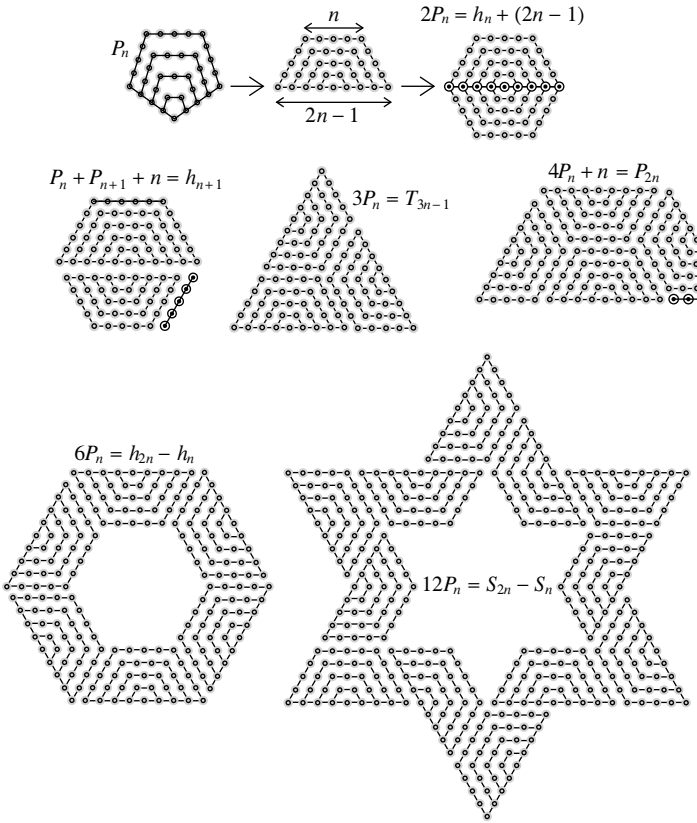
Proof: The identities are demonstrated for $n = 5$.



10.1017/mag.2022.124 © The Authors, 2022
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106.33 Fibonacci numbers and Cassini's identity

Introduction

It is well known that the Fibonacci numbers F_n, defined by $F_{n+2} = F_n + F_{n+1}$, $F_0 = 0$ and $F_1 = 1$, satisfy a large number of identities. Nevertheless, putting their popularity aside, they do not have a special role within the general theory of recurrence relations, and many (and probably all) of the these identities are just special cases of identities that hold for solutions of the general, constant coefficient, recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \qquad ab \neq 0. \tag{1}$$

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This observation suggests that we might understand these identities better if we were to establish them for the general relation (1), rather than relying on specific properties of the F_n. For example, we should know whether or not it is relevant that the F_n are integers, we should not constrain the initial values x_0 and x_1 to be 0 and 1, and we should find what the identities become in those cases that are not suggested by the Fibonacci numbers, namely where the auxiliary equation has coincident, or complex, roots. In short, we should perhaps regard an identity between the Fibonacci numbers as the *starting point* of an investigation rather than a terminal point of the theory.

Here, we illustrate this point of view with a single example and, for simplicity, we shall restrict ourselves to the recurrence relations (1) for which a and b are real, and (to avoid trivial cases) $ab \neq 0$.

The Cassini identity

If we replace F_{n-1} by $F_{n+1} - F_n$ in Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, and then solve the resulting quadratic equation for F_{n+1}, we obtain

$$F_{n+1} = \tfrac{1}{2}F_n + \tfrac{1}{2}\sqrt{5F_n^2 + 4(-1)^n}, \qquad F_1 = 1, \tag{2}$$

where we add (rather than subtract) the positive square root because $F_{n+1} > F_n$ (see [1] for a discussion of this formula). This shows that although the F_n are usually defined by the familiar second order relation, they can also be defined by a *first order* relation. So is a similar statement true for all solutions of the recurrence relations of the form (1) and, if so, what should replace the inequality $F_{n+1} > F_n$ in the proof above? We begin with the following version of Cassini's identity for the relation (1), which is surely known, but perhaps not well known.

Lemma 1: Any solution x_n of (1) satisfies the relation

$$x_{n+2}x_n - x_{n+1}^2 = (-b)^n\left(x_2x_0 - x_1^2\right). \tag{3}$$

Proof: Since

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \cdots = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}, \tag{4}$$

we have

$$\begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix}, \tag{5}$$

and the result follows by taking the determinant of both sides.

It follows that the Cassini identity does not depend on any property of the Fibonacci numbers, nor does it require the x_n to be integers. Of course, there may be interesting number theory that follows from Cassini's identity

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for the F_n, but we can now see that this is a consequence of a special case of a general identity, and is not an essential part of this general identity. The Cassini identity for the Fibonacci numbers is a special case of the Catalan identity

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2,$$

which, in turn, is a special case of the identity

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^nF_iF_j.$$

The reader may like to consider whether these identities have versions that are applicable to (1). For more information, see [2].

The formula for x_{n+1} in terms of x_n

If we combine the relation (1) with the Cassini identity (3), we find that

$$2x_{n+1} = x_n\left(a + \delta_n\sqrt{(a^2 + 4b) + \sigma_n}\right), \qquad \sigma_n = \frac{-4(-b)^n(x_2x_0 - x_1^2)}{x_n^2},$$

where, for each n, δ_n is 1 or -1, and where we are assuming that $x_n \neq 0$. We must now find δ_n for each n, and here the argument must not rely on the property $F_{n+1} > F_n$ of the Fibonacci numbers.

We shall proceed under the assumption that the auxiliary equation $t^2 = at + b$ has real, distinct, roots α and β (as in the case of the Fibonacci relation), and leave the remaining cases to the interested reader to explore. As $\alpha \neq \beta$ we see that $a^2 + 4b > 0$, and we let $\sqrt{a^2 + 4b}$ be the positive square root. Also, as $\alpha + \beta = a \neq 0$, we see that $\alpha \neq -\beta$; thus $|\alpha| \neq |\beta|$. We may now label α and β so that $|\alpha| > |\beta|$, and if we do this we find that

$$\alpha = \tfrac{1}{2}\left(a + \varepsilon\sqrt{a^2 + 4b}\right), \beta = \tfrac{1}{2}\left(a - \varepsilon\sqrt{a^2 + 4b}\right), \varepsilon = \frac{a}{|a|}, \qquad (6)$$

because a and $\varepsilon\sqrt{a^2 + 4b}$ have the same sign (and $\varepsilon = \pm 1$).

The general solution of (1) is $x_n = A\alpha^n + B\beta^n$, and we shall suppose that $A \neq 0$. Then, as $|\alpha| > |\beta|$ we see that $x_n \sim A\alpha^n$ as $n \to \infty$. Therefore, as $\alpha\beta = -b$,

$$\frac{(-b)^n}{x_n^2} \sim \frac{(\alpha\beta)^n}{A^2\alpha^{2n}} = \frac{1}{A^2}\left(\frac{\beta}{\alpha}\right)^n \to 0$$

as $n \to \infty$. This implies that $\sigma_n \to 0$ as $n \to \infty$, and as $x_{n+1}/x_n \to \alpha$ as $n \to \infty$, this shows that

$$a + \delta_n\sqrt{a^2 + 4b} \to 2\alpha = a + \varepsilon\sqrt{a^2 + 4b} \qquad (7)$$

as $n \to \infty$. We deduce that $\delta_n = \varepsilon = a/|a|$ for all sufficiently large n, and we have now found x_{n+1} as an explicit function of n. Finally, it is important to understand that our objective was not to find the explicit formula for x_{n+1} in terms of x_n (which, I suspect, has little value or interest); it was to identify the mechanism that leads to such a formula.

https://doi.org/10.1017/mag.2022.125 Published online by Cambridge University Press

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106.34 A theorem on divisibility by congruence

Introduction

Elementary Number Theory deals with various methods for checking divisibility by some types of numbers, see [1, 2, 3, 4, 5, 6, 7]. This Note discusses a method using congruence relation to check divisibility by digit wise computation. It is slightly different from the method proposed by J. Weitsman [5]. In school days we studied simple methods to check the divisibility of an integer by 2, 3, 5, 6 and 9. Using the following theorem we can check divisibility of an arbitrary integer by 7 or a two-digit integer ending with the digits 9, 7, 3 or 1.

Divisibility by Congruence Relation

Let s, t, d be integers with $d > 0$. Then $s \equiv t \pmod d$ if, and only if, $s - t$ is divisible by d.

Theorem

Let $f(x) = \sum_{i=0}^{k} a_i x^i$ and $g(x) = \sum_{i=1}^{k} a_i x^{k-i}$ be polynomials with integer coefficients and let $st \equiv 1 \pmod d$. Then $f(t) \equiv 0 \pmod d$ if, and only if, $g(s) \equiv 0 \pmod d$.

Proof:

For $i = 0, 1, \ldots, k$; we have $s^i t^i \equiv 1 \pmod d$, since $st \equiv 1 \pmod d$. Multiplying both sides of the congruence relations by $a_i s^{k-i}$ and adding the $k+1$ relations, we have $\sum_{i=0}^{k} a_i s^k t^i \equiv \sum_{i=0}^{k} a_i s^{k-i} \pmod d$. Hence $s^k \sum_{i=0}^{k} a_i t^i \equiv \sum_{i=0}^{k} a_i s^{k-i} \pmod d$. That is, $s^k f(t) \equiv g(s) \pmod d$. The relation $s^k t^k \equiv 1 \pmod d$ implies s^k and d are relatively prime. Therefore, $f(t) \equiv 0 \pmod d$ if, and only if, $s^k f(t) \equiv 0 \pmod d$ if, and only if, $g(s) \equiv 0 \pmod d$.

https://doi.org/10.1017/mag.2022.125 Published online by Cambridge University Press