East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.181016.300517h

Vol. 7, No. 4, pp. 827-836
November 2017

A Convergence Analysis of the MINRES Method for Some Hermitian Indefinite Systems

Ze-Jia Xie[1], Xiao-Qing Jin[2,*] and Zhi Zhao[3]

[1] *Department of Mathematics and Data Science, Dongguan University of Technology, Dongguan, 523808, China.*
[2] *Department of Mathematics, University of Macau, Macao.*
[3] *Department of Mathematics, School of Sciences, Hangzhou Dianzi University, Hangzhou, 310018, China.*

Received 18 October 2016; Accepted (in revised version) 30 May 2017.

Abstract. Some convergence bounds of the **min**imal **res**idual (MINRES) method are studied when the method is applied for solving Hermitian indefinite linear systems. The matrices of these linear systems are supposed to have some properties so that their spectra are all clustered around ± 1. New convergence bounds depending on the spectrum of the coefficient matrix are presented. Some numerical experiments are shown to demonstrate our theoretical results.

AMS subject classifications: 65F10, 65F15, 65L05, 65N22

Key words: MINRES, Convergence bound, Hermitian indefinite, Toeplitz system.

1. Introduction

Let $H_n \in \mathbb{C}^{n\times n}$ be a nonsingular Hermitian matrix. We consider solving the linear system

$$H_n \mathbf{x}_n = \mathbf{b}_n \tag{1.1}$$

by using the **min**imal **res**idual (MINRES) method, which has favorable properties (see, e.g., [27]):

(i) minimization of the residual;
(ii) short-term recurrences;
(iii) sharp convergence bounds depending on the spectrum of the matrix.

The MINRES method is a Krylov subspace method developed by Paige and Saunders [26]. Thanks to its favorable properties above, it is popular for solving Hermitian linear systems. A large amount of literature (see, e.g., [1–3, 5–7, 9–18, 23–25, 27–30, 33–36]

*Corresponding author. *Email addresses:* xiezejia2012@gmail.com (Z.-J. Xie), xqjincumac.mo (X.-Q. Jin), zhaozhi231@163.com (Z. Zhao)

 ©2017 Global-Science Press

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

and the references therein) exists on discussing the convergence rate of Krylov subspace methods, including the **c**onjugate **g**radient (CG) method, the **g**eneralized **m**inimal **res**idual (GMRES) method, the MINRES method, and so on.

In this paper, we consider the convergence rate of the MINRES method for solving Hermitian indefinite linear systems. Existing convergence analyses for the GMRES and MINRES methods can be found in [16,23,28,29,34] and the references therein. Different from these analyses, our motivation is to demonstrate the relation between a matrix decomposition and the superlinear convergence rate. We try to expose some convergence results for the MINRES method as those in [18, Subsection 1.3] for the CG method. Similar to the case of the CG method, the convergence bounds of the MINRES progress are largely related to the spectral distribution of the coefficient matrix. By using preconditioners and Proposition 2 in [27], we show that the MINRES method has a superlinear convergence rate when it is applied for solving a large class of Toeplitz systems.

2. Preliminaries

Denote the Euclidean norm of vectors by $\|\cdot\|_2$. Let $\mathbf{r}_n^{(k)}$ be the residual vector after kth iteration of the MINRES method applied for solving the Hermitian linear system $H_n\mathbf{x}_n = \mathbf{b}_n$, i.e., $\mathbf{r}_n^{(k)} = \mathbf{b}_n - H_n\mathbf{x}_n^{(k)}$.

Let $\mathscr{P}_k$ be the set of all real polynomials of degree not exceeding k with constant term 1, and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ (or $\lambda(H_n)$) be the spectrum of H_n. The following results can be found in [15, Subsection 3.1] and [12, Section 6].

Lemma 2.1 ([15]). *Let $\mathbf{r}_n^{(k)}$ be the residual vector after kth iteration of the MINRES method applied to the Hermitian linear system $H_n\mathbf{x}_n = \mathbf{b}_n$. Then one has*

$$\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} \le \min_{p_k\in\mathscr{P}_k} \max_{1\le j\le n} \left|p_k(\lambda_j)\right|. \tag{2.1}$$

Lemma 2.2 ([15]). *Let b_1, b_2, b_3, and b_4 be positive constants with $b_1 - b_2 = b_4 - b_3$. Then*

$$\min_{p_k\in\mathscr{P}_k} \max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]} \left|p_k(\lambda)\right| \le 2\left(\frac{\sqrt{b_1b_4}-\sqrt{b_2b_3}}{\sqrt{b_1b_4}+\sqrt{b_2b_3}}\right)^{[k/2]}, \tag{2.2}$$

where $[x]$ denotes the largest integer less than or equal to a number x.

3. Main Theorems

In this section we consider some convergence bounds of the MINRES method when it is applied for solving the Hermitian indefinite linear system $H_n\mathbf{x}_n = \mathbf{b}_n$.

Theorem 3.1. *If the eigenvalues of $H_n \in \mathbb{C}^{n\times n}$ are ordered such that*

$$\lambda_1 \le \cdots \le \lambda_p \le -b_1 \le \lambda_{p+1} \le \cdots \le \lambda_{n_1-q} \le -b_2 \le \lambda_{n_1-q+1} \le \cdots \le \lambda_{n_1} < 0,$$

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

and

$$0 < \lambda_{n_1+1} \le \cdots \le \lambda_{n_1+m} \le b_3 \le \lambda_{n_1+m+1} \le \cdots \le \lambda_{n-l} \le b_4 \le \lambda_{n-l+1} \le \cdots \le \lambda_n,$$

where b_1, b_2, b_3, *and* b_4 *are four positive constants with* $b_1 - b_2 = b_4 - b_3$, *then for* $k \ge p+q+m+l$, *the iteration residual* $\mathbf{r}_n^{(k)} = \mathbf{b}_n - H_n\mathbf{x}_n^{(k)}$ *satisfies*

$$\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} \le 2\left(\frac{\sqrt{b_1b_4}-\sqrt{b_2b_3}}{\sqrt{b_1b_4}+\sqrt{b_2b_3}}\right)^{[(k-p-q-m-l)/2]} \cdot \max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]} \left|P(\lambda)\right|, \tag{3.1}$$

where

$$P(\lambda) := \prod_{j=1}^{p}\left(\frac{\lambda-\lambda_j}{\lambda_j}\right)\cdot \prod_{j=n_1-q+1}^{n_1}\left(\frac{\lambda_j-\lambda}{\lambda_j}\right)\cdot \prod_{j=n_1+1}^{n_1+m}\left(\frac{\lambda-\lambda_j}{\lambda_j}\right)\cdot \prod_{j=n-l+1}^{n}\left(\frac{\lambda_j-\lambda}{\lambda_j}\right).$$

Proof. Let $d := k-p-q-m-l$, and $\mathscr{P}_k$ be the set of all real polynomials of degree not exceeding k such that $p_k(0) = 1$ for any $p_k \in \mathscr{P}_k$. Given a polynomial $p_d \in \mathscr{P}_d$, we construct a special polynomial as follows:

$$\widetilde{p}_k(\lambda) := (-1)^{p+m}\cdot p_d(\lambda)\cdot P(\lambda),$$

where

$$P(\lambda) := \prod_{j=1}^{p}\left(\frac{\lambda-\lambda_j}{\lambda_j}\right)\cdot \prod_{j=n_1-q+1}^{n_1}\left(\frac{\lambda_j-\lambda}{\lambda_j}\right)\cdot \prod_{j=n_1+1}^{n_1+m}\left(\frac{\lambda-\lambda_j}{\lambda_j}\right)\cdot \prod_{j=n-l+1}^{n}\left(\frac{\lambda_j-\lambda}{\lambda_j}\right).$$

It is obvious that

$$\widetilde{p}_k(\lambda_i) = 0 \text{ for } i = 1,\cdots,p,\ n_1-q+1,\cdots,n_1,\ n_1+1,\cdots,n_1+m,\ n-l+1,\cdots,n,$$

and $\widetilde{p}_k(0) = 1$. Therefore, $\widetilde{p}_k \in \mathscr{P}_k$. Then it follows from Lemma 2.1 that

$$\begin{aligned}\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} &\le \min_{p_k\in\mathscr{P}_k}\max_{1\le j\le n}\left|p_k(\lambda_j)\right| \le \max_{1\le j\le n}\left|\widetilde{p}_k(\lambda_j)\right| \\ &= \max_{\substack{p+1\le j\le n_1-q\\ n_1+m+1\le j\le n-l}}\left|\widetilde{p}_k(\lambda_j)\right| \le \max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|\widetilde{p}_k(\lambda)\right| \\ &\le \max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|p_d(\lambda)\right|\cdot\max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|P(\lambda)\right|.\end{aligned}$$

Since the above inequality holds for any $p_d \in \mathscr{P}_d$, we have from Lemma 2.2 that

$$\begin{aligned}\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} &\le \min_{p_d\in\mathscr{P}_d}\max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|p_d(\lambda)\right|\cdot\max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|P(\lambda)\right| \\ &\le 2\left(\frac{\sqrt{b_1b_4}-\sqrt{b_2b_3}}{\sqrt{b_1b_4}+\sqrt{b_2b_3}}\right)^{[d/2]}\cdot\max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]}\left|P(\lambda)\right|,\end{aligned}$$

which proves (3.1) with $d = k-p-q-m-l$. □

Moreover, we have the following theorem.

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

Theorem 3.2. *Let $0 < \epsilon < 1$, and the eigenvalues of $H_n \in \mathbb{C}^{n\times n}$ be ordered such that*

$$\lambda_1 \le \cdots \le \lambda_p \le -1-\epsilon \le \lambda_{p+1} \le \cdots \le \lambda_{n_1-q} \le -1+\epsilon \le \lambda_{n_1-q+1} \le \cdots \le \lambda_{n_1} \le -\delta < 0,$$

and

$$0 < \delta \le \lambda_{n_1+1} \le \cdots \le \lambda_{n_1+m} \le 1-\epsilon \le \lambda_{n_1+m+1} \le \cdots \le \lambda_{n-l} \le 1+\epsilon \le \lambda_{n-l+1} \le \cdots \le \lambda_n,$$

where $\delta > 0$ is a constant independent of n. Then

$$\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} < 2B\epsilon^{[(k-p-q-m-l)/2]}, \tag{3.2}$$

where $k \ge p+q+m+l$, and

$$B := \max\left\{\left(\frac{1+\epsilon}{\delta}\right)^q \cdot \left(\frac{2}{\delta}\right)^m \cdot 2^l,\ 2^p \cdot \left(\frac{2}{\delta}\right)^q \cdot \left(\frac{1+\epsilon}{\delta}\right)^m\right\}. \tag{3.3}$$

Proof. Taking $b_1 = b_4 = 1+\epsilon$ and $b_2 = b_3 = 1-\epsilon$ in Theorem 3.1 gives

$$\begin{aligned}
&\left(\frac{\sqrt{b_1 b_4} - \sqrt{b_2 b_3}}{\sqrt{b_1 b_4} + \sqrt{b_2 b_3}}\right)^{[(k-p-q-m-l)/2]} \\
=&\left(\frac{1+\epsilon-1+\epsilon}{1+\epsilon+1-\epsilon}\right)^{[(k-p-q-m-l)/2]} \\
=&\epsilon^{[(k-p-q-m-l)/2]},
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\max_{\lambda\in[-b_1,-b_2]\cup[b_3,b_4]} \left|P(\lambda)\right| \\
=&\max_{\lambda\in[-1-\epsilon,-1+\epsilon]\cup[1-\epsilon,1+\epsilon]} \left|P(\lambda)\right| \\
=&\max\left\{\max_{\lambda\in[-1-\epsilon,-1+\epsilon]} |P(\lambda)|,\ \max_{\lambda\in[1-\epsilon,1+\epsilon]} \left|P(\lambda)\right|\right\}.
\end{aligned}$$

If $\lambda \in [-1-\epsilon, -1+\epsilon]$, it follows from the spectral distribution of H_n that

$$\begin{aligned}
&\frac{|\lambda-\lambda_j|}{|\lambda_j|} < 1 \quad \text{for } j = 1,\cdots,p; \\
&\frac{|\lambda_j-\lambda|}{|\lambda_j|} < \frac{1+\epsilon}{\delta} \quad \text{for } j = n_1-q+1,\cdots,n_1; \\
&\frac{|\lambda-\lambda_j|}{|\lambda_j|} < \frac{1+\epsilon+1-\epsilon}{\delta} = \frac{2}{\delta} \quad \text{for } j = n_1+1,\cdots,n_1+m; \\
&\frac{|\lambda_j-\lambda|}{|\lambda_j|} \le \frac{1+\epsilon+\lambda_j}{\lambda_j} = 1 + \frac{1+\epsilon}{\lambda_j} \le 1 + \frac{1+\epsilon}{1+\epsilon} = 2 \quad \text{for } j = n-l+1,\cdots,n.
\end{aligned}$$

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

Therefore,

$$\max_{\lambda\in[-1-\epsilon,-1+\epsilon]}\left|P(\lambda)\right| < \left(\frac{1+\epsilon}{\delta}\right)^q \cdot \left(\frac{2}{\delta}\right)^m \cdot 2^l.$$

Similarly, if $\lambda \in [1-\epsilon, 1+\epsilon]$, we have

$$\max_{\lambda\in[1-\epsilon,1+\epsilon]}\left|P(\lambda)\right| < 2^p \cdot \left(\frac{2}{\delta}\right)^q \cdot \left(\frac{1+\epsilon}{\delta}\right)^m.$$

Hence, from the above two inequalities and (3.1) and (3.4), the inequality in (3.2) holds. □

It is well known [11, 18] that there is a relationship between the matrix decomposition and the superlinear convergence rate of the CG method. For the MINRES method, similarly, the following matrix decomposition (3.5) also implies a superlinear convergence rate. Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of Hermitian indefinite matrices, and $\lambda(H_n)$ be the spectrum of H_n. Suppose that $\min\{|\lambda(H_n)|\}$ is uniformly bounded away from zero. We consider the following matrix decomposition for H_n. Given any $\epsilon > 0$, if there exist M and $N > 0$ such that when $n > N$, H_n can be decomposed as

$$H_n = U_n + R_n + E_n, \tag{3.5}$$

where U_n is unitary with eigenvalues ± 1, R_n and E_n are Hermitian matrices with $\operatorname{rank}(R_n) \le M$, and $\|E_n\|_2 < \epsilon$. Then the spectrum of H_n is clustered around 1 and -1.

More precisely, from (3.5), the spectrum of $U_n + E_n$ is contained in the interval $(-1-\epsilon, -1+\epsilon) \cup (1-\epsilon, 1+\epsilon)$ and all the eigenvalues of R_n are equal to zero except at most M nonzero eigenvalues. Thus from Weyl's theorem, all the eigenvalues of H_n are in the interval $(-1-\epsilon, -1+\epsilon) \cup (1-\epsilon, 1+\epsilon)$ except at most M outlying eigenvalues. Moreover, if $\min\{|\lambda(H_n)|\}$ is uniformly bounded away from zero, then we see that the spectrum of H_n satisfies all the conditions in Theorem 3.2. Take $K = \max\{M, N\}$. When $n > k \ge K$, the kth residual $\mathbf{r}_n^{(k)}$ of $H_n\mathbf{x}_n = \mathbf{b}_n$ satisfies the convergence bound (3.2), which actually shows that the convergence rate of the MINRES method is superlinear (see Fig. 1) when it is applied for solving $H_n\mathbf{x}_n = \mathbf{b}_n$.

Next we show that some Toeplitz systems could have the decomposition (3.5). Consider the following Toeplitz system

$$T_n\mathbf{x}_n = \mathbf{b}_n, \tag{3.6}$$

where

$$T_n = (t_{ij}) = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \cdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is a Toeplitz matrix, i.e., $t_{ij} = t_{i-j}$. If the diagonals $\{t_k\}_{k=-n+1}^{n-1}$ of T_n are the Fourier coefficients of a function f, i.e.,

$$t_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}dx,$$

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

where $\mathbf{i} := \sqrt{-1}$, then the function f is called the generating function of T_n. If f is a real-valued function defined on $[-\pi, \pi]$, then the Toeplitz matrix T_n is Hermitian (see, e.g., [4, 9]). Some properties on the Toeplitz matrix T_n and its generating function f can be found in [18, Subsection 1.5.2]. The generating function f is said to be in the Wiener class if $\sum_{k=-\infty}^{\infty} |t_k(f)| < \infty$ (see, e.g., [18, p. 23]).

In [27], Pestana and Wathen transform a nonnormal Toeplitz matrix to a symmetric matrix by a simple device of reordering, i.e., $Y_n T_n$ is symmetric, where

$$Y_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{3.7}$$

Hence, the transformed system $Y_n T_n \mathbf{x}_n = Y_n \mathbf{b}_n$ can be solved by the MINRES method. If the eigenvalues of $Y_n T_n$ are not nicely distributed, the convergence of this method may be slow. Pestana and Wathen [27] also provide a preconditioning technique to accelerate the convergence. They use a circulant preconditioner C_n to build the positive definite preconditioner $|C_n| := (C_n^T C_n)^{1/2}$ for the MINRES method.

The following theorem given in [27] shows that the decomposition (3.5) exists for some Toeplitz matrices.

Theorem 3.3 ([27]). *Let*

$$C_n^{-1} T_n = I_n + R_n + E_n, \tag{3.8}$$

where $\operatorname{rank}(R_n) \le M$ *and* $\|E_n\| \le \epsilon$ *with any unitarily invariant norm* $\|\cdot\|$ *and any* $\epsilon > 0$. *Then*

$$|C_n|^{-1} Y_n T_n = Q_n + \widehat{R}_n + \widehat{E}_n, \tag{3.9}$$

where $\operatorname{rank}(\widehat{R}_n) = \operatorname{rank}(R_n)$ *and* $\|\widehat{E}_n\| = \|E_n\|$. *Moreover,* Q_n *is orthogonal and its eigenvalues are* 1 *or* -1.

Theorem 3.3 shows that if the preconditioned matrix $C_n^{-1} T_n$ has the decomposition (3.8), the eigenvalues of $|C_n|^{-1} Y_n T_n$ are clustered near ± 1 except M outliers. Hence the MINRES method for solving

$$|C_n|^{-1} Y_n T_n \mathbf{x}_n = \mathbf{b}_n$$

has a superlinear convergence rate.

4. Numerical Examples

In this section, we give a numerical example to demonstrate our theoretical results.

Example 4.1. Let Y_n be given by (3.7). Consider the indefinite Toeplitz system

$$Y_n T_n \mathbf{x}_n = Y_n (I_n - 8 G_n^{(\alpha)}) \mathbf{x}_n = Y_n \mathbf{b}_n,$$

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

Table 1: The number of iterations for the indefinite case.

n	Unpreconditioned	Strang	Optimal	Superoptimal
128	179	9	18	32
256	384	9	15	27
512	518	9	13	21
1024	563	9	11	18

where

$$G_n^{(\alpha)} = \begin{bmatrix} g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \cdots & 0 & 0 \\ g_2^{(\alpha)} & g_1^{(\alpha)} & g_0^{(\alpha)} & \ddots & \ddots & 0 \\ \vdots & g_2^{(\alpha)} & g_1^{(\alpha)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ g_{n-1}^{(\alpha)} & \ddots & \ddots & \ddots & g_1^{(\alpha)} & g_0^{(\alpha)} \\ g_n^{(\alpha)} & g_{n-1}^{(\alpha)} & \cdots & \cdots & g_2^{(\alpha)} & g_1^{(\alpha)} \end{bmatrix}$$

with $g_0^{(\alpha)} = 1$, and

$$g_k^{(\alpha)} = \frac{(-1)^k}{k!}\alpha(\alpha-1)\cdots(\alpha-k+1) \quad \text{for } k = 1, 2, \cdots, n.$$

The matrix $G_n^{(\alpha)}$ is obtained from a discretisation of a fractional derivative [22]. Obviously, $T_n = I_n - 8G_n^{(\alpha)}$ is a nonsymmetric Toeplitz matrix. We use the technique given by Pestana and Wathen [27] to convert T_n to a symmetric matrix, and then solve the transformed system by the MINRES method. In [27], the authors also propose a method to construct the preconditioner for the transformed system by existing circulant preconditioners. We then use their preconditioned technique in this example. The analysis on the existence of the decomposition (3.9) for this example could be found in [21] combining with Theorem 3.3. Take $\alpha = 1.8$. We compare the performance of the Strang [31], optimal [8, 20] and superoptimal [20, 32] preconditioners with the unpreconditioned case. The right hand side $\mathbf{b}_n$ is set to be the vector whose elements are all ones, and the initial guess for the MINRES method is the zero vector. The stopping criterion is that the relative residual after kth iteration satisfies

$$\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} < 10^{-10},$$

or the residual reduction is not achieved in 10000 iterations.

Fig. 1 shows the convergence curves of the MINRES method with different preconditioners. From this figure, we also see that for the MINRES methods with eigenvalues clustered around ± 1 (see the spectral distributions in Fig. 2), the reduction of the residual accelerates as the matrix size n increases. Table 1 shows the number of iterations required for convergence.

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

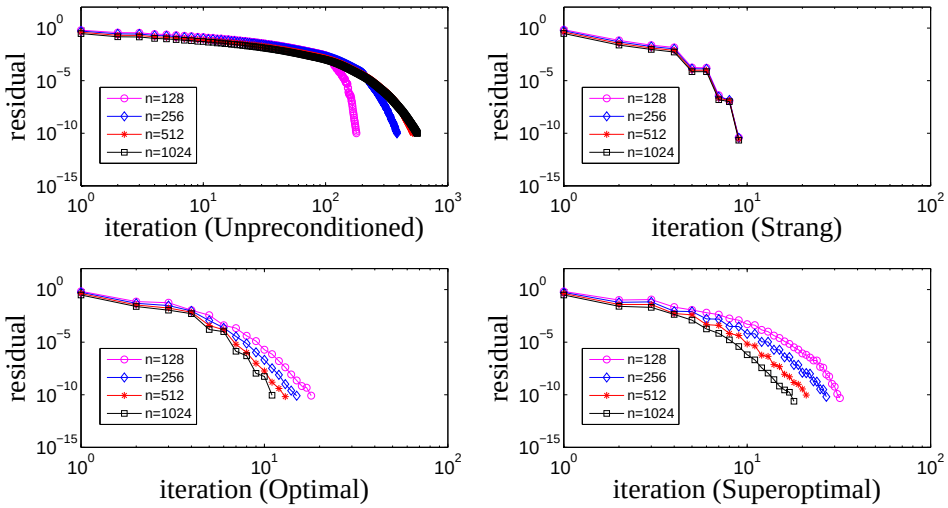


Figure 1: Convergence curves for the indefinite case.

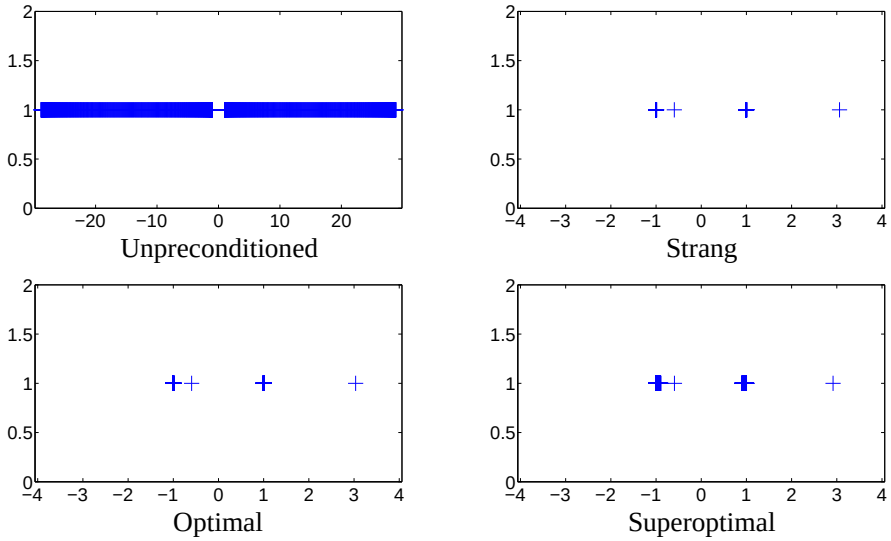


Figure 2: Spectral distributions when $n = 1024$ for the indefinite case.

Acknowledgments

The authors would like to thank the referees and Professors Zheng-Jian Bai, Chong Li, and Yi-Min Wei for their helpful suggestions and comments. The second author is supported by the research grants MYRG2016-00077-FST from University of Macau. The third author is supported by the National Natural Science Foundation of China (No. 11601112).

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

References

[1] O. Axelsson, *A class of iterative methods for finite element equations*, Comput. Methods Appl. Mech. Engrg. **9**, 123-127 (1976).

[2] O. Axelsson, G. Lindskog, *On the rate of convergence of the preconditioned conjugate gradient method*, Numer. Math. **48**, 499-523 (1986).

[3] O. Axelsson, *Solution of linear systems of equations: Iterative methods*, In V. Barker, editor, Sparse Matrix Techniques, v. 572 of Lecture Notes in Mathematics, Springer, 1997, pp. 1-51.

[4] Z. Bai, X. Jin, T. Yao, *Superoptimal preconditioners for functions of matrices*, Numerical Mathematics: Theory, Methods and Applications **8**, 515-529 (2015).

[5] B. Beckermann, A. Kuijlaars, *Superlinear convergence of conjugate gradients*, SIAM J. Numer. Anal. **39**, 300-329 (2001).

[6] B. Beckermann, *Discrete orthogonal polynomials and superlinear convergence of Krylov subspace methods in numerical linear algebra*, In Orthogonal Polynomials and Special Functions, Springer, 2006.

[7] S. Campbell, I. Ipsen, C. Kelley, C. Meyer, *GMRES and the minimal polynomial*, BIT **36**, 664-675 (1996).

[8] T. Chan, *An optimal circulant preconditioner for Toeplitz systems*, SIAM J. Sci. Stat. Comput. **9**, 766-771 (1988).

[9] R. Chan, M. Ng, *Conjugate gradient methods for Toeplitz systems*, SIAM Rev. **38**, 427-482 (1996).

[10] R. Chan, D. Potts, G. Steidl, *Preconditioners for nondefinite Hermitian Toeplitz systems*, SIAM J. Matrix Anal. Appl. **22**, 647-665 (2000).

[11] R. Chan, X. Jin, *An Introduction to Iterative Toeplitz Solvers*, SIAM, Philadelphia, 2007.

[12] H. Elman, D. Silvester, A. Wathen, *Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics*, Oxford University Press, Oxford, 2005.

[13] B. Fischer, *Polynomial Based Iteration Methods for Symmetric Linear Systems*, Willey and Teubner, Chichester, West Essex, England, and Stuttgart, 1996.

[14] T. Gergelits, Z. Strakoš, *Composite convergence bounds based on Chebyshev polynomials and finite precision conjugate gradient computations*, Numer. Algor. **65**, 759-782 (2014).

[15] A. Greenbaum, *Iterative Methods for Solving Linear Systems*, SIAM, Philadelphia, 1997.

[16] R. Herzog, E. Sachs, *Superlinear convergence of Krylov subspace methods for self-adjoint problems in Hilbert space*, SIAM J. Numer. Anal. **53**, 1304-1324 (2015).

[17] A. Jennings, *Influence of the eigenvalue spectrum on the convergence rate of the conjugate gradient method*, J. Inst. Math. Appl. **20**, 61-72 (1977).

[18] X. Jin, *Preconditioning Techniques for Toeplitz Systems*, Higher Education Press, Beijing, 2010.

[19] X. Jin, Y. Wei, Z. Zhao, *Numerical Linear Algebra and Its Applications*, 2nd edition, Science Press, Beijing, 2015.

[20] X. Jin, S. Vong, *An Introduction to Applied Matrix Analysis*, Higher Education Press, Beijing, and World Scientific Publishing, Singapore, 2016.

[21] S. Lei, H. Sun, *A circulant preconditioner for fractional diffusion equations*, J. Comput. Phys. **242**, 715-725 (2013).

[22] F. Lin, S. Yang, X. Jin, *Preconditioned iterative methods for fractional diffusion equation*, J. Comput. Phys. **256**, 109-117 (2014).

[23] I. Moret, *A note on the superlinear convergence of GMRES*, SIAM J. Numer. Anal. **34**, 513-516 (1997).

[24] M. Ng, *Iterative Methods for Toeplitz Systems*, Oxford University Press, Oxford, UK, 2004.

[25] M. Olshanskii, E. Tyrtyshnikov, *Iterative Methods for Linear Systems: Theory and Applications*,

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press

SIAM, Philadelphia, 2014.
[26] C. Paige, M. Saunders, *Solution of sparse indefinite systems of linear equations*, SIAM J. Numer. Anal. **12**, 617-629 (1975).
[27] J. Pestana, A. Wathen, *A preconditioned MINRES method for nonsymmetric Toeplitz matrices*, SIAM J. Matrix Anal. Appl. **36**, 273-288 (2015).
[28] V. Simoncini, D. Szyld, *On the occurrence of superlinear convergence of exact and inexact Krylov subspace methods*, SIAM Rev. **47**, 247-272 (2005).
[29] V. Simoncini, D. Szyld, *On the superlinear convergence of MINRES*, In Numerical Mathematics and Advanced Applications 2011, Springer, 2013.
[30] G. Sleijpen, A. van der Sluis, *Further results on the convergence behavior of conjugate-gradients and Ritz values*, Linear Algebra Appl. **246**, 233-278 (1996).
[31] G. Strang, *A proposal for Toeplitz matrix calculations*, Stud. Appl. Math. **74**, 171-176 (1986).
[32] E. Tyrtyshnikov, *Optimal and superoptimal circulant preconditioners*, SIAM J. Matrix Anal. Appl. **13**, 459-473 (1992).
[33] A. van der Sluis, H. van der Vorst, *The rate of convergence of conjugate gradients*, Numer. Math. **48**, 543-560 (1986).
[34] H. van der Vorst, C. Vuik, *The superlinear convergence behaviour of GMRES*, J. Comput. Appl. Math. **48**, 327-341 (1993).
[35] A. Wathen, *Preconditioning*, Acta Numer. **24**, 329-376 (2015).
[36] R. Winther, *Some superlinear convergence results for the conjugate gradient method*, SIAM J. Numer. Anal. **17**, 14-17 (1980).

https://doi.org/10.4208/eajam.181016.300517h Published online by Cambridge University Press