)-(3.16) *has a unique solution.*

Proof. As the number of unknowns is the same as the number of equations, then the existence of the solution is equivalent to its uniqueness. Thus we just need to confirm the uniqueness of solution in homogenous case, i.e. $\mathbf{f}=\mathbf{0}$. Let $\mathbf{v}=\mathbf{u}_h$, $q=p_h$ and subtracting (3.16) from (3.15) we arrive at

$$a(\mathbf{u}_h,\mathbf{u}_h)+s_2(p_h,p_h)=0.$$

Notice that both $a(\mathbf{u}_h,\mathbf{u}_h)$ and $s_2(p_h,p_h)$ are non-negative, then from definition (3.11) and (3.14) we have $(\kappa^{-1}\mathbf{u}_0,\mathbf{u}_0)=0$ for all $T\in\mathcal{T}_h$, $\mathbf{u}_0=\mathbf{u}_b$ and $p_0=p_b$ for all $e\in\mathcal{E}_h$. Combining with the boundary condition on $\mathbf{u}_b$, we obtain that $\mathbf{u}_h=\{\mathbf{0},\mathbf{0}\}$ and p_h continuous in Ω.

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As $\mathbf{u}_h=\{\mathbf{0},\mathbf{0}\}$, we can obtain $b(\mathbf{v},p_h)=0$ for any $\mathbf{v}\in V_h^0$. If we take $\mathbf{v}_0=\nabla p_0$, it reaches that

$$\begin{aligned}0&=b(\mathbf{v},p_h)\\&=\sum_{T\in\mathcal{T}_h}(\mathbf{v}_0,\nabla_w p_h)_T\\&=\sum_{T\in\mathcal{T}_h}-(p_0,\nabla\cdot\mathbf{v}_0)_T+\sum_{T\in\mathcal{T}_h}\langle p_b,\mathbf{v}_0\cdot\mathbf{n}\rangle_{\partial T}\\&=\sum_{T\in\mathcal{T}_h}(\mathbf{v}_0,\nabla p_0)_T=\sum_{T\in\mathcal{T}_h}(\nabla p_0,\nabla p_0)_T.\end{aligned}$$

It follows that $\nabla p_0=0$ for all $T\in\mathcal{T}_h$. Noticing that p_h is continuous and $p_h\in L_0^2(\Omega)$ we have $p_h=0$, which completes the proof. □

Next we shall derive an error equation for the WG finite element solution obtained from (3.15)-(3.16), which is critical in convergence analysis. Define the following three bilinear forms

$$l_1(\mathbf{u},\mathbf{v})=\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T},\tag{4.1}$$

$$l_2(p,\mathbf{v})=\sum_{T\in\mathcal{T}_h}\langle p-Q_b p,(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_{\partial T},\tag{4.2}$$

$$l_3(q,\mathbf{u})=\sum_{T\in\mathcal{T}_h}\langle q_0-q_b,(\mathbf{u}-\hat{Q}_0\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T},\tag{4.3}$$

for all $\mathbf{u}\in[H^1(\Omega)]^d$, $p\in H^1(\Omega)$, $\mathbf{v}\in V_h^0$ and $q\in W_h^0$.

Let

$$\mathbf{e}_h=\hat{Q}_h\mathbf{u}-\mathbf{u}_h,\quad \varepsilon_h=Q_h p-p_h.$$

We shall derive the error equations that $\mathbf{e}_h\in V_h$ and $\varepsilon_h\in W_h$ satisfy.

Lemma 4.2. *Let $(\boldsymbol{u};p)$ be the exact solution of (1.1)-(1.3), and $(\boldsymbol{u}_h;p_h)\in V_h\times W_h^0$ be the solutions of (3.15)-(3.16). Then, the error functions $\boldsymbol{e}_h$ and ε_h satisfy the following equations*

$$a(\boldsymbol{e}_h,\boldsymbol{v})+b(\boldsymbol{v},\varepsilon_h)=\varphi_{\boldsymbol{u},p}(\boldsymbol{v}),\quad\forall\boldsymbol{v}\in V_h^0,\tag{4.4}$$

$$b(\boldsymbol{e}_h,q)-s_2(p_h,q)=\phi_{\boldsymbol{u},p}(q),\quad\forall q\in W_h^0,\tag{4.5}$$

where

$$\varphi_{\mathbf{u},p}(\mathbf{v})=s_1(\hat{Q}_h\mathbf{u},\mathbf{v})-l_1(\mathbf{u},\mathbf{v})+l_2(p,\mathbf{v}),\tag{4.6}$$

$$\phi_{\mathbf{u},p}(q)=-s_2(Q_h p,q)+l_3(q,\mathbf{u}).\tag{4.7}$$

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Proof. First, applying the property (3.1), definition (2.6), and the integration by parts yields

$$\begin{aligned}(\hat{\nabla}_w(\hat{Q}_h\mathbf{u}),\hat{\nabla}_w\mathbf{v})_T &= (\hat{\mathbf{Q}}_h(\nabla\mathbf{u}),\hat{\nabla}_w\mathbf{v})_T\\ &= -(\mathbf{v}_0,\nabla\cdot\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))_T+\langle\mathbf{v}_b,\hat{\mathbf{Q}}_h(\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\mathbf{v}_0,\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))_T-\langle\mathbf{v}_0-\mathbf{v}_b,\hat{\mathbf{Q}}_h(\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\mathbf{v}_0,\nabla\mathbf{u})_T-\langle\mathbf{v}_0-\mathbf{v}_b,\hat{\mathbf{Q}}_h(\nabla\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(\Delta\mathbf{u},\mathbf{v}_0)_T+\langle\mathbf{v}_0-\mathbf{v}_b,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad+\langle\mathbf{v}_b,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

Summing over all $T\in\mathcal{T}_h$ reaches

$$-(\Delta\mathbf{u},\mathbf{v}_0)=(\hat{\nabla}_w(\hat{Q}_h\mathbf{u}),\hat{\nabla}_w\mathbf{v})-\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T}. \tag{4.8}$$

Similarly, using the property (3.2) we have

$$\begin{aligned}(\mathbf{v}_0,\nabla_w Q_h p) &= (\mathbf{v}_0,\hat{Q}_0(\nabla p))-\sum_{T\in\mathcal{T}_h}\langle p-Q_b p,\mathbf{v}_0\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\mathbf{v}_0,\nabla p)-\sum_{T\in\mathcal{T}_h}\langle p-Q_b p,\mathbf{v}_0\cdot\mathbf{n}\rangle_{\partial T},\end{aligned}$$

which leads to

$$(\mathbf{v}_0,\nabla p)=(\mathbf{v}_0,\nabla_w Q_h p)+\sum_{T\in\mathcal{T}_h}\langle p-Q_b p,(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_{\partial T}. \tag{4.9}$$

Next, testing (1.1) by $\mathbf{v}_0$ gives

$$-(\Delta\mathbf{u},\mathbf{v}_0)+(\nabla p,\mathbf{v}_0)+(\kappa^{-1}\mathbf{u},\mathbf{v}_0)=(\mathbf{f},\mathbf{v}_0). \tag{4.10}$$

Substituting (4.8) and (4.9) into (4.10) yields

$$a(\mathbf{v},\hat{\mathbf{Q}}_h\mathbf{u})+b(\mathbf{v},Q_h p)=(\mathbf{f},\mathbf{v}_0)+\varphi_{\mathbf{u},p}(\mathbf{v}). \tag{4.11}$$

Combining with the scheme (3.15), it follows that

$$a(\mathbf{e}_h,\mathbf{v})+b(\mathbf{v},\varepsilon_h)=\varphi_{\mathbf{u},p}(\mathbf{v}).$$

As to (4.5), using the definition (2.2) and the integration by parts, we obtain

$$\begin{aligned}(\hat{Q}_0\mathbf{u},\nabla_w q)_T &= -(q_0,\nabla\cdot(\hat{Q}_0\mathbf{u}))_T+\langle q_b,\hat{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla q_0,\mathbf{u})_T-\langle q_0-q_b,\hat{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

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Summing over all $T\in\mathcal{T}_h$ yields

$$(\hat{Q}_0\mathbf{u},\nabla_w q)=(\nabla q_0,\mathbf{u})-\sum_{T\in\mathcal{T}_h}\langle q_0-q_b,\hat{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}. \tag{4.12}$$

Testing (1.2) by q_0 yields

$$(\nabla\cdot\mathbf{u},q_0)=0.$$

Applying the integration by parts, we have

$$-\sum_{T\in\mathcal{T}_h}(\mathbf{u},\nabla q_0)_T+\sum_{T\in\mathcal{T}_h}\langle\mathbf{u}\cdot\mathbf{n},q_0-q_b\rangle_{\partial T}=0. \tag{4.13}$$

Substituting (4.12) into (4.13), we obtain

$$b(\hat{Q}_h\mathbf{u},q)-\sum_{T\in\mathcal{T}_h}\langle q_0-q_b,(\mathbf{u}-\hat{Q}_0\mathbf{u})\cdot\mathbf{n}\rangle_{\partial T}=0. \tag{4.14}$$

The difference between (3.16) and (4.14) is

$$b(\mathbf{e}_h,q)-s_2(p_h,q)=\phi_{\mathbf{u},p}(q).$$

This completes the proof. □

5 Preparation for error estimates

The following lemma provides some approximation properties for the projections. Observe that the underlying mesh $\mathcal{T}_h$ is assumed to be sufficiently general to allow polygons or polyhedra. A proof of the lemma can be found in [18].

Lemma 5.1. *Let $\mathcal{T}_h$ be a shape regular partition of Ω of WG method, $\boldsymbol{w}\in[H^{r+1}(\Omega)]^d$ and $\rho\in H^r(\Omega)$ with $1\le r\le k$. Then for $0\le s\le 1$, we have*

$$\sum_{T\in\mathcal{T}_h}h_T^{2s}\|\boldsymbol{w}-\hat{Q}_0\boldsymbol{w}\|_{T,s}^2\le h^{2(r+1)}\|\boldsymbol{w}\|_{r+1}^2, \tag{5.1}$$

$$\sum_{T\in\mathcal{T}_h}h_T^{2s}\|\nabla\boldsymbol{w}-\hat{\mathbf{Q}}_h(\nabla\boldsymbol{w})\|_{T,s}^2\le h^{2r}\|\boldsymbol{w}\|_{r+1}^2, \tag{5.2}$$

$$\sum_{T\in\mathcal{T}_h}h_T^{2s}\|\rho-Q_0\rho\|_{T,s}^2\le h^{2r}\|\rho\|_r^2. \tag{5.3}$$

Let T be an element with e as an edge/face. For any function $\theta\in H^1(T)$, the following trace inequality has been proved to be valid for general meshes [18]

$$\|\theta\|_e^2\le C(h_T^{-1}\|\theta\|_T^2+h_T\|\nabla\theta\|_T^2). \tag{5.4}$$

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Furthermore, if $\theta\in P_k(T)$, we have the following inverse inequality

$$\|\nabla\theta\|_T\leq Ch_T^{-1}\|\theta\|_T. \tag{5.5}$$

If we apply Lemma 5.1 and the estimates (5.4) and (5.5) directly, we can obtain

$$\sum_{T\in\mathcal{T}_h}h_T^{2s}\|\mathbf{w}-\hat{Q}_b\mathbf{w}\|_{\partial T,s}^2\leq\sum_{T\in\mathcal{T}_h}h_T^{2s}\|\mathbf{w}-\hat{Q}_0\mathbf{w}\|_{\partial T,s}^2\leq Ch^{2(r+1)}\|\mathbf{w}\|_{r+1}^2,$$
$$\|\theta\|_e^2\leq C(h_T^{-1}\|\theta\|_T^2+h_T\|\nabla\theta\|_T^2)\leq Ch_T^{-1}\|\theta\|_T^2,$$

where the notations are as noted in Lemma 5.1.

Lemma 5.2. *Assume that $\boldsymbol{w}\in[H^{r+1}(\Omega)]^d$ and $\rho\in H^r(\Omega)$, where $1\leq r\leq k$. Then for any $\boldsymbol{v}\in V_h$, $q\in W_h^0$ we have*

$$|s_1(\hat{Q}_h\boldsymbol{w},\boldsymbol{v})|\leq Ch^r\|\boldsymbol{w}\|_{r+1}\|\boldsymbol{v}\|_{1,h}, \tag{5.6}$$
$$|s_2(Q_h\rho,q)|\leq Ch^r\|\rho\|_r|q|_{0,h}, \tag{5.7}$$
$$|l_1(\boldsymbol{w},\boldsymbol{v})|\leq Ch^r\|\boldsymbol{w}\|_{r+1}\|\boldsymbol{v}\|_{1,h}, \tag{5.8}$$
$$|l_2(\rho,\boldsymbol{v})|\leq Ch^r\|\rho\|_r\|\boldsymbol{v}\|_{1,h}, \tag{5.9}$$
$$|l_3(q,\boldsymbol{w})|\leq Ch^r\|\boldsymbol{w}\|_{r+1}|q|_{0,h}, \tag{5.10}$$

where the functions $l_1(\cdot,\cdot)$, $l_2(\cdot,\cdot)$ and $l_3(\cdot,\cdot)$ are given by (4.1)-(4.3).

Proof. It has been proved in [14] that (5.6) and (5.8) are valid. For (5.7), from the definition of Q_h, trace inequality (5.4) and Lemma 5.1 we have

$$\begin{aligned}|s_2(Q_h\rho,q)|&=\left|\sum_{T\in\mathcal{T}_h}h_T\langle Q_0\rho-Q_b\rho,q_0-q_b\rangle_{\partial T}\right|\\&=\left|\sum_{T\in\mathcal{T}_h}h_T\langle Q_0\rho-\rho,q_0-q_b\rangle_{\partial T}\right|\\&\leq C\left(\sum_{T\in\mathcal{T}_h}\|Q_0\rho-\rho\|_T^2+h_T^2\|\nabla(Q_0\rho-\rho)\|_T^2\right)^{\frac{1}{2}}|q|_{0,h}\\&\leq Ch^r\|\rho\|_r|q|_{0,h}.\end{aligned}$$

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As to (5.9), using the property of projection operator we can obtain

$$\begin{aligned}
|l_2(\rho,\mathbf{v})| &= \left| \sum_{T\in\mathcal{T}_h} \langle \rho - Q_b\rho, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T} \right| \\
&\leq \left| \sum_{T\in\mathcal{T}_h} \|\rho - Q_b\rho\|_{\partial T} \sum_{T\in\mathcal{T}_h} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \right| \\
&\leq \left| \sum_{T\in\mathcal{T}_h} \|\rho - Q_0\rho\|_{\partial T} \sum_{T\in\mathcal{T}_h} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \right| \\
&\leq C \left(\sum_{T\in\mathcal{T}_h} \|\rho - Q_0\rho\|_T^2 + h_T^2 \|\nabla(\rho - Q_0\rho)\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \right)^{\frac{1}{2}} \\
&\leq Ch^r \|\rho\|_r \|\mathbf{v}\|_{1,h}.
\end{aligned}$$

Similarly, it follows from definition (3.14), trace inequality (5.4) and Lemma 5.1 that

$$\begin{aligned}
|l_3(q,\mathbf{w})| &= \left| \sum_{T\in\mathcal{T}_h} \langle q_0 - q_b, (\mathbf{w} - \hat{Q}_0\mathbf{w})\cdot\mathbf{n} \rangle_{\partial T} \right| \\
&\leq \left| \sum_{T\in\mathcal{T}_h} \|q_0 - q_b\|_{\partial T} \sum_{T\in\mathcal{T}_h} \|\mathbf{w} - \hat{Q}_0\mathbf{w}\|_{\partial T} \right| \\
&\leq C \sum_{T\in\mathcal{T}_h} \left(h_T^{-2} \|\mathbf{w} - \hat{Q}_0\mathbf{w}\|_T^2 + \|\nabla(\mathbf{w} - \hat{Q}_0\mathbf{w})\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T \|q_0 - q_b\|_{\partial T} \right)^{\frac{1}{2}} \\
&\leq Ch^r \|\mathbf{w}\|_{r+1} |q|_{0,h}.
\end{aligned}$$

The proof is complete. □

6 Error estimates

The goal of this section is to establish some error estimates for the WG finite element solution $(\mathbf{u}_h; p_h)$ arising from (3.15)-(3.16).

Theorem 6.1. *Let $(\boldsymbol{u}; p) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)] \times (L_0^2(\Omega) \cap H^k(\Omega))$ be the exact solution of (1.1)-(1.3), and $(\boldsymbol{u}_h; p_h) \in V_h \times W_h$ be the solutions of (3.15)-(3.16) with $k \geq 1$. Then, there exists a constant C such that*

$$|||\hat{Q}_h\boldsymbol{u} - \boldsymbol{u}_h||| + h|||Q_h p - p_h|||_1 \leq Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{6.1}$$

Proof. Let $\mathbf{v} = \mathbf{e}_h$, $q = \varepsilon_h$ in Lemma 4.2, then add (4.4) and (4.5) together, it reaches

$$|||\mathbf{e}_h|||^2 + |\varepsilon_h|_{0,h}^2 = \varphi_{\mathbf{u},p}(\mathbf{e}_h) + \phi_{\mathbf{u},p}(\varepsilon_h). \tag{6.2}$$

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From Lemma 5.2 we have

$$\varphi_{\mathbf{u},p}(\mathbf{e}_h)+\phi_{\mathbf{u},p}(\varepsilon_h)\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)(|||\mathbf{e}_h|||+|\varepsilon_h|_{0,h}).$$

Thus

$$|||\mathbf{e}_h|||+|\varepsilon_h|_{0,h}\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k). \tag{6.3}$$

For any $\mathbf{v}\in V_h^0$, we have

$$b(\mathbf{v},\varepsilon_h)=a(\mathbf{e}_h,\mathbf{v})-\varphi_{\mathbf{u},p}(\mathbf{v}),$$

which leads to

$$\begin{aligned}(\mathbf{v}_0,\nabla_w\varepsilon_h)&\leq |||\mathbf{e}_h|||\ |||\mathbf{v}|||+Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)|||\mathbf{v}|||\\&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)|||\mathbf{v}|||.\end{aligned} \tag{6.4}$$

In particular, taking $\mathbf{v}=\{\kappa\nabla_w\varepsilon_h,0\}\in V_h^0$ we can rewrite (6.4) as

$$\|\kappa^{\frac{1}{2}}\nabla_w\varepsilon_h\|^2\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)|||\mathbf{v}|||. \tag{6.5}$$

From the definition (2.7) we have

$$\begin{aligned}(\hat{\nabla}_w\mathbf{v},\hat{\nabla}_w\mathbf{v})&=(\nabla\mathbf{v}_0,\hat{\nabla}_w\mathbf{v})-\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,\hat{\nabla}_w\mathbf{v}\cdot\mathbf{n}\rangle_{\partial T}\\&=(\nabla\mathbf{v}_0,\nabla\mathbf{v}_0)-\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\hat{\nabla}_w\mathbf{v}+\nabla\mathbf{v}_0)\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

Using the trace inequality, the inverse inequality, and Young inequality yields

$$\begin{aligned}\|\hat{\nabla}_w\mathbf{v}\|^2&\leq\|\nabla\mathbf{v}_0\|^2+\sum_{T\in\mathcal{T}_h}\|\mathbf{v}_0\|_{\partial T}(\|\hat{\nabla}_w\mathbf{v}\|_{\partial T}+\|\nabla\mathbf{v}_0\|_{\partial T})\\&\leq Ch^{-2}\|\mathbf{v}_0\|^2+Ch^{-1}\|\mathbf{v}_0\|(\|\hat{\nabla}_w\mathbf{v}\|+\|\nabla\mathbf{v}_0\|)\\&\leq Ch^{-2}\|\mathbf{v}_0\|^2+Ch^{-1}\|\mathbf{v}_0\|\|\hat{\nabla}_w\mathbf{v}\|+Ch^{-1}\|\mathbf{v}_0\|\|\nabla\mathbf{v}_0\|\\&\leq Ch^{-2}\|\mathbf{v}_0\|^2+\frac{1}{2}\|\hat{\nabla}_w\mathbf{v}\|^2,\end{aligned}$$

which leads to

$$\|\hat{\nabla}_w\mathbf{v}\|\leq Ch^{-1}\|\kappa\nabla_w\varepsilon_h\|\leq Ch^{-1}\|\kappa^{\frac{1}{2}}\nabla_w\varepsilon_h\|\leq Ch^{-1}|||\varepsilon_h|||_1. \tag{6.6}$$

Also, from the trace inequality and the inverse inequality we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}h_T^{-\frac{1}{2}}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}&=\sum_{T\in\mathcal{T}_h}h_T^{-\frac{1}{2}}\|\kappa\nabla_w\varepsilon_h\|_{\partial T}\\&\leq h^{-1}\|\kappa\nabla_w\varepsilon_h\|\\&\leq Ch^{-1}|||\varepsilon_h|||_1.\end{aligned} \tag{6.7}$$

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For $\mathbf{v}_0=\kappa\nabla_w\varepsilon_h$, we have

$$\|\kappa^{-\frac{1}{2}}\mathbf{v}_0\|=\|\kappa^{\frac{1}{2}}\nabla_w\varepsilon_h\|\leq C|||\varepsilon_h|||_1,$$

combining with (6.6) and (6.7) yields

$$|||\mathbf{v}|||\leq Ch^{-1}|||\varepsilon_h|||_1. \tag{6.8}$$

Using (6.3), (6.5), and (6.8) yields

$$\begin{aligned}|||\varepsilon_h|||_1^2&\leq C(\|\kappa^{\frac{1}{2}}\nabla_w\varepsilon_h\|^2+h^{-2}|\varepsilon_h|_{0,h}^2)\\&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)|||\mathbf{v}|||+Ch^{2k-2}(\|\mathbf{u}\|_{k+1}+\|p\|_k)^2\\&\leq Ch^{k-1}(\|\mathbf{u}\|_{k+1}+\|p\|_k)|||\varepsilon_h|||_1+Ch^{2k-2}(\|\mathbf{u}\|_{k+1}+\|p\|_k)^2.\end{aligned}$$

From Young inequality, it reaches

$$|||\varepsilon_h|||_1\leq Ch^{k-1}(\|\mathbf{u}\|_{k+1}+\|p\|_k),$$

which completes the proof. □

Now we turn to the L^2 estimate for the error $\mathbf{e}_0$. As the classical finite element theory, we try to apply the Nitsche's technique. Consider the dual problem

$$-\Delta\psi+\kappa^{-1}\psi+\nabla\xi=\mathbf{e}_0 \quad \text{in } \Omega, \tag{6.9}$$

$$\nabla\cdot\psi=0 \quad \text{in } \Omega, \tag{6.10}$$

$$\psi=0 \quad \text{on } \partial\Omega. \tag{6.11}$$

Assume the solution $(\psi;\xi)$ can be found in $[H^2(\Omega)]^d\times H^1(\Omega)$ and satisfies the regularity assumption

$$\|\psi\|_2+\|\xi\|_1\leq C\|\mathbf{e}_0\|. \tag{6.12}$$

As claimed in classical theory, assumption (6.12) holds if Ω is convex and κ is not highly varying.

Theorem 6.2. *Let $(\boldsymbol{u};p)$ be the solution of problem of (1.1)-(1.3), and $(\boldsymbol{u}_h;p_h)$ be the solution of (3.15)-(3.16). Assume that (6.12) is true, then the following estimate holds true*

$$\|\hat{Q}_0\boldsymbol{u}-\boldsymbol{u}_0\|\leq Ch^{k+1}(\|\boldsymbol{u}\|_{k+1}+\|p\|_k). \tag{6.13}$$

Proof. From the error equations (4.4)-(4.5) we have

$$\begin{aligned}a(\mathbf{e}_h,\hat{Q}_h\psi)+b(\hat{Q}_h\psi,\varepsilon_h)&=\varphi_{\mathbf{u},p}(\hat{Q}_h\psi),\\b(\mathbf{e}_h,Q_h\xi)-s_2(p_h,Q_h\xi)&=\phi_{\mathbf{u},p}(Q_h\xi),\end{aligned}$$

where we take $\mathbf{v}=\hat{Q}_h\psi$ and $q=Q_h\xi$.

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Suppose (ψ_h, ξ_h) is the solution of the corresponding WG scheme of the dual problem (6.9)-(6.11). Similarly, it comes from the error equations (4.4)-(4.5) that

$$\begin{aligned} a(\hat{Q}_h\psi,\mathbf{e}_h)+b(\mathbf{e}_h,Q_h\xi) &= a(\hat{Q}_h\psi-\psi_h,\mathbf{e}_h)+b(\mathbf{e}_h,Q_h\xi-\xi_h)+\|\mathbf{e}_0\|^2, \\ &= \varphi_{\psi,\xi}(\mathbf{e}_h)+\|\mathbf{e}_0\|^2, \\ b(\hat{Q}_h\psi,\varepsilon_h)-s_2(Q_h\xi,\varepsilon_h) &= b(\hat{Q}_h\psi-\psi_h,\varepsilon_h)-s_2(Q_h\xi-\xi_h,\varepsilon_h) \\ &= \phi_{\psi,\xi}(\varepsilon_h). \end{aligned}$$

Combining the above four formulas together obtains

$$\|\mathbf{e}_0\|^2 = \varphi_{\mathbf{u},p}(\hat{Q}_h\psi)+\phi_{\mathbf{u},p}(Q_h\xi)-\varphi_{\psi,\xi}(\mathbf{e}_h)-\phi_{\psi,\xi}(\varepsilon_h). \tag{6.14}$$

Now we estimate the four terms separately. From the definition of $\varphi_{\mathbf{u},p}$, we have

$$\varphi_{\mathbf{u},p}(\hat{Q}_h\psi) = s_1(\hat{Q}_h\mathbf{u},\hat{Q}_h\psi)-l_1(\mathbf{u},\hat{Q}_h\psi)+l_2(p,\hat{Q}_h\psi).$$

Applying Lemma 6.1, it reaches that

$$\begin{aligned} |s_1(\hat{Q}_h\mathbf{u},\hat{Q}_h\psi)| &= \left|\sum_{T\in\mathcal{T}_h}\langle\hat{Q}_0\mathbf{u}-\hat{Q}_b\mathbf{u},\hat{Q}_0\psi-\hat{Q}_b\psi\rangle_{\partial T}\right| \\ &= \left|\sum_{T\in\mathcal{T}_h}\langle\hat{Q}_b\hat{Q}_0\mathbf{u}-\hat{Q}_b\mathbf{u},\hat{Q}_b\hat{Q}_0\psi-\hat{Q}_b\psi\rangle_{\partial T}\right| \\ &\le \left(\sum_{T\in\mathcal{T}_h}\|\hat{Q}_0\mathbf{u}-\mathbf{u}\|_{\partial T}\right)\left(\sum_{T\in\mathcal{T}_h}\|\hat{Q}_0\psi-\psi\|_{\partial T}\right) \\ &\le Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\psi\|_2. \end{aligned}$$

Notice that

$$\sum_{T\in\mathcal{T}_h}\langle\psi-\hat{Q}_b\psi,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle\psi-\hat{Q}_b\psi,\nabla\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T} = 0,$$

we deduce

$$\begin{aligned} |l_1(\mathbf{u},\hat{Q}_h\psi)| &= \left|\sum_{T\in\mathcal{T}_h}\langle\hat{Q}_0\psi-\hat{Q}_b\psi,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T}\right| \\ &= \left|\sum_{T\in\mathcal{T}_h}\langle\hat{Q}_0\psi-\psi,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h}\langle\psi-\hat{Q}_b\psi,(\nabla\mathbf{u}-\hat{\mathbf{Q}}_h(\nabla\mathbf{u}))\cdot\mathbf{n}\rangle_{\partial T}\right| \\ &\le Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\psi\|_2. \end{aligned}$$

Similarly, we obtain

$$|l_2(p,\hat{Q}_h\psi)| \le Ch^{k+1}\|p\|_k\|\psi\|_2,$$

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so that

$$\varphi_{\mathbf{u},p}(\hat{Q}_h\psi)\leq Ch^{k+1}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\psi\|_2. \tag{6.15}$$

From Lemma 5.2, we have

$$\begin{aligned}
|\phi_{\mathbf{u},p}(Q_h\xi)| &= |-s_2(Q_hp,Q_h\xi)+l_3(Q_h\xi,\mathbf{u})| \\
&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)|Q_h\xi|_{0,h} \\
&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)\sum_{T\in\mathcal{T}_h}h_T^{\frac{1}{2}}\|Q_0\xi-Q_b\xi\|_{\partial T} \\
&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)\sum_{T\in\mathcal{T}_h}h_T^{\frac{1}{2}}(\|Q_0\xi-\xi\|_{\partial T}+\|\xi-Q_b\xi\|_{\partial T}) \\
&\leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k)\sum_{T\in\mathcal{T}_h}h_T^{\frac{1}{2}}\|Q_0\xi-\xi\|_{\partial T} \\
&\leq Ch^{k+1}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\xi\|_1.
\end{aligned} \tag{6.16}$$

The rest two terms can be handled in the same way. Also from Lemma 5.2 we can get

$$|\varphi_{\psi,\xi}(\mathbf{e}_h)|\leq Ch(\|\psi\|_2+\|\xi\|_1)|||\mathbf{e}_h|||, \tag{6.17}$$

$$|\varphi_{\psi,\xi}(\varepsilon_h)|\leq Ch(\|\psi\|_2+\|\xi\|_1)|\varepsilon_h|_{0,h}. \tag{6.18}$$

Combining (6.15)-(6.18) and Theorem 6.1 with the assumption (6.12), it finally goes to

$$\begin{aligned}
\|\mathbf{e}_0\|^2 &\leq Ch^{k+1}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\mathbf{e}_0\|+Ch(|||\mathbf{e}_h|||+|\varepsilon_h|_{0,h})\|\mathbf{e}_0\| \\
&\leq Ch^{k+1}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\mathbf{e}_0\|,
\end{aligned}$$

which completes the proof. □

7 An effective implementation through variable reduction

The degree of freedoms of the WG formulation (3.15)-(3.16) is associated with $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}$ and $p_h=\{p_0,p_b\}$. In this section, we will demonstrate how $\mathbf{u}_0$ and p_0 can be eliminated from the system in order to obtain a global system that depends only on $\mathbf{u}_b$ and p_b. With such a variable reduction, the number of unknowns of the WG method is reduced significantly for an efficient practical implementation.

Let $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h^0$ and $p_h=\{p_0,p_b\}\in W_h^0$ be the solution of the WG method (3.15)-(3.16). Recall that $(\mathbf{u}_h;p_h)$ satisfies the following equations:

$$a(\mathbf{u}_h,\mathbf{v})+b(\mathbf{v},p_h)=(\mathbf{f},\mathbf{v}_0),\quad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{0}\}\in V_h^0, \tag{7.1}$$

$$b(\mathbf{u}_h,q)-s_2(p_h,q)=0,\quad \forall q=\{q_0,0\}\in W_h^0, \tag{7.2}$$

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and

$$a(\mathbf{u}_h,\mathbf{v})=0,\quad \forall \mathbf{v}=\{\mathbf{0},\mathbf{v}_b\}\in V_h^0, \tag{7.3}$$
$$b(\mathbf{u}_h,q)-s_2(p_h,q)=0,\quad \forall q=\{0,q_b\}\in W_h^0. \tag{7.4}$$

Denote by $V_k(T)$ and $W_k(T)$ the restrictions of V_h and W_h on T:

$$V_k(T)=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\mathbf{v}_0|_T\in[P_k(T)]^d,\mathbf{v}_b|_T\in[P_k(e)]^d,e\subset\partial T\}.$$

Since the testing functions $\mathbf{v}=\{\mathbf{v}_0,\mathbf{0}\}$ and $q=\{q_0,0\}$ are locally supported on each element, then the system of equations (7.1)-(7.2) is equivalent to the following system of equations defined locally on each element T:

$$a(\mathbf{u}_h,\mathbf{v})+b(\mathbf{v},p_h)=(\mathbf{f},\mathbf{v}_0),\quad \forall \mathbf{v}=\{\mathbf{v}_0,\mathbf{0}\}\in V_k(T), \tag{7.5}$$
$$b(\mathbf{u}_h,q)-s_2(p_h,q)=0,\quad \forall q=\{q_0,0\}\in W_k(T). \tag{7.6}$$

If the exact solution of $\mathbf{u}_b$ and p_b were known on ∂T, then the corresponding $\mathbf{u}_0$ and p_0 can be obtained by solving (7.5) and (7.6) locally on each element. Therefore, the key is to derive a system of equations that shall determine $\mathbf{u}_b$ and p_b.

For any given $\mathbf{u}_b$ and p_b on ∂T, let us solve (7.5) and (7.6) to obtain $\mathbf{u}_0$ and p_0 on each element T. For simplicity, we introduce the following notation

$$\mathbf{u}_0=D(\mathbf{u}_b,p_b,\mathbf{f}), \tag{7.7}$$
$$p_0=E(\mathbf{u}_b,p_b,\mathbf{f}). \tag{7.8}$$

Then the solution of $\mathbf{u}_b$ and p_b of (3.15)-(3.16) can be written as $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}=\{D(\mathbf{u}_b,p_b,\mathbf{f}),\mathbf{u}_b\}$ and $p=p_h=\{E(\mathbf{u}_b,p_b,\mathbf{f}),p_b\}$.

Let $D_1(\mathbf{u}_b,p_b)=D(\mathbf{u}_b,p_b,0)$ and $D_2(\mathbf{f})=D(0,0,\mathbf{f})$. Similarly, let $E_1(\mathbf{u}_b,p_b)=E(\mathbf{u}_b,p_b,0)$ and $E_2(\mathbf{f})=E(0,0,\mathbf{f})$. Since $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $s_2(\cdot,\cdot)$ are bilinear, then superposition implies

$$\begin{aligned}(\mathbf{u}_h;p_h)&=(\{\mathbf{u}_0,\mathbf{u}_b\};\{p_0,p_b\})\\&=(\{D(\mathbf{u}_b,p_b,\mathbf{f}),\mathbf{u}_b\};\{E(\mathbf{u}_b,p_b,\mathbf{f}),p_b\})\\&=(\{D(\mathbf{u}_b,p_b,\mathbf{0}),\mathbf{u}_b\};\{E(\mathbf{u}_b,p_b,\mathbf{0}),p_b\})+(\{D(0,0,\mathbf{f}),\mathbf{u}_b\};\{E(0,0,\mathbf{f}),p_b\})\\&=(\{D_1(\mathbf{u}_b,p_b),\mathbf{u}_b\};\{E_1(\mathbf{u}_b,p_b),p_b\})+(\{D_2(\mathbf{f}),\mathbf{u}_b\};\{E_2(\mathbf{f}),p_b\}).\end{aligned}$$

Substituting $\mathbf{u}_h=\{D(\mathbf{u}_b,p_b,\mathbf{f})\}$ and $p_h=\{E(\mathbf{u}_b,p_b,\mathbf{f}),p_b\}$ into the system (7.3)-(7.4) yields

$$a(\{D_1(\mathbf{u}_b,p_b),\mathbf{u}_b\},\mathbf{v})=\zeta_1(\mathbf{v}), \tag{7.9}$$
$$b(\{D_1(\mathbf{u}_b,p_b),\mathbf{u}_b\},q)-s_2(\{E_1(\mathbf{u}_b,p_b),p_b\},q)=\zeta_2(q), \tag{7.10}$$

for all $\mathbf{v}=\{0,\mathbf{v}_b\}\in V_h^0$ and $q=0,q_b\in W_h^0$. Here

$$\zeta_1(\mathbf{v})=-a(\{D_2(\mathbf{f}),0\},\mathbf{v}),$$
$$\zeta_2(q)=-b(\{D_2(\mathbf{f}),0\},q)+s_2(\{E_2(\mathbf{f}),0\},q).$$

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Note that the system (7.9)-(7.10) is a square matrix problem with $\mathbf{u}_b$ and p_b as unknowns, and this is the system of equations that $\mathbf{u}_b$ and p_b have to satisfy.

To summarize, our WG scheme (3.15)-(3.16) can be implemented as follows:

Step 1. Find $\mathbf{u}_b$ and p_b with $\mathbf{u}_b=\mathbf{0}$ and $p_b=0$ on $\partial\Omega$ satisfying (7.9)-(7.10).

Step 2. Recover $\mathbf{u}_0$ and p_0 by $\mathbf{u}_0=D(\mathbf{u}_b,p_b,\mathbf{f})$ and $p_0=E(\mathbf{u}_b,p_b,\mathbf{f})$ by solving (7.5) and (7.6) locally on each element.

The system of equations (7.9)-(7.10) is known as a Schur complement of the original WG finite element scheme (3.15)-(3.16).

8 Numerical experiments

The goal of this section is to report some numerical results for the weak Galerkin finite element algorithm (3.15)-(3.16) proposed and analyzed in previous sections when the lowest order of element (i.e., $k=1$) is chosen.

Let $(\mathbf{u};p)$ be the exact solution of (1.1)-(1.3) and $(\mathbf{u}_h;p_h)$ be the numerical solution of (3.15)-(3.16). Denote $\mathbf{e}_h=\hat{Q}_h\mathbf{u}-\mathbf{u}_h$ and $\varepsilon_h=Q_hp-p_h$. The error for the weak Galerkin solution is measured in three norms defined as follows:

$$
\begin{aligned}
|||\mathbf{v}|||^{\mathbf{2}} &:= \sum_{T\in\mathcal{T}_h}\left(\|\kappa^{-\frac{1}{2}}\mathbf{v}_0\|^2+\|\hat{\nabla}_w\mathbf{v}\|^2+\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\hat{\mathbf{v}}_0-\mathbf{v_b}\|^{\mathbf{2}}_{\partial\mathbf{T}}\right),\\
\|\mathbf{v}\|^2 &:= \sum_{T\in\mathcal{T}_h}\|\mathbf{v}\|_T^2,\\
\|p\|^2 &:= \sum_{T\in\mathcal{T}_h}\|p\|_T^2.
\end{aligned}
$$

The numerical simulation of this section has been discussed in [7, 20]. The first example is presented for studying the reliability of the WG method for problems with high contrast of permeability such that κ^{-1} varies from 1 to 10^4. In particular, the first example(see [20]), which has known analytical solution, is designed to demonstrate uniform convergence of the WG method. In this kind of geometry, large highly permeable media connect vugs surrounded by a quite lowly permeable material.

Example 8.1. Consider the problem (1.1)-(1.3) in the square domain $\Omega=(0,1)^2$. This example will test the accuracy and reliability of the method for given analytical solutions and greatly varying permeability tensor κ. The profile of κ^{-1} is shown in Fig. 1.

The WG finite element space $k=1$ is employed in the numerical discretization. It has the analytic solution

$$
\mathbf{u}=\begin{pmatrix}\sin(2\pi x)\cos(2\pi y)\\ -\cos(2\pi x)\sin(2\pi y)\end{pmatrix}\quad\text{and}\quad p=x^2y^2-\frac{1}{9},
$$

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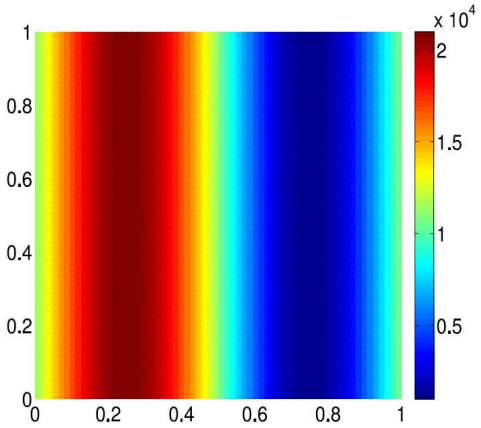


Figure 1: Geometry for κ^{-1} in Example 8.1 with $a=10^4$.

and $\kappa^{-1}=a(\sin(2\pi x)+1.1)$, where a is a given constant. The values of κ^{-1} are plotted in Fig. 1 for $a=10^4$. The right hand side function $\mathbf{f}$ in (1.1) is computed to match the exact solution. The mesh size is denoted by h.

Table 1 shows that the errors and convergence rates of Example 8.1 in $|||\cdot|||$-norm, L^2-norm for $\mathbf{u}$ and L^2-norm for p of the WG-FEM solutions are of order $\mathcal{O}(h)$, $\mathcal{O}(h^2)$, and $\mathcal{O}(h)$ when $k=1$, respectively as $\mu=1$, $a=1$.

Tables 2-4 show that the errors and orders of Example 8.1 in the above three norms as $\mu=1$, $a=10^4$; $\mu=0.01$, $a=1$; and $\mu=0.01$, $a=10^4$, respectively.

Table 1: Numerical errors and orders for Example 8.1 as $\mu=1$, $a=1$.

h	$\|\|\|\mathbf{e}_h\|\|\|$	order	$\|\mathbf{e}_0\|$	order	$\|\varepsilon_h\|$	order
1/64	3.6091e-01		1.5674e-03		9.5389e-03	
1/80	2.8873e-01	1.0000	9.9756e-04	2.0250	7.4283e-03	1.1207
1/96	2.4061e-01	1.0000	6.9064e-04	2.0167	6.0826e-03	1.0963
1/112	2.0623e-01	1.0000	5.0648e-04	2.0119	5.1508e-03	1.0787
1/128	1.8045e-01	1.0000	3.8731e-04	2.0089	4.4675e-03	1.0657

Table 2: Numerical errors and orders for Example 8.1 as $\mu=1$, $a=10^4$.

h	$\|\|\|\mathbf{e}_h\|\|\|$	order	$\|\mathbf{e}_0\|$	order	$\|\varepsilon_h\|$	order
1/64	3.5656e-01		8.9465E-04		1.1866E-01	
1/80	2.8636e-01	0.9826	6.1718E-04	1.6639	1.0147E-01	0.7015
1/96	2.3918e-01	0.9873	4.4712E-04	1.7679	8.6278E-02	0.8893
1/112	2.0532e-01	0.9905	3.3713E-04	1.8316	7.3340E-02	1.0539
1/128	1.7983e-01	0.9926	2.6254E-04	1.8727	6.2539E-02	1.1931

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Table 3: Numerical errors and orders for Example 8.1 as $\mu=0.01$, $a=1$.

h	$\|\|\|\mathbf{e}_h\|\|\|$	order	$\|\mathbf{e}_0\|$	order	$\|\varepsilon_h\|$	order
1/64	6.4479e-02		2.0572e-03		9.9475e-03	
1/80	5.1566e-02	1.0015	1.3075e-03	2.0309	7.9746e-03	0.9907
1/96	4.2965e-02	1.0009	9.0449e-04	2.0213	6.6527e-03	0.9941
1/112	3.6824e-02	1.0005	6.6294e-04	2.0155	5.7058e-03	0.9960
1/128	3.2219e-02	1.0003	5.0676e-04	2.0118	4.9945e-03	0.9971

Table 4: Numerical errors and orders for Example 8.1 as $\mu=0.01$, $a=10^4$.

h	$\|\|\|\mathbf{e}_h\|\|\|$	order	$\|\mathbf{e}_0\|$	order	$\|\varepsilon_h\|$	order
1/64	6.4309e-02		6.2028e-04		1.0891e-02	
1/80	5.1479e-02	0.9973	3.9639e-04	2.0067	8.4729e-03	1.1250
1/96	4.2914e-02	0.9980	2.7500e-04	2.0053	6.9462e-03	1.0897
1/112	3.6792e-02	0.9985	2.0191e-04	2.0042	5.8927e-03	1.0670
1/128	3.2198e-02	0.9988	1.5452e-04	2.0034	5.1206e-03	1.0517

The numerical results are consistent with theory for these cases and demonstrate that the WG method is accurate and robust.

The rest examples have the data setting as follows:

$$\Omega=(0,1)^2, \quad \mu=0.01, \quad \mathbf{f}=0, \quad \mathbf{g}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{8.1}$$

Example 8.2 is relevant to practical problems for which no analytical solutions are known.

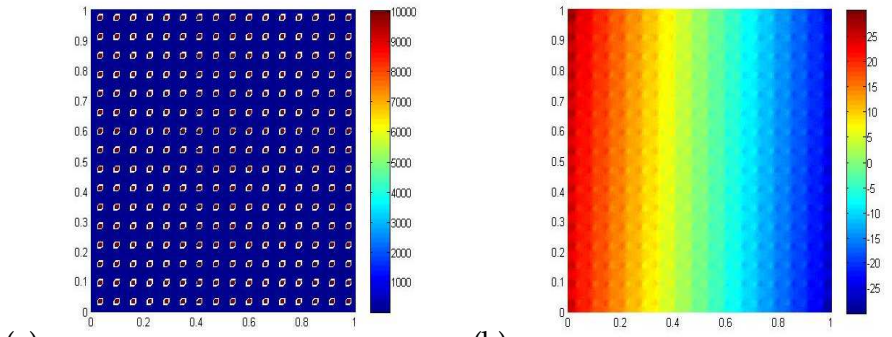
Example 8.2. The Brinkman equations (1.1)-(1.3) model flow of fluid in complex porous media with a permeability coefficient highly varying. The profile of the permeability inverse is plotted in Fig. 2(a).

For this example, a 128×128 mesh is used for plotting Fig. 2 and Fig. 3. The pressure profile of the WG method is presented in Fig. 2(b). The first and the second components of the velocity calculated by the WG method are shown in Figs. 3(a) and 3(b) respectively.

The following two examples simulate flow through porous media with different geometries which are also no analytical solutions known. Flow through vuggy media and fibrous materials are tried and their permeability inverse profiles can be found in the literatures such as [7,10].

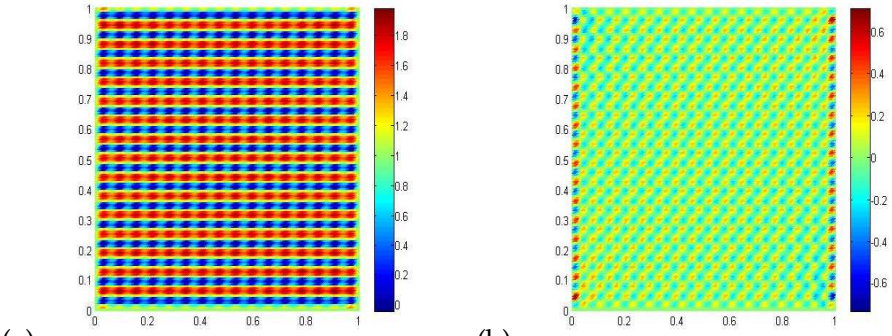
Example 8.3. This example is frequently used in a vuggy medium with the data set in (8.1). A 128×128 mesh is used for plotting Fig. 4 and Fig. 5. The inverse of permeability

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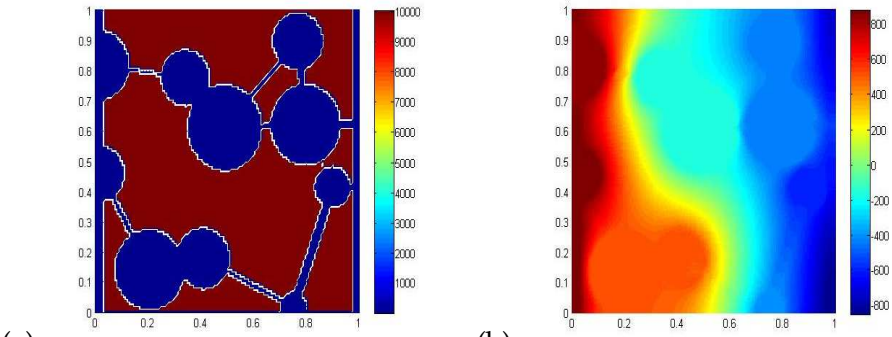
(a) (b)

Figure 2: Example 8.2: (a) Profile of κ^{-1}; (b) Pressure profile.



(a) (b)

Figure 3: Example 8.2: (a) First component of velocity u_1; (b) Second component of velocity u_2.

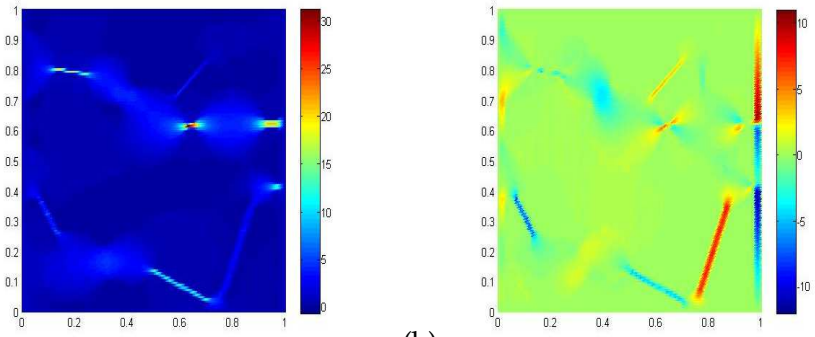


(a) (b)

Figure 4: Example 8.3: (a) Profile of κ^{-1} for vuggy medium; (b) Pressure profile.

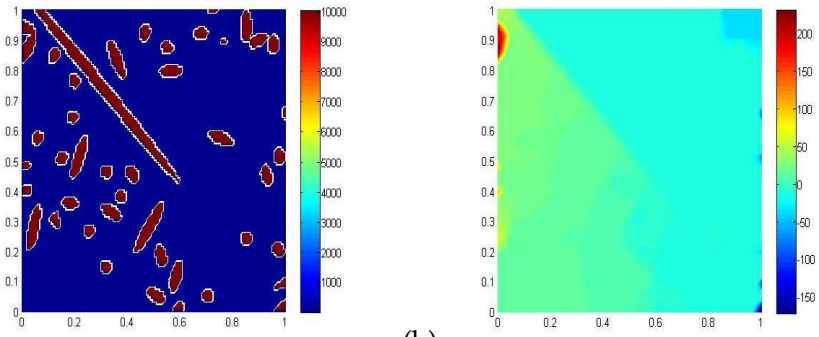
of fibrous structure is shown in Fig. 4(a). The pressure profile of the WG method is presented in Fig. 4(b). The first and the second components of the velocity calculated by the WG method are shown in Figs. 5(a) and 5(b) respectively.

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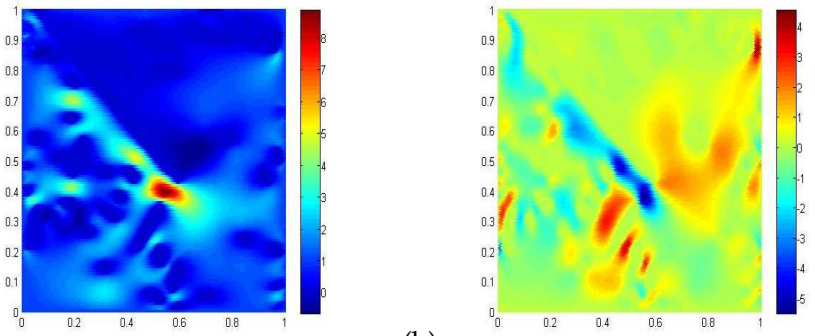
(a) (b)

Figure 5: Example 8.3: (a) First component of velocity u_1; (b) Second component of velocity u_2.



(a) (b)

Figure 6: Example 8.4: (a) Profile of κ^{-1} for fibrous structure; (b) Pressure profile.



(a) (b)

Figure 7: Example 8.4: (a) First component of velocity u_1; (b) Second component of velocity u_2.

Example 8.4. This example is frequently used in filtration and insulation materials. The inverse of permeability of fibrous structure is shown in Fig. 6(a). A 128×128 mesh is used for plotting Fig. 6 and Fig. 7. The inverse of permeability of fibrous structure is shown

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in Fig. 6(a). The pressure profile of the WG method is presented in Fig. 6(b). The first and the second components of the velocity calculated by the WG method are shown in Figs. 7(a) and 7(b) respectively.

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