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HOFFMAN PROGRAM OF LAPLACIAN MATCHING POLYNOMIALS OF GRAPHS

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Abstract Let $\mathcal{G}$ be the class of all connected simple graphs. The Hoffman program of graphs with respect to a spectral invariant $\lambda(G)$ consists of determining all the limit points of the set $\{\lambda(G) \mid G \in \mathcal{G}\}$ and characterising all G's such that $\lambda(G)$ does not exceed a fixed limit point. In this paper, we study the Hoffman program for Laplacian matching polynomials of graphs in regard to their largest Laplacian matching roots. Precisely, we determine all the limit points of the largest Laplacian matching roots of graphs less than $\tau = 2 + \omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}} (= 4.38+)$, and then characterise the connected graphs with the largest Laplacian matching roots less than $2 + \sqrt{5}$, where $\omega = \frac{1}{3}(\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1)$.

Keywords: matching polynomial; Laplacian matching polynomial; limit point; largest root; Hoffman program

2010 Mathematics subject classification: 05C31; 05C70

1. Introduction

In this paper, all graphs considered are simple and undirected. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$ and size $m = |E(G)|$. For a subset $W \subseteq V(G)$, we use $G[W]$ to denote the subgraph of G induced by W, and use $G - W$ to denote the induced subgraph $G[V(G)\backslash V(W)]$. An edge subset $M \subseteq E(G)$ is called a *matching* of G if no two distinct edges in it share a common vertex, and let $\mathcal{M}(G)$ be the set of matchings (including the empty matching) of G. To study statistical physics, Heilmann and Lieb [8] gave a formal definition of *matching polynomial* of G as follows:

$$\mathscr{M}(G, x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} x^{|V(G)\backslash V(M)|}.$$

As is known, the matching polynomial is one of the classical graph polynomials in algebraic combinatorics and has a wide range of applications. In particular, it was used to

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construct infinitely many bipartite Ramanujan graphs [13]. For more details, we refer readers to the related chapters of the monographs [5, 16].

In 2020, Mohammadian [14] introduced the *Laplacian matching polynomial* of G defined as follows:

$$\mathscr{L}(G,x)=\sum_{M\in\mathcal{M}(G)}(-1)^{|M|}\prod_{v\in V(G)\setminus V(M)}(x-d_G(v)),$$

where $d_G(v)$ is the degree of $v\in V(G)$ and $V(M)$ is the set of vertices incident to some edge in M. As pointed out in [12], the Laplacian matching polynomial of a graph is identical with its *average Laplacian polynomial* introduced by Zhang and Chen [21], who were inspired by the celebrated result of Godsil and Gutman [6] that the matching polynomial is just the average of adjacency characteristic polynomials of all signed graphs with underlying graph G. Nowadays, the Laplacian matching polynomial of G has been attracting much more attention with its fantastic properties. Mohammadian [14] proved that all roots of this polynomial are nonnegative and real. Zhang and Chen [21] revealed the relations between this polynomial and graph structures. Wan, Wang and Mohammadian [17] investigated the location of its zeros. Li and Yan [12] gave a combinatorial explanation of its coefficients. See [18, 20] for more new results. For convenience, the *Laplacian matching roots* of G are those of its Laplacian matching polynomial, where the largest one is called the *Laplacian matching index of* G.

The spectral radius $\rho(G)$ of a graph G is that of its graph matrix. A real number γ is called a *limit point* of spectral radii of graphs if there exists a sequence of graphs $\{G_k\in\mathcal{G}\,|\,k\in\mathbb{N}\}$ such that

$$\rho(G_i)\neq\rho(G_j)\,\text{whenever i}\neq\text{j, and}\lim_{k\to\infty}\rho(G_k)=\gamma.$$

Historically, it originated from A.J. Hoffman [9] on the limit points of spectral radii (or say, the largest roots of characteristic polynomials) of nonnegative symmetric integral matrices by means of adjacency matrices of graphs. Later on, the Hoffman program of graphs [2] (with respect to a fixed spectral radius) mainly determine all the limit points of the set $\{\rho(G)|G\in\mathcal{G}\}$ and characterise all G's such that $\rho(G)$ does not exceed a fixed limit point. As an important application, Jiang and Polyanskii [11] used the Hoffman program to attack a longstanding problem on equiangular lines. So far, Hoffman's program of graphs has been extensively studied in the setting of the Laplacian, the signless Laplacian, the signed adjacency, the Hermitian adjacency, the skew-adjacency and A_α-matrices of graphs as well as the adjacency tensors of hypergraphs. See a very recent survey [19] and a new paper [15] for more details.

In this paper, we will consider the Hoffman program for the Laplacian matching indices $\lambda(G)$ of graphs at the first attempt. More generally, the Laplacian matching index of G is an algebraic invariant but not a spectral radius.

We introduce more notations. Let $G\cup H$ be the disjoint union of two graphs G and H. We use $H\subseteq G$ to denote the subgraph H of G, and use $H\subset G$ if it is proper. For a graph matrix B, the *B-characteristic polynomial* of G is $\varphi_B(G,x)=\det(xI-B)$ with I being the identity matrix. The *spectral radius* $\rho_B(G)$ of G is the largest absolute value of all zeros of $\varphi_B(G,x)$. Let $A(G)=(a_{ij})$ be the *adjacency matrix* of G, where $a_{ij}=1$ if $v_iv_j\in E(G)$

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and otherwise $a_{ij} = 0$. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_G(v_1), d_G(v_2), \cdots, d_G(v_n))$ is the diagonal matrix of vertex degrees. Let $\mu(G)$ and $\lambda(G)$ denote the largest root of $\mathscr{M}(G,x)$ and $\mathscr{L}(G,x)$, respectively. As usual, set P_n and C_n to be the *path* and *cycle* with order n, respectively. Let $F_{s,r}(s>1, r\geq 1)$ denote the tree obtained from P_{s+r+1}: $v_0v_1\cdots v_su_1\cdots u_r$ and two isolated vertices w_1, w_2 by adding edges v_1w_1 and v_sw_2.

The organisation of this paper is as follows: In Section 2, we introduce some basic lemmas and useful results. In Section 3, motivated by the Hoffman program w.r.t. Laplacian matrix (see the relation (1)), we determine all the limit points of the Laplacian matching indices of graphs less than $\tau = 2 + \omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}} (= 4.38+)$, where $\omega = \frac{1}{3}(\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1)$, and then characterise the connected graphs with the Laplacian matching indices less than $2+\sqrt{5}$. Finally, we put forward two problems for further study.

2. Preliminaries

As a starter, we give the relation between matching and Laplacian matching polynomials of a graph.

Lemma 2.1. ([21]) *Let G be a graph. Then*

$$\mathscr{M}(S(G),x) = x^{|E(G)|-|V(G)|}\mathscr{L}(G,x^2),$$

where $S(G)$ is the subdivision graph of G.

For a connected graph G and $v \in V(G)$, Godsil defined the *path-tree* $T(G,v)$ of G to be a tree which has as vertices the paths in G that start at v and where two such paths are adjacent if one is a maximal proper sub-path of the other, and he showed that $\mu(G) = \rho_A(T(G,v))$ [4]. On the other side, Mohammadian [14] proved that

$$\varphi_L(G,x) = \mathscr{L}(G,x) \text{ iff G is a forest.} \tag{1}$$

So, $\rho_L(G) = \lambda(G)$. Along with the previous lemma, we immediately get the following corollary.

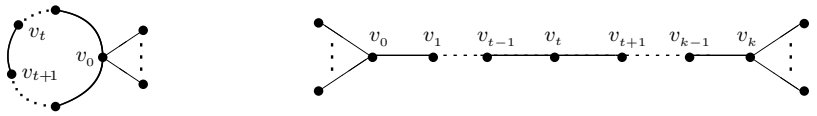
Corollary 2.2. *Let G be a connected graph and $S(G)$ be its subdivision with $v \in V(S(G))$. Then*

$$\lambda(G) = \mu^2(S(G)) = \rho_A^2(T(S(G),v)).$$

Let $\Delta(G)$ and $\delta(G)$ be, respectively, the maximum and minimum degrees of G.

Lemma 2.3. *Let G be a connected graph with $H \subset G$ and $\ell(G)$ be the length of a longest path in G. Then*

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Type A Type B

Figure 1. Two types of internal path.

(i) *[3]* $\rho_A(H) < \rho_A(G)$.
(ii) *[17]* $\lambda(H) < \lambda(G)$ *and*

$$\max\{\Delta(G)+1, \delta(G)+\sqrt{\Delta(G)}\} \leq \lambda(G) \leq \Delta(G) + 2\sqrt{\Delta(G)-1}\cos\frac{\pi}{2\ell(G)+2}$$

with the left equality holds iff G is a star, and the right one holds iff G is a cycle.

Hoffman and Smith [10] defined an *internal path* of a graph G as a walk $v_0, v_1, \cdots, v_k (k \geq 1)$ where the vertices $v_1, \cdots, v_k$ are distinct (v_0, v_k need not be distinct), $d_G(v_0) > 2$, $d_G(v_k) > 2$ and $d_G(v_i) = 2$ whenever $0 < i < k$. In terms of the definition, two types of internal paths (see Fig. 1) are introduced as follows:

Type A. $v_0 = v_k$, $d_G(v_0) \geq 3$ and $d_G(v_i) = 2$ for $0 < i < k$.

Type B. $v_0 \neq v_k$, $d_G(v_0) \geq 3$, $d_G(v_k) \geq 3$ and $d_G(v_i) = 2$ for $0 < i < k$.

In Spectral Graph Theory, it is well-known that subdividing an edge can change the spectral radius of the adjacency matrix (see Lemma 2.4). We here provide a novel conclusion for the Laplacian matching indices of graphs (see Lemma 2.5).

Lemma 2.4. ([10]) *Let uv be an edge of the connected graph G and let G_{uv} be obtained from G by subdividing uv.*

(i) *If uv is not in an internal path and $G \neq C_n$, then $\rho_A(G_{uv}) > \rho_A(G)$.*
(ii) *If uv is in an internal path and $G \neq F_{s,1}$, then $\rho_A(G_{uv}) < \rho_A(G)$.*

Lemma 2.5. *Let uv be an edge of the connected graph G and let G_{uv} be obtained from G by subdividing uv.*

(i) *If uv is not in an internal path or uv is in an internal path of Type A, then*

$$\lambda(G_{uv}) > \lambda(G).$$

(ii) *If uv is a cut edge in an internal path of Type B, then*

$$\lambda(G_{uv}) < \lambda(G).$$

Proof. (i) If uv is not in an internal path and if $G \neq C_n$, then G is a proper subgraph of G_{uv}, and so $\lambda(G_{uv}) > \lambda(G)$ by Lemma (ii). If $G = C_n$, then $S(G) = C_{2n}$ and

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$S(G_{uv}) = C_{2n+2}$. Note that the path-tree $T(C_{2n}, u) = P_{4n-1}$ is a proper subgraph of path-tree $T(C_{2n+2}, u) = P_{4n+3}$. By Lemma (i) we get $\rho_A(T(C_{2n+2}, u)) > \rho_A(T(C_{2n}, u))$, and thus $\lambda(G_{uv}) > \lambda(G)$ by Corollary 2.2.

If uv is in an internal path of Type A, as labelled in Fig. 1, we get $T(S(G), v_0) \subset T(S(G_{uv}), v_0)$. Hence, $\lambda(G_{uv}) > \lambda(G)$ follows by Lemma (i) and Corollary 2.2 again.

(ii) Since uv is a cut edge in an internal path of Type B, all edges in this internal path are cut edges. Hence, the path-tree $T(S(G_{uv}), v_0)$ is obtained from $T(S(G), v_0)$ by subdividing some edges of the internal path, as labelled in Type B of Fig. 1. Then $\lambda(G_{uv}) < \lambda(G)$ by Lemma 2.4(ii) and Corollary 2.2. □

We next derive a formula for computing Laplacian matching polynomials. For a subgraph $H \subseteq G$, the *Laplacian sub-matching polynomial* of G w.r.t. H is defined as [17]:

$$\mathscr{L}_G(H, x) = \sum_{M \in \mathcal{M}(H)} (-1)^{|M|} \prod_{v \in V(H) \setminus V(M)} (x - d_G(v)) .$$

In particular, $\mathscr{L}_G(G, x) = \mathscr{L}(G, x)$. It is easy to check the fact that for two vertex-disjoint graphs $G_1, G_2 \subseteq G$, we have

$$L_G(G_1 \cup G_2, x) = L_G(G_1, x) L_G(G_2, x).$$

We will label $\mathscr{L}_G(H, x)$ as $\mathscr{L}_G(H)$ if it is clear from the text. The following lemma will be used in the sequel.

Lemma 2.6. *Let $H \subseteq G$ and $v \in V(H)$. Then*

$$\mathscr{L}_G(H, x) = (x - d_G(v)) \mathscr{L}_G(H - v, x) - \sum_{u \in N_H(v)} \mathscr{L}_G(H - v - u, x), \tag{2}$$

where $N_H(v)$ is the set of all vertices in $V(H)$ adjacent to v.

Proof. We partition $M \in \mathcal{M}(H)$ into two parts depending on whether $v \in V(M)$ or not. Then

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$$\begin{aligned}\mathscr{L}_G(H,x) &= \sum_{M\in\mathcal{M}(H)}(-1)^{|M|}\prod_{w\in V(H)\setminus V(M)}(x-d_G(w))\\ &= \sum_{\substack{M\in\mathcal{M}(H)\\ v\notin V(M)}}(-1)^{|M|}\prod_{w\in V(H)\setminus V(M)}(x-d_G(w))\\ &\quad+\sum_{\substack{M\in\mathcal{M}(H)\\ v\in V(M)}}(-1)^{|M|}\prod_{w\in V(H)\setminus V(M)}(x-d_G(w))\\ &=(x-d_G(v))\sum_{M\in\mathcal{M}(H-v)}(-1)^{|M|}\prod_{w\in V(H-v)\setminus V(M)}(x-d_G(w))\\ &\quad+\sum_{u\in N_H(v)}(-1)\sum_{M\in\mathcal{M}(H-v-u)}(-1)^{|M|}\prod_{w\in V(H-v-u)\setminus V(M)}(x-d_G(w))\\ &=(x-d_G(v))\,\mathscr{L}_G(H-v,x)-\sum_{u\in N_H(v)}\mathscr{L}_G(H-v-u,x).\end{aligned}$$

This completes the proof of this lemma. □

Let P_n be a path of order n with its vertices labelled as $v_1v_2\cdots v_{n-1}v_n$. Set $\psi_n(x)=\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1,x)$ and $\phi_n(x)=\mathscr{L}_{P_{n+2}}(P_{n+2}-v_1-v_{n+2},x)$.

Lemma 2.7. *Under the introduced notation, the following results hold:*

(i) $\mathscr{L}(P_{n+1},x)=(x-2)\mathscr{L}(P_n,x)-\mathscr{L}(P_{n-1},x)$.
(ii) $x\psi_n(x)=\mathscr{L}(P_{n+1},x)+\mathscr{L}(P_n,x)$.
(iii) $x\phi_n(x)=\mathscr{L}(P_{n+1},x)$.

Proof. Due to (2), we get

$$\begin{aligned}\mathscr{L}(P_{n+1},x) &= (x-1)\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1,x)-\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1-v_2,x)\\ &= (x-2)\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1,x)-\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1-v_2,x)\\ &\quad+\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1,x)\\ &= \mathscr{L}_{P_{n+2}}(P_{n+2}-v_1,x)+\mathscr{L}_{P_{n+1}}(P_{n+1}-v_1,x)\\ &= \psi_{n+1}(x)+\psi_n(x),\end{aligned}\tag{3}$$

$$\begin{aligned}\psi_{n+1}(x) &= \mathscr{L}_{P_{n+2}}(P_{n+2}-v_1,x)\\ &= (x-1)\mathscr{L}_{P_{n+2}}(P_{n+2}-v_1-v_{n+2},x)\\ &\quad-\mathscr{L}_{P_{n+2}}(P_{n+2}-v_1-v_{n+1}-v_{n+2},x)\\ &= (x-1)\phi_n(x)-\phi_{n-1}(x)\\ &= (x-2)\phi_n(x)-\phi_{n-1}(x)+\phi_n(x)\\ &= \phi_{n+1}(x)+\phi_n(x).\end{aligned}\tag{4}$$

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In the light of (2), (3) and (4),

$$x\psi_n(x) = x\psi_n(x) + \mathscr{L}(P_n, x) - (\psi_n(x) + \psi_{n-1}(x)) = \mathscr{L}(P_{n+1}, x) + \mathscr{L}(P_n, x), \quad (5)$$

and

$$\begin{aligned}\mathscr{L}(P_{n+1}, x) &= (x-1)\psi_n(x) - \psi_{n-1}(x)\\ &= (x-1)\left((x-2)\psi_{n-1}(x) - \psi_{n-2}(x)\right) - \psi_{n-1}(x)\\ &= (x-2)\mathscr{L}(P_n, x) - (\psi_{n-1}(x) + \psi_{n-2}(x))\\ &= (x-2)\mathscr{L}(P_n, x) - \mathscr{L}(P_{n-1}, x).\end{aligned}$$

Owe to (4) and (5),

$$x\phi_{n+1}(x) - \mathscr{L}(P_{n+2}, x) = \mathscr{L}(P_{n+1}, x) - x\phi_n(x),$$

which is equivalent to

$$x\phi_n(x) - \mathscr{L}(P_{n+1}, x) = (-1)^{n-1}\left(x\phi_1(x) - \mathscr{L}(P_2, x)\right) = 0, \text{ i.e., } x\phi_n(x) = \mathscr{L}(P_{n+1}, x).$$

This finishes our proof. □

Lemma 2.8. *Let* $a = \frac{x-2+\sqrt{x^2-4x}}{2}$ *and* $b = \frac{x-2-\sqrt{x^2-4x}}{2}$. *Then*

(i) $\mathscr{L}(P_n, x) = \frac{x}{\sqrt{x^2-4x}}(a^n - b^n)$.
(ii) $\psi_n(x) = \frac{1}{\sqrt{x^2-4x}}(a^{n+1} + a^n - b^{n+1} - b^n)$.
(iii) $\phi_n(x) = \frac{1}{\sqrt{x^2-4x}}(a^{n+1} - b^{n+1})$.

Proof. For (i), we employ induction on n. It is easy to verify the result for $n = 1, 2$. Suppose that the result holds if $1 \le k \le n$ and $n \ge 3$. When $k = n+1$, in terms of Lemma 2.7(i) we get

$$\begin{aligned}\mathscr{L}(P_{n+1}, x) &= (x-2)\mathscr{L}(P_n, x) - \mathscr{L}(P_{n-1}, x)\\ &= \frac{x}{\sqrt{x^2-4x}}\left((x-2)(a^n - b^n) - (a^{n-1} - b^{n-1})\right)\\ &= \frac{x}{\sqrt{x^2-4x}}\left((x-2)a^n - a^{n-1} - \left((x-2)b^n - b^{n-1}\right)\right)\\ &= \frac{x}{\sqrt{x^2-4x}}\left(a^{n+1} - b^{n+1}\right).\end{aligned}$$

Further, (ii) and (iii) follow from Lemma 2.7(ii) and (iii). □

Remark 2.9. By Mohammadian's result in (1), Lemma 2.8(i) can be directly deduced by [[7], Lemma 3.3]. The same Mohammadian's result, together with $\lim_{i\to\infty} \rho_L(P_i)$ noted in [7], gives an alternative proof to the subsequent Corollary 2.10(i).

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Corollary 2.10. *Let $\{G_i\}_{i=1}^{\infty}$ be a graph sequence.*

(i) *If $G_i = P_i$, then $\lim_{i\to\infty} \lambda(P_i) = 4$.*
(ii) *If $G_i = C_i$, then $\lim_{i\to\infty} \lambda(C_i) = 4$.*

Proof. From $P_i \subseteq P_{i+1}$ and $\Delta(P_i) \leq 2$ it follows by Lemma (ii) that $\lambda(P_i) < \lambda(P_{i+1}) < 4$. Since P_i is a tree, by (1) we obtain $\lambda(P_i) = \rho_L(P_i) = 4\cos^2\frac{\pi}{2i}$ (see [1]) showing $\lim_{i\to\infty} \lambda(P_i) = 4$. In terms of Lemma (ii) again, $\lambda(C_i) = 2 + 2\cos\frac{\pi}{2i} < 4$ and thus $\lim_{i\to\infty} \lambda(C_i) = 4$. □

We now pay attention to the limit points of the Laplacian matching indices of graphs.

Lemma 2.11. *Let G be a connected graph and $v \in V(G)$. $Q_v(G; kP_n)$ is obtained from vertex-disjoint graphs G and k paths P_n by adding k new edges vv_i with $i = 1, \cdots, k$, respectively, where v_i is an end vertex of i-th path P_n. Then*

(i) *$\lim_{n\to\infty} \lambda(Q_v(G; kP_n)) = r$ exists and $r \geq 4$.*
(ii) *If $r > 4$, then r is the largest positive root of the equation:*

$$\frac{x + \sqrt{x^2 - 4x}}{2kx}\mathscr{L}(G, x) - \mathscr{L}_G(G - v, x) = 0.$$

Proof. Since $Q_v(G; kP_n) \subset Q_v(G; kP_{n+1})$, then $\lim_{n\to\infty} \lambda(Q_v(G; kP_n)) = r$ exists by Lemma (ii), and thus $r \geq 4$ by $P_n \subset Q_v(G; kP_n)$ and Corollary 2.10(i).

For simplicity, set $G' = Q_v(G; kP_n)$. In the light of the equality in (2),

$$\begin{aligned}
\mathscr{L}(G') &= (x - d_G(v) - k)\mathscr{L}_{G'}(G' - v) - \sum_{u\in N_{G'}(v)} \mathscr{L}_{G'}(G' - v - u) \\
&= (x - d_G(v))\mathscr{L}_G(G - v)\psi_n^k(x) - \sum_{u\in N_G(v)} \mathscr{L}_G(G - v - u)\psi_n^k(x) \\
&\quad - k\mathscr{L}_G(G - v)\psi_n^k(x) - k\mathscr{L}_G(G - v)\psi_{n-1}(x)\psi_n^{k-1}(x) \\
&= \mathscr{L}(G)\psi_n^k(x) - k\mathscr{L}_G(G - v)\psi_n^k(x) - k\mathscr{L}_G(G - v)\psi_{n-1}(x)\psi_n^{k-1}(x) \\
&= \mathscr{L}^k(P_n)\left(\frac{\psi_n^k(x)}{\mathscr{L}^k(P_n)}\left[\mathscr{L}(G) - k\mathscr{L}_G(G - v)\right] - k\frac{\psi_{n-1}(x)\psi_n^{k-1}(x)}{\mathscr{L}^k(P_n)}\mathscr{L}_G(G - v)\right).
\end{aligned}$$

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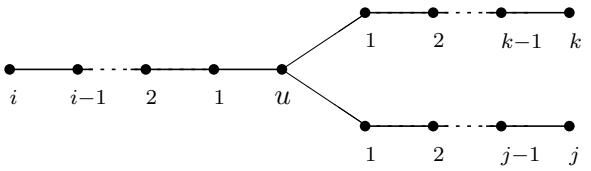


Figure 2. The tree $T_{i,j,k}$ $(i,j,k \geq 1)$.

For $x > 4$, easily to verify that $\frac{b}{a} < 1$. By Lemma 2.8 we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{\psi_n^k(x)}{\mathscr{L}^k(P_n)}\left[\mathscr{L}(G) - k\mathscr{L}_G(G-v)\right] - k\frac{\psi_{n-1}(x)\psi_n^{k-1}(x)}{\mathscr{L}^k(P_n)}\mathscr{L}_G(G-v)\right)\\
&= \left(\frac{a+1}{x}\right)^k\left(\mathscr{L}(G) - k\mathscr{L}_G(G-v)\right) - k\frac{b+1}{x}\left(\frac{a+1}{x}\right)^k\mathscr{L}_G(G-v)\\
&= k\left(\frac{a+1}{x}\right)^{k-1}\left[\frac{a+1}{x}\left(\frac{1}{k}\mathscr{L}(G) - \mathscr{L}_G(G-v)\right) - \frac{b+1}{x}\mathscr{L}_G(G-v)\right]\\
&= k\left(\frac{a+1}{x}\right)^{k-1}\left(\frac{x+\sqrt{x^2-4x}}{2kx}\mathscr{L}(G) - \mathscr{L}_G(G-v)\right).
\end{aligned}$$

Since $r > 4$, we conclude that r is the largest positive root of the equation:

$$\frac{x+\sqrt{x^2-4x}}{2kx}\mathscr{L}(G,x) - \mathscr{L}_G(G-v,x) = 0.$$

This finishes the proof. □

Next, we apply the above lemma to a class of specific graphs. Let $T_{i,j,k}$ be the tree with exactly one vertex u of maximum degree three such that $T_{i,j,k} - u = P_i \cup P_j \cup P_k$, which is shown in Fig. 2.

Corollary 2.12. *For the tree* $T_{1,i,i}$, $\lim_{i\to\infty}\lambda(T_{1,i,i}) = \tau$.

Proof. Note that $T_{1,i,i} \subset T_{1,i+1,i+1}$, then $\lambda(T_{1,i,i}) < \lambda(T_{1,i+1,i+1})$ and the sequence $\{\lambda(T_{1,i,i})\}_{i=1}^{\infty}$ is strictly increasing from Lemma (ii). Hence, $\lim_{i\to\infty}\lambda(T_{1,i,i}) = r > 4$ exists.

Let v be an end vertex of P_2. In the light of Lemma 2.11(ii), then r is the largest root of the equation:

$$\frac{x+\sqrt{x^2-4x}}{4x}\mathscr{L}(P_2,x) - \mathscr{L}_{P_2}(P_2-v,x) = 0.$$

Since $\mathscr{L}(P_2,x) = x^2 - 2x$ and $\mathscr{L}_{P_2}(P_2-v,x) = x-1$, the equation is equivalent to

$$\frac{x+\sqrt{x^2-4x}}{4x}(x^2-2x) - (x-1) = 0, \text{ i.e., } x^3 - 6x^2 + 8x - 4 = 0,$$

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whose largest root is τ. Hence, the result follows. □

Proposition 2.13. *Let G_1 and G_2 be two vertex-disjoint connected graphs with $u \in V(G_1)$ and $v \in V(G_2)$. $\tilde{Q}_{uv}(G_1, G_2; P_n)$ is the graph obtained from G_1, G_2 and a new path P_n: $v_1v_2\cdots v_{n-1}v_n$ by adding two new edges uv_1 and vv_n, respectively. Then*

$$\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) = \max\left\{\lim_{n\to\infty} \lambda(Q_u(G_1; P_n)), \lim_{n\to\infty} \lambda(Q_v(G_2; P_n))\right\}.$$

Proof. From Lemma 2.11(i), both $\lim_{n\to\infty} \lambda(Q_u(G_1; P_n))$ and $\lim_{n\to\infty} \lambda(Q_v(G_2; P_n))$ exist. If $uv_1v_2\cdots v_nv$ is not in an internal path, then either $Q_u(G_1; P_n)$ or $Q_v(G_2; P_n)$ is a path. Obviously, $\tilde{Q}_{uv}(G_1, G_2; P_n) \subset \tilde{Q}_{uv}(G_1, G_2; P_{n+1})$, and thus we obtain that $\lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) < \lambda(\tilde{Q}_{uv}(G_1, G_2; P_{n+1}))$ by Lemma (ii). Therefore, the sequence $\{\lambda(\tilde{Q}_{uv}(G_1, G_2; P_n))\}_{n=1}^{\infty}$ is bounded, and so

$$\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) = \max\left\{\lim_{n\to\infty} \lambda(Q_u(G_1; P_n)), \lim_{n\to\infty} \lambda(Q_v(G_2; P_n))\right\}.$$

We next consider that $uv_1v_2\cdots v_nv$ is in an internal path. Since uv_1 is a cut edge, it follows from Lemma 2.5(ii) that

$$\lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) > \lambda(\tilde{Q}_{uv}(G_1, G_2; P_{n+1})).$$

Hence, $\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n))$ exists. Note that $\Delta(Q_u(G_1; P_n)), \Delta(Q_v(G_2; P_n)) \geq 3$ and $Q_u(G_1; P_n), Q_v(G_2; P_n) \subset \tilde{Q}_{uv}(G_1, G_2; P_n)$. Along with Lemmas (ii) and 2.11, we have

$$\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) \geq \lim_{n\to\infty} \lambda(Q_u(G_1; P_n)) > 4$$

and

$$\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) \geq \lim_{n\to\infty} \lambda(Q_v(G_2; P_n)) > 4.$$

Set $G = \tilde{Q}_{uv}(G_1, G_2; P_n), G_1 = Q_u(G_1; P_n)$ and $G_2 = Q_v(G_2; P_n)$. In the light of (2),

$$\begin{aligned}
\mathscr{L}(G) &= (x - d_{G_1}(u) - 1)\mathscr{L}_G(G - u) - \sum_{w\in N_G(u)} \mathscr{L}_G(G - u - w) \\
&= (x - d_{G_1}(u) - 1)\mathscr{L}_{G_1}(G_1 - u)\mathscr{L}_G(Q_v(G_2; P_n)) \\
&\quad - \sum_{w\in N_{G_1}(u)} \mathscr{L}_{G_1}(G_1 - u - w)\mathscr{L}_G(Q_v(G_2; P_n)) \\
&\quad - \mathscr{L}_{G_1}(G_1 - u)\mathscr{L}_G(Q_v(G_2; P_n) - v_1)
\end{aligned}$$

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$$
\begin{aligned}
&= (\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u))\, \mathscr{L}_G(Q_v(G_2; P_n)) \\
&\quad - \mathscr{L}_{G_1}(G_1 - u)\mathscr{L}_G(Q_v(G_2; P_n) - v_1) \\
&= (\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u))\, [\phi_n(x)\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) \\
&\quad - \phi_{n-1}(x)\mathscr{L}_{G_2}(G_2 - v)] \\
&\quad - \mathscr{L}_{G_1}(G_1 - u)\, [\phi_{n-1}(x)\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) - \phi_{n-2}(x)\mathscr{L}_{G_2}(G_2 - v)] \\
&= \left[(\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u)) \left(\frac{\phi_n(x)}{\mathscr{L}(P_n)}\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) \right.\right. \\
&\quad \left. - \frac{\phi_{n-1}(x)}{\mathscr{L}(P_n)}\mathscr{L}_{G_2}(G_2 - v)\right) - \mathscr{L}_{G_1}(G_1 - u) \\
&\quad \left. \times \left(\frac{\phi_{n-1}(x)}{\mathscr{L}(P_n)}\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) - \frac{\phi_{n-2}(x)}{\mathscr{L}(P_n)}\mathscr{L}_{G_2}(G_2 - v)\right)\right] \mathscr{L}(P_n) \\
&\triangleq \mathscr{L}(P_n) f_n(x).
\end{aligned}
$$

For $x > 4$, from Lemma 2.8 we have

$$
\begin{aligned}
\lim_{n\to\infty} f_n(x) &= (\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u)) \left(\lim_{n\to\infty} \frac{\phi_n(x)}{\mathscr{L}(P_n)}\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) \right. \\
&\quad \left. - \lim_{n\to\infty} \frac{\phi_{n-1}(x)}{\mathscr{L}(P_n)}\mathscr{L}_{G_2}(G_2 - v)\right) \\
&\quad - \mathscr{L}_{G_1}(G_1 - u) \left(\lim_{n\to\infty} \frac{\phi_{n-1}(x)}{\mathscr{L}(P_n)}\, (\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) \right. \\
&\quad \left. - \lim_{n\to\infty} \frac{\phi_{n-2}(x)}{\mathscr{L}(P_n)}\mathscr{L}_{G_2}(G_2 - v)\right) \\
&= (\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u)) \left(\frac{a}{x}(\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) - \frac{1}{x}\mathscr{L}_{G_2}(G_2 - v)\right) \\
&\quad - \mathscr{L}_{G_1}(G_1 - u) \left(\frac{1}{x}(\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)) - \frac{b}{x}\mathscr{L}_{G_2}(G_2 - v)\right) \\
&= \left(\frac{x + \sqrt{x^2 - 4x}}{2x}\mathscr{L}(G_1) - \mathscr{L}_{G_1}(G_1 - u)\right) \\
&\quad \times \left(\frac{x + \sqrt{x^2 - 4x}}{2x}\mathscr{L}(G_2) - \mathscr{L}_{G_2}(G_2 - v)\right) \\
&\triangleq f(x).
\end{aligned}
$$

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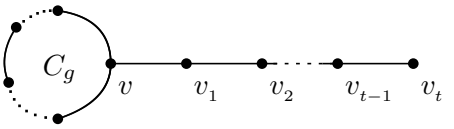


Figure 3. The graph $H_{g,t}$ $(g \geq 3, t \geq 1)$.

Since $\lim_{n\to\infty} \lambda(G) > 4$, we conclude that $\lim_{n\to\infty} \lambda(G)$ is the largest positive root of the equation $f(x) = 0$. Thereby, it follows from Lemma 2.11 that

$$\lim_{n\to\infty} \lambda(\tilde{Q}_{uv}(G_1, G_2; P_n)) = \max\left\{\lim_{n\to\infty} \lambda(Q_u(G_1; P_n)), \lim_{n\to\infty} \lambda(Q_v(G_2; P_n))\right\}.$$

This finishes our proof. □

Let $H_{g,t}$ be the lollipop graph shown in Fig. 3. At the end of this section, we focus on the limit points of the largest Laplacian matching root of $H_{g,t}$ when g and t tend to infinity, respectively.

Proposition 2.14. *Let $H_{g,t}$ be defined as above. Then*

$$\lim_{g\to\infty} \lambda(H_{g,1}) = \tau.$$

Proof. Due to Lemma 2.5(i), $\{\lambda(H_{g,1})\}_{g=3}^{\infty}$ is a strictly increasing sequence. Along with Lemma (ii) we get that $\lim_{g\to\infty} \lambda(H_{g,1}) = r > 4$ exists.

For simplicity, let $H' = H_{g,1}$ and v be the vertex of C_g with $d_{H'}(v) = 3$. In the light of (2),

$$\begin{aligned}
\mathscr{L}(H') &= (x-3)\mathscr{L}_{H'}(H'-v) - \sum_{u\in N_{H'}(v)} \mathscr{L}_{H'}(H'-v-u) \\
&= (x-3)\phi_{g-1}(x)\psi_1(x) - 2\phi_{g-2}(x)\psi_1(x) - \phi_{g-1}(x) \\
&= (x^2-4x+2)\phi_{g-1}(x) - 2(x-1)\phi_{g-2}(x) \\
&= \mathscr{L}(P_g)\left[\left(\frac{\phi_{g-1}(x)}{\mathscr{L}(P_g)}\right)(x^2-4x+2) - 2\frac{\phi_{g-2}(x)}{\mathscr{L}(P_g)}(x-1)\right].
\end{aligned}$$

From Lemma 2.8, we have for $x > 4$

$$\begin{aligned}
&\lim_{g\to\infty}\left[\left(\frac{\phi_{g-1}(x)}{\mathscr{L}(P_g)}\right)(x^2-4x+2) - 2\frac{\phi_{g-2}(x)}{\mathscr{L}(P_g)}(x-1)\right] \\
=&(x^2-4x+2)\frac{1}{x} - (x-1)\frac{2b}{x} \\
=&\frac{1}{x}\left((x-1)\sqrt{x^2-4x} - x\right),
\end{aligned}$$

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which implies that $\lim_{g\to\infty} \lambda(H_{g,1}) = r$ is the largest positive root of the equation:

$$(x-1)\sqrt{x^2-4x} - x = 0, \text{ i.e., } x^3 - 6x^2 + 8x - 4 = 0.$$

Thus, we finally conclude that

$$\lim_{g\to\infty} \lambda(H_{g,1}) = \tau.$$

This finishes the proof. □

Proposition 2.15. *Let $H_{g,t}$ be defined as above. Then for every $g \geq 3$,*

$$\lim_{t\to\infty} \lambda(H_{g,t}) > \tau.$$

Proof. Since $H_{g,t} \subset H_{g,t+1}$, by Lemma (ii) we get $\lambda(H_{g,t}) < \lambda(H_{g,t+1})$, and thus the sequence $\{\lambda(H_{g,t})\}_{t=1}^{\infty}$ is strictly increasing. Hence, $\lim_{t\to\infty} \lambda(H_{g,t}) = r_g > 4$ exists by Lemma (ii) again. On the other hand, from Lemma 2.5(i) it follows that $\lambda(H_{g+1,t}) > \lambda(H_{g,t})$ which indicates

$$\lim_{t\to\infty} \lambda(H_{g,t}) \geq \lim_{t\to\infty} \lambda(H_{3,t}). \tag{6}$$

We now consider the sequence $\{\lambda(H_{3,t})\}_{t=1}^{\infty}$. Let $v \in V(C_3)$ with $d_{H_{3,t}}(v) = 3$. In the light of Lemma 2.11 (ii), r_3 is the largest root of the equation:

$$\frac{x+\sqrt{x^2-4x}}{2x}\mathscr{L}(C_3, x) - \mathscr{L}_{C_3}(C_3 - v, x) = 0. \tag{7}$$

Substituting $\mathscr{L}(C_3, x) = x^3 - 6x^2 + 9x - 2$ and $\mathscr{L}_{C_3}(C_3 - v, x) = x^2 - 4x + 3$ into (7), then

$$\frac{x+\sqrt{x^2-4x}}{2x}(x^3 - 6x^2 + 9x - 2) - (x^2 - 4x + 3) = 0,$$

which is equivalent to

$$x^5 - 10x^4 + 34x^3 - 46x^2 + 21x - 4 = 0.$$

Thereby, $\lim_{t\to\infty} \lambda(H_{3,t}) = r_3 \approx 4.47 > \tau$, and hence the result follows from (6). □

3. Main results

We are now in the stage to obtain the main results, where the method used in Theorem 3.1 is inspired by Hoffman [9] and Guo [7].

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Theorem 3.1 *Let $\beta_0 = 1$ and β_n be the largest positive root of*

$$h_n(x) = x^{n+1} - (1 + x + ... + x^{n-1})(\sqrt{x} + 1)^2.$$

And let $\alpha_n = 2 + \beta_n^{\frac{1}{2}} + \beta_n^{-\frac{1}{2}}$. Then

$$4 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$$

are all the limit points of the Laplacian matching indices of graphs smaller than $\lim\limits_{n\to\infty} \alpha_n = \tau$.

Proof. Obviously, we suppose each one in the graph sequence $\{G_i\}_{i=1}^{\infty}$ to be connected from the fact that $\lambda(G) = \max\{\lambda(\tilde{G}_1), \lambda(\tilde{G}_2)\}$ for $G = \tilde{G}_1 \cup \tilde{G}_2$. Then for $i \neq j$, set $\lambda(G_i) \neq \lambda(G_j)$ and $\lim\limits_{i\to\infty} \lambda(G_i) = r < \tau$. If $\Delta(G_i) \geq 4$, by Lemma (ii), we get $\lambda(G_i) \geq 5$ and so $r \geq 5$, a contradiction. Thus, $\Delta(G_i) \leq 3$. If $\Delta(G_i) = 2$ for infinitely many $i's$, then G_i is a path or a cycle. It follows from Corollary 2.10 that

$$\lim_{i\to\infty} \lambda(G_i) = \lim_{i\to\infty} \lambda(P_i) = \lim_{i\to\infty} \lambda(C_i) = 4.$$

Thereby, in what follows, we always set $\Delta(G_i) = 3$.

Let $\mathcal{S} = \{g_i \mid i \geq 1\}$ where g_i is the girth of G_i. The following cases are to be considered:

Case 1. $\mathcal{S}$ is unbounded. Assume that the cycle $C_{g_i} \subseteq G_i$ with $g_i > g_j$ for $i > j$. Since $\Delta(G_i) = 3$, then $H_{g_i,1} \subseteq G_i$ and so $\lambda(G_i) \geq \lambda(H_{g_i,1})$. Along with Proposition 2.14 we get

$$\lim_{i\to\infty} \lambda(G_i) \geq \lim_{i\to\infty} \lambda(H_{g_i,1}) = \tau,$$

a contradiction.

Case 2. $\mathcal{S}$ is bounded and there are infinitely many G_i's which are not trees. If $C_g \subset G_i$, by $\Delta(G_i) = 3$ we obtain $H_{g,k_i} \subseteq G_i$ with $k_i > k_j$ for $i > j$. Similarly, by Proposition 2.15 it follows that

$$\lim_{i\to\infty} \lambda(G_i) \geq \lim_{i\to\infty} \lambda(H_{g,k_i}) > \tau,$$

a contradiction again.

Case 3. Infinitely many G_i's are trees. If G_i has at least three vertices of degree three, then $R_{p_i,q_i} \subseteq G_i$ (see Fig. 4). When k_i is sufficiently large, by Lemma 2.5(ii) we get $\lambda(G_i) \geq \lambda(R_{p_i,q_i}) \geq \lambda(T_{1,k_i,k_i})$. Due to Corollary 2.12, we obtain that

$$\lim_{i\to\infty} \lambda(G_i) \geq \lim_{i\to\infty} \lambda(T_{1,k_i,k_i}) = \tau,$$

a contradiction.

If G_i has two vertices of degree three and their distance is bounded, then $F_{s,r_i} \subseteq G_i$ (see Fig. 5) with $r_i > r_j$ for $i > j$. In terms of Lemma 2.5(ii), for a sufficiently large

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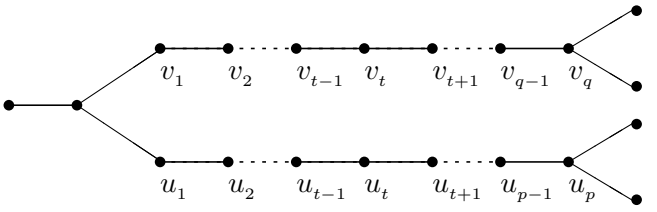


Figure 4. The graph $R_{p,q}$ $(p, q \geq 1)$.

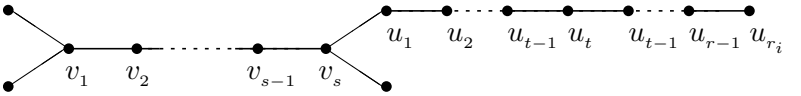


Figure 5. The graph F_{s,r_i} $(s > 1, r_i \geq 1)$.

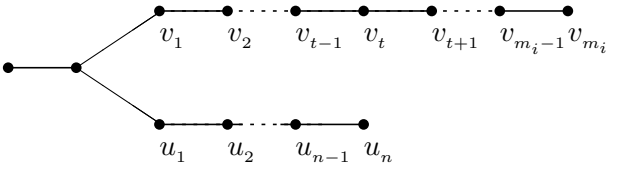


Figure 6. The graph T_{1,n,m_i}.

integer k_i, it follows that $\lambda(G_i) \geq \lambda(F_{s,r_i}) \geq \lambda(T_{1,k_i,k_i})$. Along with Corollary 2.12, we have

$$\lim_{i\to\infty} \lambda(G_i) \geq \lim_{i\to\infty} \lambda(T_{1,k_i,k_i}) = \tau,$$

a contradiction again.

If G_i has two vertices of degree three and the distance between them is unbounded, then by Proposition 2.13 we can construct a tree sequence $\{T_{k,n,m_i}\}_{i=1}^{\infty}$ with $m_i > m_j$ for $i > j$ such that $\lim_{i\to\infty} \lambda(T_{k,n,m_i}) = \lim_{i\to\infty} \lambda(G_i)$. Without loss of generality, set $m_i \geq n \geq k \geq 1$. If $k \geq 2$, then $T_{2,2,2} \subseteq T_{k,n,m_i}$, and thus $\lambda(T_{2,2,2}) = 3 + \sqrt{2} > \tau$, a contradiction. So, $k = 1$ and T_{1,n,m_i} is shown in Fig. 6.

As stated above, the limit points we want are the numbers $\lim_{i\to\infty} \lambda(T_{1,n,m_i}) = \alpha_n$. Due to Lemmas (ii) and 2.11, α_n is the largest root of the equation:

$$\frac{x + \sqrt{x^2 - 4x}}{2x} \mathscr{L}(P_{n+2}, x) - \psi_1(x)\psi_n(x) = 0.$$

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Let $x=\gamma+\frac{1}{\gamma}+2$ and $\gamma^2=z$. From Lemma 2.8 we arrive at

$$\begin{aligned}
&\frac{x+\sqrt{x^2-4x}}{2x}\mathscr{L}(P_{n+2},x)-\psi_1(x)\psi_n(x)\\
=&\frac{x+\sqrt{x^2-4x}}{2x}\mathscr{L}(P_{n+2},x)-(x-1)\psi_n(x)\\
=&\frac{x+\sqrt{x^2-4x}}{2\sqrt{x^2-4x}}(a^{n+2}-b^{n+2})-\frac{1}{\sqrt{x^2-4x}}(x-1)(a^{n+1}+a^n-b^{n+1}-b^n)\\
=&(\gamma+1)\frac{\gamma}{\gamma^2-1}\left(\gamma^{n+2}-\frac{1}{\gamma^{n+2}}\right)-\frac{\gamma}{\gamma^2-1}\left(\gamma+\frac{1}{\gamma}+1\right)\left(\gamma^{n+1}+\gamma^n-\frac{1}{\gamma^{n+1}}-\frac{1}{\gamma^n}\right)\\
=&\frac{1}{\gamma^n}\left[\frac{\gamma}{\gamma-1}\left(\gamma^{2n+2}-\frac{1}{\gamma^2}\right)-\frac{\gamma}{\gamma-1}\left(\gamma+\frac{1}{\gamma}+1\right)\left(\gamma^{2n}-\frac{1}{\gamma}\right)\right]\\
=&\frac{1}{\gamma^n}\left(\gamma^{2n+2}+\frac{\gamma^{2n}+\gamma^{2n+1}-\gamma-1}{1-\gamma}\right)\\
=&\frac{1}{\gamma^n}\left(\gamma^{2n+2}+\frac{\gamma^{2n}-1}{1-\gamma^2}(\gamma+1)^2\right)\\
=&\frac{1}{\gamma^n}\left(\gamma^{2n+2}-(1+\gamma^2+\cdots+\gamma^{2n-2})(\gamma+1)^2\right)\\
=&\frac{1}{\gamma^n}\left(z^{n+1}-(1+z+\cdots+z^{n-1})(\sqrt{z}+1)^2\right).
\end{aligned}$$

Since $h_n(x)-xh_{n-1}(x)=-(\sqrt{x}+1)^2<0$ for $x\geq 0$, then $\beta_n>\beta_{n-1}$, and hence the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is strictly increasing. From Corollary 2.12 it follows that $\lim\limits_{n\to\infty}\alpha_n=\tau$. This completes the proof. □

Theorem 3.2 *The first two smallest limit points of the Laplacian matching indices of connected graphs are* 4 *and* $2+\sqrt{5}$*, respectively. Moreover, for a connected graph* G *with order* n*,*

(i) $\lambda(G)<4$ *if and only if* $G\in\{P_n|n\geq 1\}\cup\{C_n|n\geq 3\}$.
(ii) $\lambda(G)=4$ *if and only if* $G=T_{1,1,1}$.
(iii) $4<\lambda(G)\leq 2+\sqrt{5}$ *if and only if* $G\in\{T_{1,1,n-3}|n\geq 5\}$.

Proof. From Theorem 3.1, we know that the smallest limit point of the Laplacian matching indices is $\alpha_0=4$ and the second smallest one is $\alpha_1=2+\beta_1^{\frac{1}{2}}+\beta_1^{-\frac{1}{2}}$ with $\beta_1=\frac{3+\sqrt{5}}{2}$ being the largest positive root of the equation $x^2-(\sqrt{x}+1)^2=0$. Thus, $\alpha_1=2+\sqrt{5}$.

We now characterise the connected graphs with $\lambda(G)\leq 2+\sqrt{5}$. When $G=C_n$, by Corollary 2.10(ii) we get $\lambda(C_n)=2+2\cos\frac{\pi}{2n}<4$. In what follows, set $G\neq C_n$. We claim that G must be a tree with $\Delta(G)\leq 3$. Otherwise, if $\Delta(G)\geq 4$ then $\lambda(G)\geq 5$ by Lemma (ii), a contradiction. If G is not a tree, then $H_{g,1}\subseteq G$ and hence

$$\lambda(G)\geq\lambda(H_{g,1})\geq\lambda(H_{3,1})\approx 4.34>2+\sqrt{5}$$

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by Lemma 2.5(i), a contradiction.

For $\Delta(G) \leq 2$, then $G = P_n$ with $\lambda(P_n) = 4\cos^2\frac{\pi}{2n} < 4$ by Corollary 2.10(i). For $\Delta(G) = 3$, we claim that G has exactly one vertex of degree three. Otherwise, G has at least two vertices of degree three. So $F_{s,1} \subseteq G$. In terms of Lemma 2.5(ii) and Proposition 2.13, the graph sequence $\{F_{s,1}\}_{s=1}^{\infty}$ is strictly decreasing and $\lim_{s\to\infty}\lambda(F_{s,1}) = \lim_{s\to\infty}\lambda(T_{1,1,s}) = 2+\sqrt{5}$. Thus, it follows from Lemma (ii) that

$$\lambda(G) \geq \lambda(F_{s,1}) > 2+\sqrt{5},$$

a contradiction. Thereby, $G \cong T_{i,j,k}$ with $k \geq j \geq i \geq 1$. Owe to $\lambda(T_{1,2,2}) = \frac{5+\sqrt{13}}{2} > 2+\sqrt{5}$, then $i = j = 1$ and $k \geq 1$, i.e., $G \cong T_{1,1,k}$. Consequently, $\lambda(T_{1,1,1}) = 4$ and $\lambda(T_{1,1,k}) \leq \lim_{k\to\infty}\lambda(T_{1,1,k}) = 2+\sqrt{5}$ by Theorem 3.1.

This finishes the proof. □

We finally propose the following problems to end this paper.

Problem 1. *Characterise the connected graphs whose Laplacian matching indices lie in the interval* $(2+\sqrt{5}, \tau]$.

Problem 2. *Determine the limit points of the Laplacian matching indices where these indices are larger than* τ.

Data availability. There is no data associated with this article.

Conflict of interest. The authors declare that they have no competing interests.

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