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Roll-spun knots

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Introduction

In this paper we will study 2-knots in the 4-sphere S^4 which are obtained by roll-spinning 1-knots in the 3-sphere S^3. The process of spinning was introduced by Artin[**2**] and generalized to twist-spinning by Fox[**7**] and Zeeman[**20**]. In [**7**] Fox introduced another variation of the spinning process, called roll-spinning. He only showed that the roll-spun figure-eight knot cannot be obtained by twist-spinning the figure-eight knot. Finally Litherland[**14**] described the general process of deform-spinning, and he gave a precise definition of roll-spinning. We remark that Fox's roll-spinning is not the same as Litherland's: Fox's roll-spun figure-eight knot is the symmetry-spun figure-eight knot in terms of [**14**].

In Section 1, we define roll-spinning, following Litherland[**14**]. We show that a roll-spun knot of a non-trivial knot is always non-trivial, by using the results of Gabai[**8**] on Property R. We also show that the 2-knot group of a non-trivial roll-spun knot is not isomorphic to a 1-knot group. In Section 2, we describe the Seifert surface for a roll-spun knot. In Section 3, we examine the case of roll-spun knots of fibred 1-knots in some detail. If K is a fibred 1-knot, then any roll-spun knot of K is again fibred. We show that a non-trivial fibred roll-spun knot is not a ribbon knot. In Section 4, we prove that a non-trivial roll-spun knot cannot be obtained by twist-spinning a 1-knot.

We will work in the piecewise linear category. An n-knot is a locally flat embedding of S^n into S^{n+2}. Two knots K_1 and K_2 are called equivalent, and we write $K_1 \sim K_2$, if there exists a homeomorphism $h: S^{n+2} \to S^{n+2}$ such that $h(K_1) = K_2$. The unit interval $[0, 1]$ is denoted by I. The circle is taken to be the quotient space $S^1 \cong \mathbb{R}/\mathbb{Z}$. The commutator subgroup of a group G is denoted by G'. For a subset S of a group G, $\langle\langle S \rangle\rangle$ denotes the normal closure of S in G, the smallest normal subgroup of G which contains S.

1. *Roll-spinning*

Let K be a knot in S^3 and $K \times D^2$ be a tubular neighbourhood. Let $E(K) = \mathrm{cl}(S^3 \backslash K \times D^2)$ be the exterior of K. We always assume that $K \times \theta\, (\theta \in \partial D^2)$ is null-homologous in $E(K)$. Let $\partial E(K) \times I$ be a collar of $\partial E(K) = K \times \partial D^2$ in $E(K)$, where $\partial E(K)$ is identified with $\partial E(K) \times \{0\}$. Define $g: (S^3, K) \to (S^3, K)$ by

$$g(x, \theta, \phi) = (x + \phi, \theta, \phi) \quad \text{for } (x, \theta, \phi) \in K \times \partial D^2 \times I,$$
$$g(y) = y \quad \text{for } y \notin \partial E(K) \times I.$$

Recall that K is identified with $\mathbb{R}/\mathbb{Z}$. Note that g is the identity on $K \times D^2$. Let ρ be the isotopy class of $g(\mathrm{rel}\, K \times D^2)$.

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Fix a point z on K. Take a ball neighbourhood K_- of z in K, and set $B_- = K_- \times D^2$. Then (B_-, K_-) is a standard ball pair. Let (B_+, K_+) be the complementary ball pair of (B_-, K_-). For each integer m, we construct $(\partial(B_+, K_+) \times B^2) \cup_\partial ((B_+, K_+) \times_{g^m} \partial B^2)$, where

$$(B_+, K_+) \times_{g^m} \partial B^2 = (B_+, K_+) \times I/((x, 0) \sim (g^m(x), 1) \text{ for all } x \in B_+).$$

This is a locally flat sphere pair depending only on the isotopy class ρ^m of g^m: see [**14**]. We denote this 2-knot in S^4 by $\rho^m K$ and call it the *m-roll spun knot* of K.

Let $\lambda \in \pi_1(S^3 \backslash K)$ be the longitude of K. By [**14**], the 2-knot group of the m-roll spun knot of K is obtained from $\pi_1(S^3 \backslash K)$ by centralizing λ^m, i.e. $\pi_1(S^4 \backslash \rho^m K) \cong \pi_1(S^3 \backslash K)/\langle\langle \{[a, \lambda^m]; a \in \pi_1(S^3 \backslash K)\} \rangle\rangle$.

THEOREM 1. *For any integer m, the m-roll spun knot $\rho^m K$ of K has the same Alexander module as K.*

Proof. We consider a Wirtinger presentation $\pi = \pi_1(S^3 \backslash K) = \langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$. Since the longitude λ is in the second commutator subgroup π'' of π, we have $\lambda = \Pi_k [u_k, v_k]$, where $u_k, v_k \in \pi'$. From the above presentation for π, we get

$$\pi_1(S^4 \backslash \rho^m K) = \langle x_1, \ldots, x_n | r_1, \ldots, r_n, [x_1, \lambda^m], \ldots, [x_n, \lambda^m] \rangle.$$

Fox's free differential calculus [**5**] on this presentation shows that $\pi_1(S^3 \backslash K)$ and $\pi_1(S^4 \backslash \rho^m K)$ have equivalent Alexander matrices.

Applying the work of Gabai [**8**], we establish the following result.

THEOREM 2. *If K is a non-trivial knot in S^3, then the m-roll spun knot $\rho^m K$ in S^4 is non-trivial for each integer m.*

Proof. Let $G = \pi_1(S^4 \backslash \rho^m K)$. If $m = 0$, then $\rho^0 K$ is the Artin spun knot of K. Hence $G \cong \pi_1(S^3 \backslash K)$, which is not infinite cyclic. Thus $\rho^0 K$ is non-trivial.

Assume that $m \neq 0$. There is a natural map of G onto $\pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle$ where λ is the longitude of K. This group $\pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle$ is isomorphic to the fundamental group of the closed orientable 3-manifold $K(0)$ obtained by zero Dehn surgery on K. Gabai [**8**] has shown that $K(0)$ is always prime. Suppose that $\rho^m K$ is trivial. Then

$$G \cong \pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle \cong \pi_1 K(0) \cong \mathbb{Z}.$$

By [**10**], theorem 5·2, this implies that $K(0) = S^2 \times S^1$. This contradicts the Property R conjecture [**8**, **16**] that zero Dehn surgery on a non-trivial knot in S^3 does not yield $S^2 \times S^1$.

A group G is a 1-knot group if there exists a 1-knot K in S^3 such that $G \cong \pi_1(S^3 \backslash K)$.

THEOREM 3. *If K is a non-trivial knot in S^3, then $\pi_1(S^4 \backslash \rho^m K)$ is not a 1-knot group for any $m \neq 0$.*

Proof. Let $G = \pi_1(S^4 \backslash \rho^m K)$. Suppose that $G \cong \pi_1(S^3 \backslash K^*)$ for some 1-knot K^* in S^3.

Case 1. $\lambda^m \neq 1$ *in* G. Then G has a non-trivial centre. Thus K^* must be a torus knot. In G we have $\lambda^m \in G'$. But this contradicts the fact that for a torus knot group the intersection of the centre with the commutator subgroup is trivial (cf. [**3**]).

Case 2. $\lambda^m = 1$ *in* G. Then $G \cong \pi_1(S^3 \backslash K)/\langle\langle \lambda^m \rangle\rangle$. If the order of λ in G is 1, then $G \cong \pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle$. Since K is non-trivial, $K(0)$ is irreducible (see [**8**]). As $H_1(K(0); \mathbb{Z}) \cong \mathbb{Z}$, $K(0)$ contains an incompressible surface [**18**]. Thus $K(0)$ is Haken,

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and in particular aspherical. Therefore $H_2(G;\mathbb{Z}) \cong H_2(K(0);\mathbb{Z}) \cong \mathbb{Z}$. This contradicts the fact that $H_2(\pi_1(S^3\backslash K^*);\mathbb{Z}) = 0$. If the order of λ is greater than 1, then G has torsion, contradicting the fact that $\pi_1(S^3\backslash K^*)$ is torsion-free. This completes the proof.

2. *Seifert surfaces*

In this section, we study the Seifert surfaces of roll-spun knots.

Let K be an n-knot in S^{n+2}. A projection for K is a pair $(p, K\times D^2)$ where $K\times D^2$ is a tubular neighbourhood of K with exterior $E(K)$ and p is a map $E(K)\to\partial D^2$ extending the canonical projection $\partial E(K) = K\times\partial D^2\to\partial D^2$. Using obstruction theory, it may be shown that every knot has a projection (cf. [**11**], or [**15**], chapter 5). We may assume that the inverse image $p^{-1}(0)$ is a bicollared, proper codimension 1 submanifold of $E(K)$, that is, a Seifert surface for K.

An n-knot K is fibred if there is a projection $(p, K\times D^2)$ for K such that p is the projection map of a fibre bundle.

Let K be a knot in S^3 with projection $(p, K\times D^2)$. Then $F = p^{-1}(0)$ is a Seifert surface for K. A tubular neighbourhood of the m-roll spun knot $\rho^m K$ of K is $(\partial K_+\times D^2\times B^2)\cup(K_+\times D^2\times\partial B^2)$ and so $\rho^m K$ has exterior $E(\rho^m K)$ equal to $(K_-\times\partial D^2\times B^2)\cup(E(K)\times_{g^m}\partial B^2)$. The space $W = (K_-\times\{0\}\times B^2)\cup(F\times_{g^m}\partial B^2)$ gives a Seifert surface for $\rho^m K$. We use the notation $\hat{W}$ for the closed 3-manifold obtained from W by a capping off a 2-sphere ∂W with a 3-ball.

Let $T(\alpha)$ denote the closed orientable surface of genus α.

PROPOSITION 1. *Suppose that F is a genus $\alpha(\geqslant 1)$ Seifert surface for K.*

(i) *If $m = 0$, then $\hat{W} = \#^{2\alpha}S^2\times S^1$.*

(ii) *If $m = \pm 1$, then $\hat{W}$ is an S^1-bundle over $T(\alpha)$.*

(iii) *Otherwise, $\hat{W}$ is a Seifert fibred manifold with orbit manifold $T(\alpha)$ and one exceptional fibre of index $|m|$.*

Furthermore, in Cases (ii) *and* (iii) *a Seifert fibration of $\hat{W}$ is unique up to homeomorphism.*

Proof. By the definition of $\hat{W}$, it is easy to see that $\hat{W}$ is homeomorphic to the 3-manifold obtained from $F\times S^1$ by attaching a solid torus J to $F\times S^1$ so that the geometric intersection number of $x\times S^1$ ($x\in\partial F$) with the image of a meridian of J is $|m|$. First, $F\times S^1$ has a Seifert fibration in which there are no exceptional fibres, and the orbit manifold is F.

Assume that $m = 0$. The image of a meridian of J is homologous (on $\partial F\times S^1$) to a regular fibre of $F\times S^1$ and we use proposition 1 in [**9**] to conclude that $\hat{W} = \#^{2\alpha}S^2\times S^1$.

Assume that $m \neq 0$. Then we can suppose that the attaching solid torus J is a fibred solid torus with exceptional fibre of index $|m|$, and the Seifert fibration of $F\times S^1$ is extended to that of $\hat{W}$. A closed Seifert fibred manifold which admits non-unique Seifert fibrings is either a lens space (including $S^2\times S^1$ and S^3), a prism manifold or the double of a twisted I-bundle over the Klein bottle (see [**12**]). The uniqueness follows from this. (Note that $H_1(\hat{W};\mathbb{Z}) \cong \mathbb{Z}^{2\alpha}$.)

We remark that $\hat{W}$ is a Haken-manifold if $m \neq 0$.

Let K be a fibred knot in S^3 with fibre bundle projection $(p, K\times D^2)$. Let F and W be as above.

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PROPOSITION 2 (cf. [13]). *If K is a fibred knot in S^3 with fibre F, then $\rho^m K$ is a fibred knot in S^4 with fibre W for all $m \in \mathbb{Z}$.*

Andrew and Sumners [1] proved that $\rho^0 K$ is fibred if and only if K is fibred. It is probable that $\rho^m K$ is not fibred if K is not fibred for any $m \in \mathbb{Z}$ in general.

3. *Fibred roll-spun knots*

The Artin spun knot of a 1-knot is a ribbon 2-knot (see [19]). Cochran [4] has shown that a non-trivial twist-spun 2-knot is not a ribbon knot.

First, we show that a similar result holds for roll-spun knots if K is fibred.

THEOREM 4. *If K is a non-trivial fibred knot in S^3, then the m-roll spun knot $\rho^m K$ is not ribbon for any $m \neq 0$.*

Proof. By Propositions 1 and 2, $\rho^m K$ is a fibred knot whose capped-off fibre $\hat{W}$ is Haken, in particular aspherical. But Cochran [4] shows that no fibred 2-knot with fibre a punctured aspherical manifold is a ribbon knot.

Since $\rho^m K \sim \rho^{-m} K$, we may assume $m \geqslant 0$.

THEOREM 5. *Let K be a non-trivial fibred knot in S^3. Let m and n be non-negative integers. Then $\rho^m K \not\sim \rho^n K$ if $m \neq n$.*

Proof. Suppose that $\rho^m K \sim \rho^n K$. By Theorem 4, $m = 0$ implies $n = 0$, and vice versa. Assume that $m > 0, n > 0$. By Proposition 2, $\rho^m K$ and $\rho^n K$ are fibred 2-knots with capped-off fibres M and N, respectively. Since $\pi_1(S^4 \backslash \rho^m K) \cong \pi_1(S^4 \backslash \rho^n K)$, we have $\pi_1(M) \cong \pi_1(N)$ by considering the commutator subgroup of both sides. As M and N are Haken, $M \cong N$ (see [17]), and so $m = n$ by Proposition 1.

PROPOSITION 3. *Let K_1 and K_2 be fibred knots in S^3, and suppose that $K_1 \not\sim K_2$. Let m and n be non-negative integers. If either* genus $(K_1) \neq$ genus (K_2) *or $m \neq n$, then $\rho^m K_1 \not\sim \rho^n K_2$.*

Proof. Suppose that $\rho^m K_1 \sim \rho^n K_2$. By Theorem 4, $m = 0$ implies $n = 0$, and vice versa. Then $\pi_1(S^3 \backslash K_1) \cong \pi_1(S^3 \backslash K_2)$ in either case. Since the genus of a knot is a knot group invariant [6], we have genus $(K_1) =$ genus (K_2).

For $m > 0, n > 0$, the same argument as in the proof of Theorem 5 shows that $m = n$ and genus $(K_1) =$ genus (K_2).

One might expect that no non-trivial roll-spun knot of K_1 can be obtained from K_2 by roll-spinning, but we have been unable to prove this.

4. *Roll-spun knots and twist-spun knots*

In this section, we show that if K is not a torus knot then no non-trivial roll-spun knot of K can be obtained by twist-spinning a 1-knot. We shall use the notation of Section 1.

Let K be a knot in S^3 and take a collar $\partial E(K) \times I$ of $\partial E(K)$ in $E(K)$. Define $t : (S^3, K) \to (S^3, K)$ by

$$t(x, \theta, \phi) = (x, \theta + \phi, \phi) \quad \text{for } (x, \theta, \phi) \in K \times \partial D^2 \times I,$$
$$t(y) = y \quad \text{for } y \notin \partial E(K) \times I.$$

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Let τ be the isotopy class of $t(\mathrm{rel}\, K \times D^2)$. For each integer n, we can construct a 2-knot $\tau^n K$, which is called the n-twist spun knot of K, similarly to $\rho^m K$.

It is shown in [**14**] that any roll-spun knot (indeed, any deform-spun knot) of a torus knot K is equivalent to a twist-spun knot of K. Therefore we need to consider the case of non-torus knots.

THEOREM 6. *Let K be a knot in S^3, and suppose that K is not a torus knot. For any non-zero integer m, the m-roll spun $\rho^m K$ of K cannot be obtained as $\tau^n K^*$ for any knot K^* and integer n.*

Proof. Suppose that $\rho^m K \sim \tau^n K^*$ for some knot K^*. If $n = 0$, then

$$\pi_1(S^4 \backslash \rho^m K) \cong \pi_1(S^4 \backslash \tau^0 K^*) \cong \pi_1(S^3 \backslash K^*).$$

Thus we have a contradiction by Theorem 3. Assume that $n \neq 0$. By Zeeman [**20**], the nth power of a meridian is central in $\pi_1(S^4 \backslash \tau^n K^*)$. Since $\pi_1(S^4 \backslash \rho^m K) \cong \pi_1(S^4 \backslash \tau^n K^*)$, there exists a central element w for $\pi_1(S^4 \backslash \rho^m K)$ such that w is mapped to a non-zero element under the abelianizer $\pi_1(S^4 \backslash \rho^m K) \to \mathbb{Z}$. Let $\lambda \in \pi_1(S^3 \backslash K)$ be the longitude. There is a natural map of $\pi_1(S^4 \backslash \rho^m K)$ onto $\pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle$. The image of w is still non-trivial. Recall that $\pi_1(S^3 \backslash K)/\langle\langle \lambda \rangle\rangle$ is isomorphic to the fundamental group of $K(0)$, the closed 3-manifold obtained by zero Dehn surgery on K. Thus $\pi_1 K(0)$ has a non-trivial centre. Since K is non-trivial, $K(0)$ is irreducible by Gabai [**8**]. It follows from [**18**], Satz 4·1 that $K(0)$ is a Seifert fibred manifold with orientable orbit manifold.

The Lemma below now shows that K is a torus knot. This contradicts the hypothesis.

LEMMA. *Let K be a knot in S^3. Let $K(0)$ denote the closed orientable 3-manifold obtained by zero Dehn surgery on K. Then $K(0)$ is a Seifert fibred manifold if and only if K is a torus knot.*

Proof. If K is a torus knot of type (p, q), then the exterior $E(K)$ is a Seifert fibred manifold, and the regular fibres on $\partial E(K)$ are presented by $\lambda \mu^{pq}$, where (λ, μ) is the longitude-meridian system of $E(K)$. The intersection number of two simple closed curves $\lambda \mu^{pq}$ and λ is $pq\,(\neq 0)$. Hence the Seifert fibration of $E(K)$ extends to one of $K(0)$.

Conversely, suppose that $K(0)$ is a Seifert fibred manifold. Let S be a minimal genus Seifert surface for K in $E(K)$. In $K(0)$, we can cap off S by a disk to create $\hat{S}$. Then $\hat{S}$ is a 2-sided non-separating orientable closed surface in $K(0)$ and incompressible (see [**8**]). By theorem VI·34 of [**12**] there are two cases.

Case 1. *$\hat{S}$ is a fibre in a fibration of $K(0)$ as a surface bundle over S^1.* Then the sewing map of $K(0)$ is periodic (see [**12**]). By corollary 8·19 of [**8**], K is a fibred knot. In fact, the fibration of $E(K)$ is obtained by restricting that of $K(0)$ to $E(K)$. Hence the sewing map of $E(K)$ is still periodic, and so $E(K)$ is a Seifert fibred manifold. Therefore K is a torus knot.

Case 2. *$\hat{S}$ is a torus and $\hat{S}$ is saturated (i.e. a union of fibres) in some Seifert fibration of $K(0)$.* Let O denote the orbit manifold of $K(0)$. Since $H_1(K(0); \mathbb{Z}) \cong \mathbb{Z}, H_1(O; \mathbb{Z})$ is cyclic. Therefore $O = S^2$ or P^2. But it is easy to see that any saturated torus in a Seifert fibred manifold with orbit manifold S^2 or P^2 must be separating. This contradicts the fact that $\hat{S}$ is non-separating. This completes the proof of the Lemma.

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Remark. If $K(0)$ is a Seifert fibred manifold, then the torus knot K is determined uniquely by $K(0)$.

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