Bull. Aust. Math. Soc. **113** (2026), 483–491
doi:10.1017/S0004972725100506

A NOTE ON GROUPS WITH FINITENESS CONDITIONS ON COMMUTATOR SUBGROUPS

ROSA CASCELLA

(Received 14 July 2025; accepted 29 July 2025; first published online 7 November 2025)

Abstract

The structure of groups for which certain sets of commutator subgroups are finite is investigated, with a particular focus on the relationship between these groups and those with finite derived subgroup.

2020 *Mathematics subject classification*: primary 20F24.

Keywords and phrases: *FC*-group, commutator subgroup, finiteness condition.

1. Introduction

A finiteness condition is a group theoretical property satisfied by every finite group. In many cases, the imposition of a finiteness condition produces strong restrictions on the structure of a group. For instance, this is the case for the property *FC*, whose definition requires that each element of the group has only finitely many conjugates. The behaviour of *FC*-groups has been investigated by several authors over the last seventy years; among the basic properties of *FC*-groups, it is well known that the commutator subgroup of any *FC*-group is locally finite and that finitely generated groups with the property *FC* are central-by-finite. A special subclass of *FC*-groups is formed by finite-by-abelian groups. A relevant theorem of B. H. Neumann [6] states that a group G is finite-by-abelian if and only if each subgroup of G is nearly normal, where a subgroup X of a group G is called *nearly normal* if it has finite index in its normal closure. Moreover, B. H. Neumann [5] proved that a group G has finite derived subgroup if and only if it has boundedly finite conjugacy classes.

Our aim is to study new finiteness conditions related to the derived subgroup of a group. Groups with finitely many derived groups of subgroups or infinite subgroups are described in [3], while groups with finitely many derived groups of nonnormal subgroups or of infinite nonnormal subgroups are studied in [4]. We study groups G in which, for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G\}$ is finite and groups in which the set $\{[X, H] \mid H \leq G \text{ and } H \text{ is infinite}\}$ is finite. Groups with the first property need not have a finite derived subgroup, since any Tarski group (that is, an

© The Author(s), 2025. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

infinite group whose proper nontrivial subgroups have prime orders) has this property. However, it turns out that if G is a soluble-by-finite group, requiring that for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G\}$ is finite is equivalent to the finiteness of the derived subgroup of G.

THEOREM 1.1. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G\}$ is finite. Then, G' is finite.*

It has recently been proved that a group G is soluble-by-finite and for each subgroup X of G, the set $\{[X,H] \mid H \leq G, H \text{ is infinite}\}$ is finite if and only if it has finite commutator subgroup or it is a finite extension of a group of type p^∞ for some prime number p [1]. In the second part of this paper, we consider groups such that for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G, H \text{ is infinite}\}$ is finite. This property does not imply the finiteness of the derived subgroup, even in the case of soluble groups, as the locally dihedral 2-group shows. Our second theorem improves the result in [1].

THEOREM 1.2. *Let G be a soluble-by-finite group. Then, for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G, H \text{ is infinite}\}$ is finite if and only if it has a finite commutator subgroup or it is a finite extension of a group of type p^∞ for some prime number p.*

Finally, we mention that the behaviour of groups in which the set $\{[x,H] \mid H \leq G\}$ (or $\{[x,H] \mid H \leq G, H \text{ is infinite}\}$) is finite for each element x of G is studied in [2].

Most of the notation is standard and can be found in [7].

2. Proof of Theorem 1.1

Clearly, the properties analysed in this paper are inherited by subgroups and homomorphic images.

LEMMA 2.1. *Let G be a group such that for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G \text{ and } H \text{ is infinite}\}$ is finite. Let A be a torsion-free abelian normal subgroup of G. Then, A is contained in the centre $Z(G)$ of G.*

PROOF. Assume by contradiction that there exists an element x of G such that $[A,x] \neq \{1\}$ and suppose first that $\langle x\rangle \not\trianglelefteq G$. Let $a \in A$. From [7, Part 1, Lemma 2.12],

$$[\langle x\rangle, \langle a^n\rangle] = [x, a^n]^{\langle x\rangle\langle a^n\rangle}.$$

Moreover, since A is an abelian normal subgroup of G,

$$[x, a^n]^{\langle x\rangle\langle a^n\rangle} = [x, a^n]^{\langle x\rangle} = ([x,a]^n)^{\langle x\rangle} = ([x,a]^{\langle x\rangle})^n.$$

It follows that the set $\{([x,a]^{\langle x\rangle})^{r!} \mid r \in \mathbb{N}_0\}$ is finite. Consequently, there exists $k \in \mathbb{N}_0$ such that

$$([x,a]^{\langle x\rangle})^{k!} = ([x,a]^{\langle x\rangle})^{(k+1)!} = \cdots.$$

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

Thus, the subgroup $([x,a]^{\langle x\rangle})^{k!}$ is divisible. However, $\langle x,a\rangle$ is a finitely generated metabelian group, so that it is residually finite (see [7, Part 2, Theorem 9.51]) and hence, $[x,a]^{k!}=1$. Therefore, $[x,a]=1$ and $A\leq C_G(x)$.

Suppose now that $\langle x\rangle \lhd G$. Then, there exists an integer n such that $\langle x^n\rangle = [x,A]\leq Z(\langle x,A\rangle)$. Thus, $[A,x,x]=\{1\}=[[A,x],x]$. Then, $[A,x^n]=[A,x]^n=\{1\}$ and $[A,x]=\{1\}$. Therefore, $A\leq C_G(x)$ for each $x\in G$. □

LEMMA 2.2. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X,H]\mid H\leq G$ and H is infinite$\}$ is finite. Then, the commutator subgroup G' of G is periodic.*

PROOF. Assume for a contradiction that the statement is false and let T be the largest periodic normal subgroup of G. Clearly, the factor group G/T is also a counterexample, so we may suppose without loss of generality that G has no periodic nontrivial normal subgroups. Among all such counterexamples, choose G in such a way that its soluble radical S has minimal derived length m. Let U be the smallest nontrivial term of the derived series of S and let A be a maximal abelian normal subgroup of G containing U. Clearly, A is torsion-free and so it is contained in $Z(G)$ by Lemma 2.1. If L/A is the largest periodic normal subgroup of G/A, the factor group $L/Z(L)$ is locally finite, so that L' is locally finite by Schur's theorem (see [7, Part 1, Corollary to Theorem 4.12]) and hence, $L'=\{1\}$. Thus, L is abelian, so that $L=A$, and hence, G/A has no periodic nontrivial normal subgroups. However, the soluble radical of G/A has derived length less then m and so the minimal choice of G implies that G/A is abelian. Therefore, G is nilpotent and hence, $A=C_G(A)=G$. This contradiction completes the proof of the lemma. □

LEMMA 2.3. *Let G be a periodic soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X,H]\mid H\leq G\}$ is finite. Then, G is an FC-group.*

PROOF. Let x be an element of G. If $\langle x\rangle \lhd G$, then x is an FC-element of G. Otherwise, if $\langle x\rangle \not\lhd G$, then there exist $y_1,\dots,y_t\in G$ such that

$$\{[\langle x\rangle,\langle y\rangle]\mid y\in G\}=\{[\langle x\rangle,\langle y_1\rangle],\dots,[\langle x\rangle,\langle y_t\rangle]\}.$$

It follows that $[\langle x\rangle,G]$ is finitely generated and so, it is finite. Therefore, x is an FC-element. □

COROLLARY 2.4. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X,H]\mid H\leq G\}$ is finite. Then, G' is a periodic FC-group.*

LEMMA 2.5. *Let G be a residually finite group such that for each nonnormal subgroup X of G, the set $\{[X,H]\mid H\leq G, H$ is infinite$\}$ is finite. Then, G is an FC-group.*

PROOF. Let $\mathcal{L}$ be the set of all normal subgroups of finite index of G. If x is any element of G such that $\langle x\rangle \lhd G$, then x is an FC-element. Consider $x\in G$ such that $\langle x\rangle \not\lhd G$. By hypothesis, there exists a finite subset $\mathcal{F}$ of $\mathcal{L}$ such that

$$\{[\langle x\rangle,H]=[x,H]\mid H\in\mathcal{L}\}=\{[x,H]\mid H\in\mathcal{F}\}.$$

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

Then,

$$\left[x, \bigcap_{H\in\mathcal{F}} H\right] \leq \bigcap_{H\in\mathcal{F}} [x, H] = \bigcap_{H\in\mathcal{L}} [x, H] \leq \bigcap_{H\in\mathcal{L}} H = \{1\}$$

and hence, the subgroup $\bigcap_{H\in\mathcal{F}} H$ is contained in $C_G(x)$. As the index

$$\left|G : \bigcap_{H\in\mathcal{F}} H\right|$$

is finite, it follows that x has finitely many conjugates in G. Therefore, G is an FC-group. □

COROLLARY 2.6. *Let G be a metabelian nonperiodic group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. Then, G' is finite.*

PROOF. Let a be an element of infinite order of G. Let x be an element of G. If $\langle x\rangle \lhd G$, then x is an FC-element. Assume that $\langle x\rangle \not\lhd G$. Then, there exist $y_1, \ldots, y_k \in G$ such that

$$\{[\langle x\rangle, \langle y, a\rangle] \mid y \in G\} = \{[\langle x\rangle, \langle y_1, a\rangle], \ldots, [\langle x\rangle, \langle y_k, a\rangle]\}.$$

For a fixed i in the set $\{1, \ldots, k\}$, the subgroup $\langle x, y_i, a\rangle$ is residually finite and satisfies the maximal condition on normal subgroups. Thus, $\langle x, y_i, a\rangle'$ is finite by Lemma 2.5. Therefore, $[x, G]$ is finite and x is an FC-element. Let X be a nonnormal subgroup of G. Then, there exist $g_1, \ldots, g_t \in G$ such that

$$\{[X, \langle g, a\rangle] \mid g \in G\} = \{[X, \langle g_1, a\rangle], \ldots, [X, \langle g_t, a\rangle]\},$$

where $[X, \langle g_j, a\rangle]$ is finitely generated for all $j \in \{1, \ldots, t\}$, since G is an FC-group. Thus, $[X, G]$ is finite. Consider now $X^G = X[X, G]$. We have

$$|X^G : X| = |X[X, G] : X| = |[X, G] : (X \cap [X, G])| < \infty.$$

Thus, each subgroup of G is nearly normal and so G' is finite by Neumann's theorem [6]. □

LEMMA 2.7. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G\}$ is finite. Then, G is an FC-group.*

PROOF. Let x be an element of G. If $\langle x\rangle \lhd G$, then x is an FC-element of G. Otherwise, if $\langle x\rangle \not\lhd G$, then there exist $g_1, \ldots, g_t \in G$ such that

$$\{[\langle x\rangle, \langle g\rangle] \mid g \in G\} = \{[\langle x\rangle, \langle g_1\rangle], \ldots, [\langle x\rangle, \langle g_t\rangle]\}.$$

For a fixed i in the set $\{1, \ldots, t\}$, let us consider $E_i = \langle x, g_i\rangle$. Let X be a nonnormal subgroup of E_i'. Then, there exist $y_1, \ldots, y_k \in E_i'$ such that

$$\{[X, \langle y\rangle] \mid y \in E_i'\} = \{[X, \langle y_1\rangle], \ldots, [X, \langle y_k\rangle]\}.$$

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

By Corollary 2.4, $\langle y_j \rangle^{E_i'}$ is finite for each $j \in \{1, \dots, k\}$. It follows that $[X, E_i']$ is finitely generated and so it is finite. Observe that

$$|X^{E_i'} : X| = |X[X, E_i'] : X| = |[X, E_i'] : (X \cap [X, E_i'])| < \infty.$$

Therefore, each subgroup of E_i' is nearly normal and so E_i'' is finite [6]. For a fixed i, since E_i/E_i'' is finitely generated, it is residually finite and satisfies the maximal condition on normal subgroups. Thus, E_i/E_i'' is an FC-group by Lemma 2.5 and, by Dietzmann's lemma (see [7, Part 1, Corollary 2 to Lemma 2.14]), E_i'/E_i'' is finite; it follows that E_i' is finite. Therefore, $[x, G]$ is finite and G is an FC-group. □

We are ready to prove Theorem 1.1, the main result of this section.

PROOF OF THEOREM 1.1. Assume X is not a normal subgroup of G. Then, there exist $g_1, \dots, g_t \in G$ such that

$$\{[X, \langle g \rangle] \mid g \in G\} = \{[X, \langle g_1 \rangle], \dots, [X, \langle g_t \rangle]\}.$$

Since G is an FC-group, $[X, G]$ is finitely generated and thus it is finite. Consider now $X^G = X[X, G]$. We have

$$|X^G : X| = |X[X, G] : X| = |[X, G] : (X \cap [X, G])| < \infty.$$

Thus, each subgroup of G is nearly normal and so G' is finite by Neumann's theorem [6]. □

3. Proof of Theorem 1.2

LEMMA 3.1. *Let G be a group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. If $Z(G)$ is infinite, then for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G\}$ is finite.*

PROOF. Let X be a nonnormal subgroup of G, then $[X, H] = [X, HZ(G)]$ for every subgroup H of G and hence, the set $\{[X, H] \mid H \leq G\} = \{[X, HZ(G)] \mid H \leq G\}$ is finite. □

COROLLARY 3.2. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. If $Z(G)$ is infinite, then G' is finite.*

PROOF. The result is an immediate consequence of Lemma 3.1 and Theorem 1.1. □

As a consequence of Lemmas 2.1 and 3.1, for groups containing an infinite torsion-free abelian normal subgroup, the two properties analysed in this paper are equivalent.

LEMMA 3.3. *Let G be a soluble-by-finite group in which for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. If N is an infinite normal subgroup of G, then G/N has a finite commutator subgroup.*

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

PROOF. Consider $\bar{G} = G/N$. Then, for each nonnormal subgroup $\bar{X}$ of $\bar{G}$, the set $\{[\bar{X}, \bar{H}] \mid \bar{H} \leq \bar{G}\}$ is finite and hence, by Theorem 1.1, $\bar{G}'$ is finite. □

COROLLARY 3.4. *Let G be an infinite nilpotent group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. Then, G' is finite.*

PROOF. Let i be the smallest positive integer such that the ith term $Z_i(G)$ of the upper central series of G is infinite. Then, $G/Z_{i-1}(G)$ satisfies the hypothesis and has an infinite centre. It follows from Corollary 3.2 that $G'Z_{i-1}(G)/Z_{i-1}(G)$ is finite, so G' is finite since $Z_{i-1}(G)$ is finite. □

LEMMA 3.5. *Let G be a group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. Then, G is an FC-group if and only if G' is finite.*

PROOF. To prove that G' is finite, first suppose that G is a periodic FC-group. Let X be a nonnormal subgroup of G. Let us consider $\bar{G} = G/X_G$ and $\bar{X} = X/X_G$. If $\bar{X}$ is finite, then it is nearly normal in $\bar{G}$. It follows that X is nearly normal in G since

$$|\bar{X}^{\bar{G}} : \bar{X}| = |X^G/X_G : X/X_G| = |X^G : X| < \infty.$$

Suppose that $\bar{X}$ is infinite. Then, the set $\{[\bar{H}, \bar{K}] \mid \bar{K} \leq \bar{X}, \bar{K} \text{ is infinite}\}$ is finite for each subgroup $\bar{H}$ of $\bar{X}$. Since $\bar{X}$ is an FC-group, $|\bar{X}'|$ is finite (see [1, Theorem B]). Since $\bar{X} \trianglelefteq \bar{G}$, by hypothesis, there exist $\bar{g}_1, \ldots, \bar{g}_t \in \bar{G}$ such that

$$\{[\bar{X}, \langle \bar{X}, \bar{g}\rangle] \mid \bar{g} \in \bar{G}\} = \{[\bar{X}, \langle \bar{X}, \bar{g}_1\rangle], \ldots, [\bar{X}, \langle \bar{X}, \bar{g}_t\rangle]\}.$$

For a fixed $\bar{g} \in \{\bar{g}_1, \ldots, \bar{g}_t\}$, the subgroup $[\bar{X}, \langle \bar{X}, \bar{g}\rangle] = \langle \bar{X}', [\bar{X}, \bar{g}]\rangle^{\langle \bar{X}, \bar{g}\rangle}$ is finite by Dietzmann's lemma (see [7, Part 1, Corollary 2 to Lemma 2.14]). Thus, $[\bar{X}, \bar{G}]$ is finite. Let us consider $\bar{X}^{\bar{G}} = \bar{X}[\bar{X}, \bar{G}]$. Then,

$$|\bar{X}^{\bar{G}} : \bar{X}| = |\bar{X}[\bar{X}, \bar{G}] : \bar{X}| = |[\bar{X}, \bar{G}] : [\bar{X}, \bar{G}] \cap \bar{X}| < \infty$$

and again, X is nearly normal in G. Therefore, each subgroup of G is nearly normal and so G' is finite by Neumann's theorem [6].

Suppose now that G is a nonperiodic FC-group. Then, by Lemma 2.2, G' is a periodic FC-group and G'' is finite and, by applying Corollary 2.6 to G/G'', it follows that G' is finite. □

LEMMA 3.6. *Let G be a group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite and let A be an abelian normal subgroup of G of prime exponent p. If g is any element of finite order of G, the subgroup $[g, A]$ is finite.*

PROOF. Assume for a contradiction that the statement is false, so there exists an element g of G with prime-power order q^n such that the subgroup $[g, A]$ is infinite. Observe that $\langle g\rangle \trianglelefteq G$. If $q = p$, the subgroup $\langle g, A\rangle$ is nilpotent (see [7, Part 2,

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

Lemma 6.34]), so that, by Corollary 3.4, $\langle A, g\rangle'$ is finite and hence, $[g, A]$ is finite. This contradiction shows that $q \neq p$. Thus, A is a completely reducible $\langle g\rangle$-module and so

$$A = \operatorname*{Dr}_{i\in I} A_i,$$

where each A_i is a finite $\langle g\rangle$-invariant subgroup of A on which g acts irreducibly. Let I' be the subset of I consisting of all elements i such that $[\langle g\rangle, A_i] = [g, A_i] \neq \{1\}$. Then, I' is infinite and $[\langle g\rangle, A_i] = [g, A_i] = A_i$ for each $i \in I'$. It follows that

$$[\langle g\rangle, \langle A_j \mid j \in J\rangle] = \operatorname*{Dr}_{j\in J} A_j$$

for every subset J of I'. However, I' contains infinitely many infinite subsets, giving a contradiction to the condition that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. □

LEMMA 3.7. *Let G be a group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite. Let A be an abelian normal subgroup of G such that G/A is an FC-group. If the subgroup $[g, A]$ is finite for each element g of G, then G' is finite.*

PROOF. By Lemma 3.5, it can be assumed that the subgroup A is infinite. Let x be any element of G. If $\langle x\rangle \lhd G$, then x is an FC-element. Suppose that $\langle x\rangle \not\lhd G$. Since G/A is an FC-group, there exist finitely many elements $y_1, \ldots, y_k$ of G such that $\langle x\rangle^G A = E_x A$, where $E_x = \langle x^{y_1}, \ldots, x^{y_k}\rangle$. For each $i = 1, \ldots, k$, the subgroup $[x^{y_i}, A]$ is finite and hence, the centraliser $C_A(x^{y_i})$ has finite index in A. Thus, the index $|A : C_A(E_x)|$ is also finite and so the normal subgroup $C_A(E_x) = C_A(E_x A)$ of G is infinite. By hypothesis, the set

$$\{[\langle x\rangle, H] \mid H \leq G\} = \{[\langle x\rangle, HC_A(E_x)] \mid H \leq G\}$$

is finite. Let g be an element of G'. If $\langle g\rangle \lhd G$, then g is an FC-element. Otherwise, there exist $l_1, \ldots, l_s \in G'$ such that

$$\{[\langle g\rangle, \langle l\rangle] \mid l \in G'\} = \{[\langle g\rangle, \langle l\rangle C_A(E_g)] \mid l \in G'\} = \{[\langle g\rangle, \langle l_1\rangle], \ldots, [\langle g\rangle, \langle l_s\rangle]\}.$$

It follows that $[\langle g\rangle, G']$ is finitely generated and, thus, it is finite. Therefore, G' is an FC-group and, by Lemma 3.5, G'' is finite, so that, replacing G by G/G'', it is possible to suppose that G is metabelian. Let $\bar{x}$ be an element of G. If $\langle \bar{x}\rangle \lhd G$, $\bar{x}$ is an FC-element of G. Otherwise, if $\langle \bar{x}\rangle \not\lhd G$, then there exist $h_1, \ldots, h_r \in G$ such that

$$\{[\langle \bar{x}\rangle, \langle h\rangle] \mid h \in G\} = \{[\langle \bar{x}\rangle, \langle h\rangle C_A(E_{\bar{x}})] \mid h \in G\} = \{[\langle \bar{x}\rangle, \langle h_1\rangle], \ldots, [\langle \bar{x}\rangle, \langle h_r\rangle]\}.$$

For a fixed j in the set $\{1, \ldots, r\}$, each $\langle \bar{x}, h_j\rangle$ is residually finite and so, by Lemma 2.5, $[\bar{x}, G]$ is finite. Thus, G is an FC-group and, from Lemma 3.5, G' is finite. □

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

LEMMA 3.8. *Let G be a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G, H \text{ is infinite}\}$ is finite. Then, either G' is finite or every periodic abelian normal subgroup of G satisfies the minimal condition on subgroups.*

PROOF. Suppose that G contains a periodic abelian normal subgroup A which does not satisfy the minimal condition on subgroups, and assume for a contradiction that G' is infinite. In particular, by Lemma 3.5, G is not an FC-group. Then, it follows from Lemma 3.3 that G does not contain two infinite normal subgroups with trivial intersection. In particular, the set of primes $\pi(A)$ is finite and so there exists a prime number p such that the socle B of the p-component A_p of A is infinite. As G/B is an FC-group, Lemma 3.7 implies that the subgroup $[g,B]$ is infinite for some $g \in G$, and such an element g must have infinite order by Lemma 3.6. Then, $\langle g,B\rangle$ is not an FC-group, contradicting Corollary 2.6. □

We are now in a position to prove our main result about groups in which for each nonnormal subgroup X of G, the set $\{[X,H] \mid H \leq G, H \text{ is infinite}\}$ is finite.

PROOF OF THEOREM 1.2. The condition of the statement is obviously sufficient. To prove the converse statement, first suppose that the group is soluble and assume for a contradiction that it is neither finite-by-abelian nor a finite extension of a group of type p^∞ for some prime number p. Let G be a counterexample with minimal derived length and let A be the smallest nontrivial term of the derived series of G. Then, the factor group G/A is not a counterexample and hence, A must be infinite. Moreover, since A is periodic by Lemma 2.2, it follows from Lemma 3.8 that A satisfies the minimal condition on subgroups. In particular, A is a finite extension of a divisible nontrivial subgroup D, and D is not contained in $Z(G)$ by Corollary 3.2. If g is any element of infinite order of G, the metabelian subgroup $\langle g,D\rangle$ is an FC-group by Corollary 2.6 and so $[D,g] = \{1\}$. It follows that G cannot be generated by its element of infinite order, and so it is periodic by Lemma 2.2. Let F be the Fitting subgroup of G and consider an element x of G such that $[D,x] \neq \{1\}$. Then, the centraliser $C_D(x)$ has infinite index in D and so the index $|F : C_F(x)|$ is also infinite. Let B be a maximal abelian normal subgroup of F containing A. Since F/A is hypercentral, it is easy to see that $C_F(B) = B$. As G is periodic, the normal closure $K = B^{\langle x,F\rangle}$ is generated by finitely many conjugates of B and so it is nilpotent. Let L be the subgroup generated by all elements of prime order of K. Then, the Sylow subgroups of L have finite exponent and so $L \cap D$ is finite. However, it follows from Lemma 3.3 that $\langle x\rangle K$ cannot contain two infinite normal subgroups with finite intersection, so that L is finite and hence, all abelian subgroups of K have finite socle. Therefore, K satisfies the minimal condition on abelian subgroups and so it is a Černikov group by a result of Šunkov [8]. In particular, B is a Černikov group and then F itself is a Černikov group, since periodic groups of automorphisms of Černikov groups are likewise Černikov groups (see [7, Part 1, Theorem 3.29]). As $C_G(F) \leq F$, a second application of the same result shows that G is a Černikov group. Since G is a counterexample, its largest divisible subgroup

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press

J is not of type p^∞ for any prime p and so there exists, in J, a strictly ascending chain of infinite subgroups

$$J_1 < J_2 < \cdots < J_n < J_{n+1} < \cdots .$$

Let g be an element of G. If $\langle g\rangle \lhd G$, then g is an FC-element of G. If $\langle g\rangle$ is nonnormal in G and $C_J(g) = J$, then again g is an FC-element. By Lemma 3.5, G cannot be an FC-group, so there exists an element x of G such that $\langle x\rangle \not\lhd G$ and $C_J(x) < J$. It follows from the hypothesis that there exists a positive integer m such that $[\langle x\rangle, J_m] = [\langle x\rangle, J_n]$ for every positive integer $n \geq m$. In particular, for a fixed $n \geq m$, since J is an abelian normal subgroup of G, it follows that $[x, J_n^{\langle x\rangle}] = [x, J_m^{\langle x\rangle}] = \{[x, a] \mid a \in J_m^{\langle x\rangle}\}$. Then, $J_n \leq \langle J_m, x\rangle C_{\langle x\rangle J}(x)$ for each $n \geq m$ and hence, the subgroup $C_{\langle x\rangle J}(x)$ is infinite. Consider now $C_J(x)$. Then, $|C_{\langle x\rangle J}(x) : C_J(x)|$ is finite and so $C_J(x)$ is infinite. Let H be the subgroup $\langle x, C_G(J)\rangle$. Thus, $C_J(x) \leq Z(H)$, so H' is finite by Corollary 3.2 and hence, $[x, J] = \{1\}$. Then, $J \leq C_G(x)$ and, by Lemma 3.5, G' is finite.

Suppose finally that G is a soluble-by-finite group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite and let S be the largest soluble normal subgroup of G. Obviously, it can be assumed that S is not a finite extension of a group of type p^∞ for any prime p. If x is any element of G, then the subgroup $\langle x, S\rangle$ is a soluble group such that for each nonnormal subgroup X of G, the set $\{[X, H] \mid H \leq G, H \text{ is infinite}\}$ is finite and hence, it follows from the previous part of the proof that $\langle x, S\rangle$ is an FC-group. Then, x has finitely many conjugates in $\langle x, S\rangle$ and so also in G. Therefore, G is an FC-group and G' is finite by Lemma 3.5. □

Acknowledgement

The author is a member of both GNSAGA (INdAM) and AGTA–Advances in Group Theory and Applications (www.advgrouptheory.com).

References

[1] R. Cascella, 'Groups with finiteness conditions on commutator subgroups', *Communications in Algebra*, to appear, doi: 10.1080/00927872.2025.2544294.
[2] M. De Falco, F. de Giovanni and C. Musella, 'Groups with finiteness conditions on commutators', *Algebra Colloq.* **19**(1) (2012), 1197–1204.
[3] F. de Giovanni and D. J. S. Robinson, 'Groups with finitely many derived subgroups', *J. Lond. Math. Soc. (2)* **71** (2005), 658–668.
[4] F. De Mari and F. de Giovanni, 'Groups with finitely many derived subgroups of non-normal subgroups', *Arch. Math.* **86** (2006), 310–316.
[5] B. H. Neumann, 'Groups covered by permutable subsets', *J. Lond. Math. Soc. (2)* **29** (1954), 236–248.
[6] B. H. Neumann, 'Groups with finite classes of conjugate subgroups', *Math. Z.* **63** (1955), 76–96.
[7] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups* (Springer, Berlin, 1972).
[8] V. P. Šunkov, 'On the minimality problem for locally finite groups', *Algebra Logic* **9** (1970), 137–151.

ROSA CASCELLA, Dipartimento di Matematica e Applicazioni,
Università di Napoli Federico II, Napoli, Italy
e-mail: rosa.cascella@unina.it

https://doi.org/10.1017/S0004972725100506 Published online by Cambridge University Press