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UNIFORM ORDER STATISTICS PROPERTY AND ℓ_∞-SPHERICAL DENSITIES

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In this article, we observe that processes with the uniform order statistics property (UOSP) can be characterized by the condition that their first n epoch times have a joint $\ell_\infty^{\leq}$-spherical density, $n \geq 1$. Some related results, and some further properties of $\ell_\infty^{\leq}$-spherical densities, are also given. We also extend some of the results regarding the UOSP to the more general (not necessarily uniform) order statistics property. Finally, we develop a theory of discrete-time discrete-state processes with the UOSP, where the need to consider multiple jumps, at a single time point, arises.

1. INTRODUCTION AND PRELIMINARIES

An n-dimensional density function f of the form

$$f(x_1, x_2, \ldots, x_n) = \phi\Big(\max_{1 \leq i \leq n} x_i\Big), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n \tag{1.1}$$

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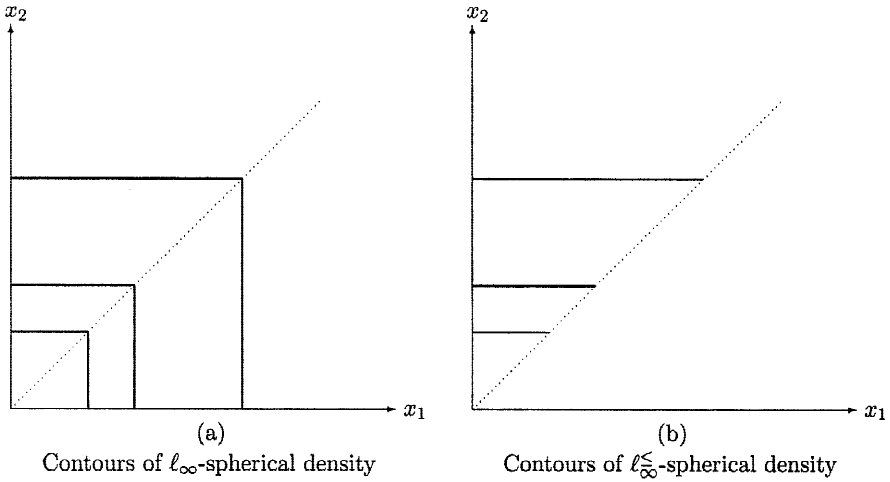


FIGURE 1. Equiprobability contours.

(where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is some function) is called ℓ_∞-spherical. Note that if $X_1, X_2, \ldots, X_n$ have an ℓ_∞-spherical density, then $X_1, X_2, \ldots, X_n$ must be exchangeable. Some typical equiprobability contours of a bivariate ℓ_∞-spherical density are depicted in Figure 1a. The density function g of the corresponding order statistics is given by

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \psi(x_n) & \text{if } 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where

$$\psi \equiv n! \phi.$$

In this article, any density of the form of g in (1.2), for some ψ, will be called $\ell_\infty^{\leq}$-spherical. Note that if $X_1, X_2, \ldots, X_n$ have an $\ell_\infty^{\leq}$-spherical density, then $X_1 \leq X_2 \leq \cdots \leq X_n$ with probability 1. Some typical equiprobability contours of a bivariate $\ell_\infty^{\leq}$-spherical density are depicted in Figure 1b.

Remark 1.1: From Figure 1b, it is obvious that if X_1 and X_2 have a bivariate $\ell_\infty^{\leq}$-spherical density, then the conditional distribution of X_1, given that $X_2 = t$ (for any $t > 0$), is uniform$(0, t)$. Similarly, also the conditional distribution of X_1, given that $X_1 \leq t$ and $X_2 > t$, is uniform$(0, t)$. Later in this article, it will be shown that these results hold also in the more general multivariate case.

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With any set of nonnegative random variables that satisfy $X_1 \le X_2 \le \cdots \le X_n$ with probability 1, we may associate the nonnegative increments

$$Z_1 = X_1, \qquad Z_2 = X_2 - X_1, \ \ldots, \qquad Z_n = X_n - X_{n-1}.$$

The densities $f_{\boldsymbol{X}}$ and $f_{\boldsymbol{Z}}$, of $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$, respectively, are related by the equation

$$f_{\boldsymbol{X}}(x_1, x_2, \ldots, x_n) = f_{\boldsymbol{Z}}(x_1, x_2 - x_1, \ldots, x_{n-1} - x_n), \qquad 0 \le x_1 \le x_2 \le \cdots \le x_n.$$

It follows that $\boldsymbol{X}$ has an $\ell_\infty^{\le}$-spherical density if and only if $f_{\boldsymbol{Z}}$ is of the form

$$f_{\boldsymbol{Z}}(z_1, z_2, \ldots, z_n) = \varphi\left(\sum_{i=1}^{n} z_i\right), \qquad (z_1, z_2, \ldots, z_n) \in \mathbb{R}_+^n, \tag{1.3}$$

for some function φ; densities of this form are called Schur-constant (see Spizzichino [17]). This equivalence is useful for the purpose of identifying random vectors with $\ell_\infty^{\le}$-spherical densities.

A counting process $N = \{N(t), t \in [0, \tilde{t})\}$ is said to be a *nonhomogeneous pure birth process* with intensity (or rate) functions $r_i \ge 0$ if

$$\{N(t), t \in [0, \tilde{t})\} \text{ has the Markov property,} \tag{1.4}$$

$$P\{N(t + \Delta t) = i + 1 | N(t) = i\} = r_i(t)\Delta t + o(\Delta t), \qquad i \ge 0, \tag{1.5}$$

$$P\{N(t + \Delta t) > i + 1 | N(t) = i\} = o(\Delta t), \qquad i \ge 0. \tag{1.6}$$

Here and in the following, $\tilde{t}$ may be ∞. Nonhomogeneous pure birth processes have been extensively used recently as a modeling tool in reliability theory and statistics; see, for instance, some references in Shaked, Spizzichino, and Suter [16, pp. 275–276]. Also, sequential order statistics, as described in Kamps [13, Sect. 1.3], can be easily seen to have the same distribution as the first epoch times of a nonhomogeneous pure birth process.

Shaked et al. [16] showed that if a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ has an $\ell_\infty^{\le}$-spherical density function of the form (1.2), then there exists a nonhomogeneous pure birth process whose first n epoch times $T_1 \le T_2 \le \cdots \le T_n$ satisfy

$$(T_1, T_2, \ldots, T_n) =_{\text{st}} (X_1, X_2, \ldots, X_n).$$

Recall that a process $\{M(t), t \in [0, \tilde{t})\}$ is called a stationary *mixed Poisson process* if there exists a unit rate homogeneous Poisson process $\{K(t), t \in [0, \tilde{t})\}$ and a nonnegative random variable W, independent of K, such that

$$\{M(t), t \in [0, \tilde{t})\} =_{\text{st}} \{K(Wt), t \in [0, \tilde{t})\}.$$

Grandell [7] provides a detailed coverage of the theory and applications of mixed Poisson processes.

We will also use in the sequel the notion of a *mixed uniform sample process*. In order to define it, let $\tilde{t} < \infty$ and let $U_1, U_2, \ldots$ be a sequence of independent

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uniform$(0, \tilde{t})$ random variables. Let L be a positive integer-valued random variable, independent of the U_i's. Given $L = l$, let $\{M(t), t \in [0, \tilde{t})\}$ be the empirical distribution function based on $U_1, U_2, \ldots, U_l$. The unconditional process $\{M(t), t \in [0, \tilde{t})\}$ will be called a mixed uniform sample process. Here, and in the rest of the article, "empirical distribution function" means the *nonnormalized* version of it; that is, if we call $\{M(t), t \in [0, \tilde{t})\}$ an empirical distribution function, we mean that for every $t \in [0, \tilde{t})$, the value $M(t)$ is the number of observations in the underlying sample that are less than or equal to t.

We now define the *uniform order statistics property* (UOSP). We will say that a (finite or infinite) sequence of nonnegative random variables $X_1 \leq X_2 \leq \cdots$ has the UOSP if for any $k \geq 1$ and for any $t > 0$ such that $P\{X_k \leq t, X_{k+1} > t\} > 0$, the conditional distribution of $X_1, X_2, \ldots, X_k$, given that $X_k \leq t$ and $X_{k+1} > t$, is the joint distribution of k order statistics from a uniform$(0, t)$ distribution. Furthermore, consider a point process $\{M(t), t \in [0, \tilde{t})\}$ with right-continuous sample paths, such that almost surely $M(0) = 0$, and $M(t) < \infty$ for all $t \in [0, \tilde{t})$. Denote by $T_1 \leq T_2 \leq \cdots$ the jump times of the process. The process M is said to have the UOSP if for every $t \in [0, \tilde{t})$ and $k \geq 1$ such that $P\{M(t) = k\} > 0$, the conditional distribution of $T_1, T_2, \ldots, T_k$, given that $M(t) = k$, is the joint distribution of k order statistics from a uniform$(0, t)$ distribution.

From Remark 1.1 it is seen that if X_1 and X_2 have a bivariate $\ell_\infty^{\leq}$-spherical density, then they have the UOSP. Later, in Section 2, it will be shown that this is the case also in the more general multivariate case.

The interest of processes with the UOSP was pointed out in Crump [3], where it was shown that they have the Markov property. From Puri [15], it follows that if a process $\{M(t), t \in [0, \infty)\}$ (note that here $\tilde{t} = \infty$) has the UOSP, then it has to be a stationary mixed Poisson process. On the other hand, for a finite $\tilde{t}$, if a process $\{M(t), t \in [0, \tilde{t})\}$ has the UOSP, then it has to be a mixed uniform sample process. It is worthwhile to note that a mixed sample process on $[0, \tilde{t})$ (here $\tilde{t}$ is finite) might also be a stationary mixed Poisson process. However, in Deffner and Haeusler [4], it is shown that not all mixed sample processes can be represented as stationary mixed Poisson processes.

In the present article, we observe that processes with the UOSP can be characterized by the condition that their first n epoch times have a joint $\ell_\infty^{\leq}$-spherical density, $n \geq 1$. Some related results are also given. We also extend some of the results regarding the UOSP to the more general (not necessarily uniform) order statistics property; in particular, we characterize such processes in terms of the joint density of the first n epoch times (for any n). Finally, we develop a theory of discrete-time discrete-state processes with the UOSP, where the need to consider multiple jumps, at a single time point, arises. Because of the need to consider both unit jumps and multiple jumps (at each discrete time point), we are led to study two different notions of discrete UOSP.

Other results that are related to those in the present article can be found in Benczur [1], in a recent paper by Iglesias, Matúš, Pereira, and Tanaka [10], and in references cited therein.

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2. CHARACTERIZATIONS OF PROCESSES WITH THE UOSP BY MEANS OF ℓ_∞-SPHERICAL DENSITIES

We start this section with an interesting property of random vectors with ℓ_∞-spherical densities. It will be seen later that this property is closely related to the UOSP of the underlying random variables.

PROPOSITION 2.1: *If $X_1, X_2, \ldots, X_n$ have an ℓ_∞-spherical density f of the form (1.1), then any k-dimensional marginal of f is also an ℓ_∞-spherical density, $k \le n$.*

PROOF: By the exchangeability of f and backward induction, it suffices to prove that the marginal density f_{n-1} of $(X_1, X_2, \ldots, X_{n-1})$ is ℓ_∞-spherical. Now, we write

$$\begin{aligned} f_{n-1}(x_1, x_2, \ldots, x_{n-1}) &= \int_0^\infty \phi\Big(\max_{1\le i\le n} x_i\Big)\, dx_n \\ &= \int_0^{\max_{1\le i\le n-1} x_i} \phi\Big(\max_{1\le i\le n} x_i\Big)\, dx_n + \int_{\max_{1\le i\le n-1} x_i}^\infty \phi\Big(\max_{1\le i\le n} x_i\Big)\, dx_n \\ &= \Big(\max_{1\le i\le n-1} x_i\Big)\phi\Big(\max_{1\le i\le n-1} x_i\Big) + \int_{\max_{1\le i\le n-1} x_i}^\infty \phi(x_n)\, dx_n; \end{aligned}$$

that is, for $(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$, we have that f_{n-1} is a function of $\max_{1\le i\le n-1} x_i$ only. ■

In a similar manner, the following result can be obtained.

PROPOSITION 2.2: *If $X_1 \le X_2 \le \cdots \le X_n$ have an $\ell_\infty^{\le}$-spherical density as in (1.2), then so do $X_1 \le X_2 \le \cdots \le X_k$ for any $k \le n$.*

Proposition 2.2 leads to the following natural question of "extendibility": If $X_1 \le X_2 \le \cdots \le X_n$ have an $\ell_\infty^{\le}$-spherical density as in (1.2), can they be the first n variables of a collection $X_1 \le X_2 \le \cdots \le X_n \le X_{n+1}$ that have an $\ell_\infty^{\le}$-spherical density in $\mathbb{R}_+^{n+1}$? In order to answer this question, denote the joint density of $X_1 \le X_2 \le \cdots \le X_n$ as

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \psi_n(x_n), \qquad 0 \le x_1 \le x_2 \le \cdots \le x_n,$$

and supposing that $X_1 \le X_2 \le \cdots \le X_n \le X_{n+1}$ also have an $\ell_\infty^{\le}$-spherical density, denote it by

$$f_{X_1, X_2, \ldots, X_n, X_{n+1}}(x_1, x_2, \ldots, x_n, x_{n+1}) = \psi_{n+1}(x_{n+1}), \qquad 0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1}.$$

Obviously,

$$\psi_n(x_n) = \int_{x_n}^\infty \psi_{n+1}(x_{n+1})\, dx_{n+1}, \tag{2.1}$$

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from which

$$\psi_{n+1}(x) = -\frac{d}{dx}\,\psi_n(x).$$

It follows that $X_1 \le X_2 \le \cdots \le X_n$ is $\ell_\infty^{\le}$-spherically extendable if and only if ψ_n is differentiable and nonincreasing.

Regarding the function ψ_n, it is worthwhile to note that it is related to the density function f_{X_n} of X_n; it is easy to verify that

$$f_{X_n}(x) = \frac{x^{n-1}}{(n-1)!}\,\psi_n(x), \qquad x \ge 0.$$

Next, we show that any random vector with an $\ell_\infty^{\le}$-spherical density has the UOSP, and vice versa.

PROPOSITION 2.3: *The random variables $X_1 \le X_2 \le \cdots \le X_n$ have an $\ell_\infty^{\le}$-spherical density as in (1.2) if and only if they have the UOSP.*

PROOF: First, suppose that $X_1 \le X_2 \le \cdots \le X_n$ have an $\ell_\infty^{\le}$-spherical density as in (1.2). Fix a $t > 0$. By Proposition 2.2, it is seen that it suffices to show that the conditional distribution of $X_1, X_2, \ldots, X_{n-1}$, given that $X_{n-1} \le t$ and $X_n > t$, is the joint distribution of $n-1$ order statistics from a uniform$(0, t)$ distribution.

Denote by $f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}$ the conditional density of $X_1, X_2, \ldots, X_{n-1}$, given that $X_{n-1} \le t$ and $X_n > t$. Let $0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le t$. Then,

$$f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}(x_1, x_2, \ldots, x_{n-1}) = \frac{\displaystyle\int_t^\infty \psi(x_n)\, dx_n}{P\{X_{n-1} \le t, X_n > t\}};$$

that is, for each t, it is seen that $f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}(x_1, x_2, \ldots, x_{n-1})$ is constant over the region $\{(x_1, x_2, \ldots, x_{n-1}) | 0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le t\}$. In other words, the conditional density $f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}$ is the density of $n-1$ order statistics from a uniform$(0, t)$ distribution.

Conversely, suppose that $X_1 \le X_2 \le \cdots \le X_n$ have the UOSP. Fix a $t > 0$. Let $0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le t$. Then, on one hand,

$$f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}(x_1, x_2, \ldots, x_{n-1}) = \frac{\displaystyle\int_t^\infty f(x_1, x_2, \ldots, x_{n-1}, x_n)\, dx_n}{P\{X_{n-1} \le t, X_n > t\}}$$

and, on the other hand,

$$f_{X_1, X_2, \ldots, X_{n-1} | X_{n-1} \le t, X_n > t}(x_1, x_2, \ldots, x_{n-1}) = \frac{(n-1)!}{t^{n-1}};$$

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that is,

$$\int_t^\infty f(x_1, x_2, \ldots, x_{n-1}, x_n)\, dx_n = \frac{(n-1)!}{t^{n-1}} P\{X_{n-1} \le t, X_n > t\}. \tag{2.2}$$

Differentiating (2.2) with respect to t, it is seen that when $0 \le x_1 \le x_2 \le \cdots \le x_{n-1} \le t$, then $f(x_1, x_2, \ldots, x_{n-1}, t)$ does not depend on the actual values of $x_1, x_2, \ldots, x_{n-1}$. ■

A straightforward modification of the proof of Proposition 2.3 yields the following result.

PROPOSITION 2.4: *The random variables $X_1 \le X_2 \le \cdots \le X_n$ have an $\ell_\infty^{\le}$-spherical density as in (1.2) if and only if the conditional distribution of $X_1, X_2, \ldots, X_k$, given that $X_{k+1} = t$, is the joint distribution of k order statistics from a uniform$(0, t)$ distribution, for all $k \le n-1$ and $t > 0$ for which the above conditional distribution is defined.*

From Proposition 2.3 we get the following characterization of processes that have the UOSP property.

THEOREM 2.5: *A point process $N = \{N(t),\, t \in [0, \tilde{t})\}$ has the UOSP if and only if its first n epoch times $T_1 \le T_2 \le \cdots \le T_n$ have a joint $\ell_\infty^{\le}$-spherical density for every $n \ge 1$ such that $P\{T_n < \infty\} = 1$.*

We thus see that for a point process $N = \{N(t),\, t \in [0, \infty)\}$, the following properties are equivalent:

(i) N is a stationary mixed Poisson process.
(ii) N has the UOSP.
(iii) The first n epoch times of N have a joint $\ell_\infty^{\le}$-spherical density, $n \ge 1$.

The equivalence of properties (i) and (ii) has already been known in the literature for a long time (see, for instance, references in Puri [15]). It is worthwhile to mention that in a recent proof of the equivalence of properties (i) and (ii) in Hayakawa [8], the equivalence of the first two conditions with property (iii) is essentially implicit. Namely, in the proof of her Theorem 3.2, Hayakawa [8] obtained the fact that the joint density function of the first n interepoch intervals of N, evaluated at the point $(y_1, y_2, \ldots, y_n) \ge \mathbf{0}$, depends only on $\sum_{i=1}^n y_i$; in other words, the density is Schur-constant; see (1.3). As we have mentioned earlier, a sequence of interepoch intervals satisfies the above property if and only if the corresponding first n epoch times have a joint $\ell_\infty^{\le}$-spherical density, $n \ge 1$.

A sequence of positive random variables $\{Y_1, Y_2, \ldots,\}$ is called ℓ_1-isotropic (see, for example, Berman [2] or Hayakawa [8]) if and only if for every $n \ge 1$, the random variables

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$$\frac{Y_1}{\sum_{i=1}^{n} Y_i}, \frac{Y_2}{\sum_{i=1}^{n} Y_i}, \ldots, \frac{Y_n}{\sum_{i=1}^{n} Y_i}$$

are distributed as a set $U_1, U_2, \ldots, U_n$ of independent uniform$(0,1)$ random variables, conditioned on $\sum_{i=1}^{n} U_i = 1$. From a result of Berman [2] (given as Theorem 2.1 in Hayakawa [8]), it follows that a sequence of positive random variables $\{Y_1, Y_2, \ldots\}$ is ℓ_1-isotropic if and only if the joint density function of any finite number of the Y_i's is Schur-constant. We can therefore restate Theorem 2.5 as follows.

THEOREM 2.6: *A point process $N = \{N(t), t \in [0, \tilde{t})\}$ has the UOSP if and only if for every $n \geq 1$ such that $P\{T_n < \infty\} = 1$, the first n epochs times $T_1 \leq T_2 \leq \cdots \leq T_n$ of N satisfy that*

$$\frac{T_1}{T_n}, \frac{T_2}{T_n}, \ldots, \frac{T_{n-1}}{T_n}$$

are distributed as the order statistics of $n-1$ independent uniform$(0,1)$ random variables.

3. SOME COMMENTS ON THE GENERAL OSP

In this section we extend some of the UOSP results in Section 2 to the more general order statistics property (OSP) that will be defined shortly. These extensions entail generalizations of the notions of ℓ_∞-spherical and $\ell_\infty^{\leq}$-spherical densities.

Let $a : [0,\infty) \to [0,\infty)$ be a differentiable, strictly increasing function such that $a(0) = 0$. This function will be fixed throughout this section. Denote by a^{-1} the inverse of a, and by a', its derivative. If $X_1, X_2, \ldots, X_n$ have the joint density given in (1.1), then

$$(\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n) = (a^{-1}(X_1), a^{-1}(X_2), \ldots, a^{-1}(X_n)) \tag{3.1}$$

has the density given by

$$\tilde{f}(x_1, x_2, \ldots, x_n) = \phi\Big(a\Big(\max_{1\leq i\leq n} x_i\Big)\Big)\prod_{i=1}^{n} a'(x_i), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n, \tag{3.2}$$

which is a generalization of (1.1). The density function $\tilde{g}$ of the corresponding order statistics is given by

$$\tilde{g}(x_1, x_2, \ldots, x_n) = \begin{cases} \psi(a(x_n)) \prod_{i=1}^{n} a'(x_i) & \text{if } 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

which is a generalization of (1.2).

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We now define the *order statistics property* (OSP), which is a generalization of the UOSP. Let $\{F_t\}_{t\geq 0}$ be a family of distribution functions such that F_t has the support $[0,t]$. We will say that a (finite or infinite) sequence of nonnegative random variables $\widetilde{X}_1 \leq \widetilde{X}_2 \leq \cdots$ has the OSP with respect to $\{F_t\}_{t\geq 0}$ if for any $k \geq 1$ and for any $t > 0$ such that $P\{\widetilde{X}_k \leq t, \widetilde{X}_{k+1} > t\} > 0$, the conditional distribution of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_k$, given that $\widetilde{X}_k \leq t$ and $\widetilde{X}_{k+1} > t$, is the joint distribution of k order statistics based on a sample from F_t. Furthermore, consider a point process $\{\widetilde{M}(t),\ t \in [0,\tilde{t})\}$ with right-continuous sample paths, such that almost surely $\widetilde{M}(0) = 0$ and $\widetilde{M}(t) < \infty$ for all $t \in [0,\tilde{t})$. Denote by $\widetilde{T}_1 \leq \widetilde{T}_2 \leq \cdots$ the jump times of the process. The process $\widetilde{M}$ is said to have the OSP with respect to $\{F_t\}_{t\geq 0}$ if for every $t \in [0,\tilde{t})$ and $k \geq 1$, such that $P\{\widetilde{M}(t) = k\} > 0$, the conditional distribution of $\widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_k$, given that $\widetilde{M}(t) = k$, is the joint distribution of k order statistics based on a sample from F_t.

Let $\widetilde{M} = \{\widetilde{M}(t),\ t \in [0,\infty)\}$ be a point process with right-continuous sample paths, such that almost surely $\widetilde{M}(0) = 0$ and $\widetilde{M}(t) < \infty$ for all $t \in [0,\infty)$. Denote $m(t) = E[\widetilde{M}(t)]$, $t \geq 0$, and $\gamma = \lim_{t\to\infty} m(t)$. From Puri [15] it follows that if the process $\widetilde{M}$ has the OSP with respect to some $\{F_t\}_{t\geq 0}$ and $\gamma = \infty$, then $\widetilde{M}$ has to be a nonstationary mixed Poisson process; that is,

$$\{\widetilde{M}(t),\ t \in [0,\infty)\} =_{\text{st}} \{K(Wm(t)),\ t \in [0,\infty)\},$$

where K is a unit rate homogeneous Poisson process and W is a positive random variable, independent of K. On the other hand, if $\gamma < \infty$, then $\widetilde{M}$ has to be a mixed sample process with the kernel function

$$F(x) = \frac{m(t)}{\gamma}, \qquad t \geq 0;$$

that is, $\widetilde{M}$ can be represented as the empirical distribution of a sample of a random size from the above distribution F.

Using the transformation (3.1) or its inverse, it is possible to obtain results about the OSP from results about the UOSP and vice versa. For example, if $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ have a joint density function $\tilde{f}$ of the form (3.2), then any k-dimensional marginal of $\tilde{f}$ is also of that form, $k \leq n$ (this statement is a generalization of Proposition 2.1). A generalization of Proposition 2.3 is the following.

PROPOSITION 3.1: *The random variables $\widetilde{X}_1 \leq \widetilde{X}_2 \leq \cdots \leq \widetilde{X}_n$ have a density of the form (3.3) if and only if they have the OSP with respect to $\{F_t\}_{t\geq 0}$, where*

$$F_t(s) = \frac{a(s)}{a(t)}, \qquad 0 \leq s \leq t. \tag{3.4}$$

From Proposition 3.1 we get the following generalization of Theorem 2.5.

THEOREM 3.2: *Let $a:[0,\infty) \to [0,\infty)$ be a differentiable, strictly increasing function such that $a(0) = 0$, and for $t \geq 0$, let F_t be defined by (3.4). A point process $\{\widetilde{M}(t),\ t \in [0,\tilde{t})\}$ has the OSP with respect to $\{F_t\}_{t\geq 0}$ if and only if for every $n \geq 1$*

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such that $P\{\tilde{T}_n < \infty\} = 1$, *its first n epoch times* $\tilde{T}_1 \le \tilde{T}_2 \le \cdots \le \tilde{T}_n$ *have a joint density of the form (3.3).*

Example 3.3: The linear birth (Yule) process $N = \{N(t), t \ge 0\}$, with initial population size k, is a nonhomogeneous pure birth process (see (1.4)–(1.6)) with intensity functions

$$r_i(t) = \lambda(k+i), \qquad i = 0,1,\ldots,$$

where $\lambda > 0$ is some constant. Let $T_1 \le T_2 \le \cdots \le T_n$ be the first n jump times of N. The joint density function of $T_1, T_2, \ldots, T_n$, evaluated at $x_1 \le x_2 \le \cdots \le x_n$, can be computed as follows (see, e.g., (2.5) in Shaked et al. [16]; there is there an easily correctable mistake in the indexing):

$$f_{T_1,T_2,\ldots,T_n}(x_1,x_2,\ldots,x_n) = \left[\prod_{i=1}^{n} \lambda(k+i-1)\right] \exp\left\{-\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \lambda(k+i-1)\,du\right\},$$

where $x_0 \equiv 0$. A straightforward algebra yields (for $x_1 \le x_2 \le \cdots \le x_n$)

$$f_{T_1,T_2,\ldots,T_n}(x_1,x_2,\ldots,x_n) = \left[\prod_{i=1}^{n} (k+i-1)\right] \exp\{-\lambda(k+n)x_n\} \left[\prod_{i=1}^{n} \lambda \exp\{\lambda x_i\}\right];$$

that is, $f_{T_1,T_2,\ldots,T_n}$ is of the form (3.3) with $a(x) = e^{\lambda x} - 1$ and $\psi(a) = [\Pi_{i=1}^{n}(k+i-1)]/(1+a)^{k+n}$. Now, from Proposition 3.1 it follows that N has the OSP with respect to $\{F_t\}_{t\ge 0}$, where

$$F_t(x) = \frac{e^{\lambda x} - 1}{e^{\lambda t} - 1}, \qquad 0 \le x \le t.$$

Applying the inverse of the transformation (3.1), it follows that $e^{\lambda T_1} - 1, e^{\lambda T_2} - 1, \ldots$ have the UOSP. The latter result was obtained in Neuts and Resnick [14] by means of a straightforward, but lengthy, computation.

An interesting application in Bayesian inference in which densities of the form (3.2) arise naturally is described next. Let $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_n)$ be a vector of independent and identically distributed positive parameters, and let $\boldsymbol{V}$ (which may be a random variable or a random vector) be an observation whose distribution depends on $\boldsymbol{\Theta}$ only through $\max_{1\le i\le n} \Theta_i$; that is, suppose that the joint density of $\boldsymbol{\Theta}$ is of the form

$$f_{\boldsymbol{\Theta}}(\theta_1,\theta_2,\ldots,\theta_n) = \prod_{i=1}^{n} h(\theta_i), \qquad (\theta_1,\theta_2,\ldots,\theta_n) \in \mathbb{R}_+^n, \tag{3.5}$$

where h is some density function on $(0,\infty)$, and that the conditional density of $\boldsymbol{V}$ given $\boldsymbol{\Theta} = (\theta_1,\theta_2,\ldots,\theta_n)$ is of the form

$$f_{\boldsymbol{V}|\boldsymbol{\Theta}}(\boldsymbol{v}|\theta_1,\theta_2,\ldots,\theta_n) = \varphi\left(\boldsymbol{v}; \max_{1\le i\le n} \theta_i\right)$$

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for some function φ. Then, by Bayes' formula, the conditional density of $\boldsymbol{\Theta}$, given $\boldsymbol{V} = \boldsymbol{v}$, is given by

$$f_{\boldsymbol{\Theta}|\boldsymbol{V}}(\theta_1, \theta_2, \ldots, \theta_n | \boldsymbol{v}) \propto \varphi\Big(\boldsymbol{v}; \max_{1 \le i \le n} \theta_i\Big) \prod_{i=1}^{n} h(\theta_i); \tag{3.6}$$

that is, given $\boldsymbol{V} = \boldsymbol{v}$, this density is of form (3.2). A particular application in reliability theory is described in the following example.

Example 3.4: Consider r components $D_1, D_2, \ldots, D_r$ whose lifetimes are denoted by $V_1, V_2, \ldots, V_r$, respectively. Suppose that a parallel system of n identical components, with independent lifetimes $\Theta_1, \Theta_2, \ldots, \Theta_n$, supports the performance of $D_1, D_2, \ldots, D_r$; that is, suppose that when the supporting parallel system is up, then the failure rate of each D_j is λ_0, whereas when the supporting parallel system is down, the failure rate of each D_j is λ_1, where $\lambda_0 < \lambda_1$. Assume that the joint density of $\Theta_1, \Theta_2, \ldots, \Theta_n$ is given by (3.5). Then, the conditional density of $(\Theta_1, \Theta_2, \ldots, \Theta_n)$, given $(V_1, V_2, \ldots, V_r)$, is of the form (3.6). It follows from Proposition 3.1 that the conditional distribution of the ordered Θ_i's, given $(V_1, V_2, \ldots, V_r)$, has the OSP with respect to $\{F_t\}_{t \ge 0}$, where

$$F_t(s) = \frac{H(s)}{H(t)}, \qquad 0 \le s \le t$$

(H is the distribution function associated with h).

4. DISCRETE UOSP AND DISCRETE ℓ_∞-SPHERICAL DENSITIES

Inspired by the UOSP for processes with continuous interepoch intervals, Huang and Shoung [9] have introduced a corresponding property for discrete-time point processes (i.e., processes with discrete interepoch intervals and unit step jumps). In this section, we will describe this property and a modification of it that is useful in some applications. We will then derive some discrete analogs of the results in Section 2. It turns out that in the discrete case, it is also of interest to consider processes that can have jumps larger than unity at the jump epochs; thus, we will also obtain further results that have no continuous analogs.

First, we introduce some notation and terminology that are used in this section and point out some simple observations. In this section, we use the notation $\mathbb{N}$ for $\{1, 2, \ldots\}$.

Let $Z_1, Z_2, \ldots, Z_n$ be $\mathbb{N}$-valued random variables and set

$$\begin{aligned} X_1 &= Z_1, \\ X_2 &= Z_1 + Z_2, \\ &\vdots \\ X_n &= Z_1 + Z_2 + \cdots + Z_n. \end{aligned} \tag{4.1}$$

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Thus, $X_1, X_2, \ldots, X_n$ are also $\mathbb{N}$-valued and they satisfy

$$P\{X_1 < X_2 < \cdots < X_n\} = 1.$$

If we denote by $p_{\mathbf{Z}}$ and $p_{\mathbf{X}}$ the discrete densities of $(Z_1, Z_2, \ldots, Z_n)$ and $(X_1, X_2, \ldots, X_n)$, respectively, then, obviously,

$$p_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = p_{\mathbf{Z}}(x_1, x_2 - x_1, \ldots, x_n - x_{n-1}), \qquad 0 < x_1 < x_2 < \cdots < x_n,$$

and

$$p_{\mathbf{Z}}(z_1, z_2, \ldots, z_n) = p_{\mathbf{X}}(z_1, z_1 + z_2, \ldots, z_1 + z_2 + \cdots + z_n), \qquad z_i > 0,\ i = 1,2,\ldots,n.$$

If, on the other hand, $Z_1, Z_2, \ldots, Z_n$ are $\mathbb{N} \cup \{0\}$-valued random variables and $X_1, X_2, \ldots, X_n$ are as defined in (4.1), then $X_1, X_2, \ldots, X_n$ are also $\mathbb{N} \cup \{0\}$-valued and they satisfy

$$P\{X_1 \le X_2 \le \cdots \le X_n\} = 1.$$

In this case, the discrete density functions of $(Z_1, Z_2, \ldots, Z_n)$ and $(X_1, X_2, \ldots, X_n)$ satisfy

$$p_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = p_{\mathbf{Z}}(x_1, x_2 - x_1, \ldots, x_n - x_{n-1}), \qquad 0 \le x_1 \le x_2 \le \cdots \le x_n,$$

and

$$p_{\mathbf{Z}}(z_1, z_2, \ldots, z_n) = p_{\mathbf{X}}(z_1, z_1 + z_2, \ldots, z_1 + z_2 + \cdots + z_n), \qquad z_i \ge 0,\ i = 1,2,\ldots,n.$$

We will say that a discrete joint density p on $(\mathbb{N} \cup \{0\})^n$ [respectively $\mathbb{N}^n$] is *Schur-constant* if

$$p(z_1, z_2, \ldots, z_n) = \varphi\left(\sum_{i=1}^{n} z_i\right), \qquad z_i \ge 0 \,[\text{respectively } z_i > 0],\ i = 1,2,\ldots,n, \tag{4.2}$$

for some function φ, and that it is ℓ_∞-*spherical* if

$$p(z_1, z_2, \ldots, z_n) = \phi\left(\max_{i=1}^{n} z_i\right), \qquad z_i \ge 0 \,[\text{respectively } z_i > 0],\ i = 1,2,\ldots,n,$$

for some function ϕ. A discrete joint density p on $(\mathbb{N} \cup \{0\})^n$ will be called $\ell_\infty^{\le}$-spherical if

$$p(z_1, z_2, \ldots, z_n) = \begin{cases} \psi(z_n) & \text{if } 0 \le z_1 \le z_2 \le \cdots \le z_n \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

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for some function ψ. A discrete joint density p on $\mathbb{N}^n$ will be called $\ell_\infty^<$-spherical if

$$p(z_1, z_2, \ldots, z_n) = \begin{cases} \psi(z_n) & \text{if } 0 < z_1 < z_2 < \cdots < z_n \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

for some function ψ. Obviously, the above $p_{\boldsymbol{Z}}$ is Schur-constant on $(\mathbb{N} \cup \{0\})^n$ [respectively $\mathbb{N}^n$] if and only if $p_{\boldsymbol{X}}$ is $\ell_\infty^\leq$-spherical [respectively $\ell_\infty^<$-spherical].

The following result is a discrete analog of Proposition 2.1. Its proof is a straightforward modification of the proof of Proposition 2.1 and is therefore omitted.

PROPOSITION 4.1: *If the discrete random variables $X_1, X_2, \ldots, X_n$ have an ℓ_∞-spherical density f on $(\mathbb{N} \cup \{0\})^n$ [respectively $\mathbb{N}^n$], then any k-dimensional marginal of f is an ℓ_∞-spherical density on $(\mathbb{N} \cup \{0\})^k$ [respectively $\mathbb{N}^k$], $k \leq n$.*

We will now define the UOSP for a (finite or infinite) sequence of discrete nonnegative random variables $X_1, X_2, \ldots$. Two different definitions are needed depending on whether the X_i's can or cannot be equal to each other. If $P\{0 < X_1 < X_2 < \cdots\} = 1$, then we say that they have the UOSP $(<)$ if for discrete $0 < x_1 < x_2 < \cdots < x_k \leq t$ we have

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k | X_k \leq t, X_{k+1} > t\} = \binom{t}{k}^{-1};$$

that is, conditional on $X_k \leq t$ and $X_{k+1} > t$, the random variables $X_1 < X_2 < \cdots < X_k$ are distributed as the order statistics of a sample of size k drawn from the set $\{1, 2, \ldots, t\}$ *without replacement*. If, on the other hand, $P\{0 \leq X_1 \leq X_2 \leq \cdots\} = 1$, then we say that these X_i's have the UOSP $(\leq)$ if for discrete $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq t$ we have

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k | X_k \leq t, X_{k+1} > t\} = \binom{t+k}{k}^{-1};$$

that is, conditional on $X_k \leq t$ and $X_{k+1} > t$, the random variables $X_1 \leq X_2 \leq \cdots \leq X_k$ are distributed as the order statistics of a sample of size k drawn from the set $\{0, 1, \ldots, t\}$ *with replacement*. (The latter claim follows from the fact that $\binom{t+k}{k}$ is the number of ways that k objects can be put into $t+1$ cells; see, e.g., Feller [6, p. 38].)

The following result is a discrete analog of Proposition 2.3. Again we omit the proof, as it is a straightforward modification of the proof of Proposition 2.3.

PROPOSITION 4.2: *The discrete random variables $0 < X_1 < X_2 < \cdots < X_n$ [respectively $0 \leq X_1 \leq X_2 \leq \cdots \leq X_n$] have an $\ell_\infty^<$-spherical density as in (4.4) [respectively $\ell_\infty^\leq$-spherical density as in (4.3)] if and only if they have the UOSP $(<)$ [respectively UOSP $(\leq)$].*

An analog of Proposition 2.4 can also be stated and proven.

In the following, we will now study two kinds of discrete-time discrete-state process: those that have only unit step jumps and those that are more general. We start with discrete-time point processes that have only unit step jumps. Let $\{A(t),$

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$t = 0,1,\ldots\}$ be such a process with $A(0) \equiv 0$. We assume that $P\{\lim_{t\to\infty} A(t) = \infty\} = 1$. The distributional properties of $\{A(t), t = 0,1,\ldots\}$ can be equivalently described by means of either one of three sequences of random variables:

- The times of the unit jumps $T_1, T_2, \ldots$; that is, $T_k = t \Leftrightarrow A(t-1) = k-1$ and $A(t) = k$, $1 \le k \le t$ (here $0 < T_1 < T_2 < \cdots$ with probability 1).
- The interepoch intervals $Z_1, Z_2, \ldots$; that is, $Z_k = T_k - T_{k-1}$, $k \ge 1$, where $T_0 \equiv 0$ (here $Z_k > 0$, $k \ge 1$, with probability 1).
- The jump indicators $I_1, I_2, \ldots$; that is, $I_t = A(t) - A(t-1)$, $t \ge 0$.

A process $\{A(t), t = 0,1,\ldots\}$ as above will be said to have the UOSP $(<)$ if $T_1, T_2, \ldots$ have this property.

We now recall a definition and a result from Huang and Shoung [9].

DEFINITION 4.3 [9, p. 72]: *A discrete-time discrete-state process* $\{A(t), t = 0,1,\ldots\}$ *is called a* mixed geometric process *if there exists a random variable* Θ *that takes on values in* $(0,1)$ *such that, given that* $\Theta = \theta$, *the interepoch intervals* $Z_1, Z_2, \ldots$ *of* $\{A(t), t = 0,1,\ldots\}$ *are independent and identically distributed with probabilities* $P_\theta\{Z_k = z\} = \theta(1-\theta)^{z-1}$, $z \ge 1$, $k \ge 1$.

THEOREM 4.4 [9, p. 73]: *A discrete-time discrete-state process* $\{A(t), t = 0,1,\ldots\}$ *with unit step jumps, such that* $A(0) = 0$ *and* $P\{\lim_{t\to\infty} A(t) = \infty\} = 1$, *has the UOSP* $(<)$ *if and only if it is a mixed geometric process.*

We can now obtain the following result; part of it is a discrete analog of Theorem 2.5.

THEOREM 4.5: *Let* $\{A(t), t = 0,1,\ldots\}$ *be a discrete-time discrete-state process with unit step jumps such that* $A(0) = 0$ *and* $P\{\lim_{t\to\infty} A(t) = \infty\} = 1$. *The following statements are equivalent:*

(i) *The process* $\{A(t), t = 0,1,\ldots\}$ *has the UOSP* $(<)$.
(ii) *The process* $\{A(t), t = 0,1,\ldots\}$ *is a mixed geometric process.*
(iii) *The interepoch intervals* $Z_1, Z_2, \ldots, Z_k$ *have a Schur-constant density on* $\mathbb{N}^k$ *for every* $k \ge 1$.
(iv) *The epoch times* $T_1, T_2, \ldots, T_k$ *have a joint* $\ell_\infty^<$*-spherical density on* $\mathbb{N}^k$ *for every* $k \ge 1$.
(v) *The jump indicators* $I_1, I_2, \ldots, I_t$ *are exchangeable for every* $t \ge 1$.

PROOF: The equivalence (i) $\Leftrightarrow$ (ii) is Theorem 4.4. The equivalence (i) $\Leftrightarrow$ (iv) follows from Proposition 4.2. The equivalence (iii) $\Leftrightarrow$ (iv) follows from the comment after (4.4).

(i) $\Rightarrow$ (v): Fix a $t \ge 1$. Let $(i_1, i_2, \ldots, i_t) \in \{0,1\}^t$. Suppose that k of the i_j's are one and that the other $t-k$ indicators are zero. Then, for some $0 < t_1 < t_2 < \cdots < t_k \le t$, we have

$$\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\} = \{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k, T_{k+1} > t\}.$$

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So,

$$\begin{aligned} P\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\} &= P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k | T_k \le t, T_{k+1} > t\} P\{T_k \le t, T_{k+1} > t\} \\ &= \frac{P\{T_k \le t, T_{k+1} > t\}}{\binom{t}{k}}, \end{aligned}$$

where the second equality follows from the assumption that $T_1, T_2, \ldots, T_k$ have the UOSP ($<$); that is, $P\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\}$ does not depend on the actual values $i_1, i_2, \ldots, i_t$, but only on the related k. Thus, $I_1, I_2, \ldots, I_t$ are exchangeable.

(v) $\Rightarrow$ (ii): From the de Finetti theorem there exists a probability measure Π on (0,1) such that

$$P\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\} = \int_0^1 \theta^k (1-\theta)^{t-k} \, d\Pi(\theta), \tag{4.5}$$

for any $t \ge 1$, where k is the number of i_j's that are one.

Now, fix a $k \ge 1$ and a $(z_1, z_2, \ldots, z_k) \in \mathbb{N}^k$. Denote $t = \sum_{j=1}^k z_j$. Then, from (4.5) we get

$$\begin{aligned} P\{Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k\} &= \int_0^1 \theta^k (1-\theta)^{t-k} \, d\Pi(\theta) \\ &= \int_0^1 \left[\prod_{j=1}^k [\theta (1-\theta)^{z_j - 1}] \right] d\Pi(\theta); \end{aligned}$$

that is, the process $\{A(t), t = 0,1,\ldots\}$ is a mixed geometric process. ■

The equivalence (ii) $\Leftrightarrow$ (iii) is a well-known de Finetti-type theorem (see, for example, Diaconis and Freedman [5, p. 414]). Also, it can be viewed as a discrete analog of Theorem 1.44 in Spizzichino [17].

Using Theorem 4.5, one can obtain interesting discrete Schur-constant, $\ell_\infty^{\le}$-spherical, and exchangeable densities. The following example illustrates this.

Example 4.6: Let Θ in Definition 4.3 have the beta density with parameters $\alpha > 0$ and $\beta > 0$. Then, the Schur-constant joint discrete density of the first $k \ge 1$ inter-epoch intervals is given by

$$\begin{aligned} P\{Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k\} &= \int_0^1 \theta^k (1-\theta)^{\sum_{i=1}^k z_i - k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \, d\theta \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+k)\Gamma\left(\beta + \sum_{i=1}^k z_i - k\right)}{\Gamma\left(\alpha + \beta + \sum_{i=1}^k z_i\right)}, \qquad z_i \ge 1, \, i = 1,2,\ldots,k. \end{aligned}$$

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This density (actually the density of $(Z_1 - 1, Z_2 - 1, \ldots, Z_k - 1)$) is a special case of the beta-compound negative multinomial density (see Johnson, Kotz, and Balakrishnan [11, p. 104]); see also Example 4.9. The marginal density of Z_1 (which, by exchangeability, is the marginal density of each Z_i) is given by

$$P\{Z_1 = z\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta+z-1)}{\Gamma(\alpha+\beta+z)}, \qquad z \geq 1. \tag{4.6}$$

This density (actually the density of $Z_1 - 1$) is a special case of the beta-negative-binomial density, which is also called the Markov–Pólya density; see Johnson, Kotz, and Kemp [12, pp. 242–243], but note the different notation; for example, for $a > 1$, they denote $(-a)! = 1/(a-1)!$, and for any nonzero a and b, they denote $\binom{a}{b} = a!/(b!(a-b)!)$, where for $a > 0$, they use the notation $a! = \Gamma(a+1)$.

The $\ell_\infty^<$-spherical joint discrete density of the first $k \geq 1$ epoch times is given by

$$P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+k)\Gamma(\beta+t_k-k)}{\Gamma(\alpha+\beta+t_k)}, \qquad 1 \leq t_1 < t_2 < \cdots < t_k.$$

Finally, the exchangeable joint density of the first $t \geq 1$ indicators can be computed as follows:

$$P\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\}$$
$$= \int_0^1 \theta^{\sum_{j=1}^t i_j} (1-\theta)^{t - \sum_{j=1}^t i_j} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}\, d\theta$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma\left(\alpha + \sum_{j=1}^t i_j\right)\Gamma\left(\beta + t - \sum_{j=1}^t i_j\right)}{\Gamma(\alpha+\beta+t)}, \qquad (i_1, i_2, \ldots, i_t) \in \{0,1\}^t.$$

Marginally, each I_j is a Bernoulli random variable with parameter $\alpha/(\alpha+\beta)$.

In the case when Θ is a uniform$(0,1)$ random variable (i.e., $\alpha = \beta = 1$), we have

$$P\{Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k\} = \frac{k!\left(\sum_{i=1}^k z_i - k\right)!}{\left(\sum_{i=1}^k z_i + 1\right)!}, \qquad z_i \geq 1,\ i = 1,2,\ldots,k;$$

in particular,

$$P\{Z_1 = z\} = \frac{1}{z} - \frac{1}{z+1}, \qquad z \geq 1.$$

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The $\ell_\infty^{\leq}$-spherical joint discrete density of the first $k \geq 1$ epoch times is then

$$P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\} = \frac{k!(t_k - k)!}{(t_k + 1)!}, \qquad 1 \leq t_1 < t_2 < \cdots < t_k,$$

and the exchangeable density of the first $t \geq 1$ indicators is then

$$P\{I_1 = i_1, I_2 = i_2, \ldots, I_t = i_t\} = \frac{\left(\sum_{j=1}^{t} i_j\right)!\left(t - \sum_{j=1}^{t} i_j\right)!}{(t+1)!}, \qquad (i_1, i_2, \ldots, i_t) \in \{0,1\}^t.$$

Marginally, each I_j is then a Bernoulli random variable with parameter $\frac{1}{2}$.

We now proceed to the study of nondecreasing discrete-time discrete-state processes that can have jumps larger than unity at the jump epochs. These can be described by allowing the interepoch intervals between the "unit jumps" to be equal to zero with a positive probability so that an actual jump of size $j > 0$ corresponds to a sequence of j consecutive interepoch intervals of size 0. Thus, let $\{B(t), t = 0,1,\ldots\}$ be such a process (here, we allow jumps at time 0, so we do not require that $B(0) = 0$ here). The distributional properties of $\{B(t), t = 0,1,\ldots\}$ can be equivalently described by means of either one of three sequences of random variables:

- The interepoch intervals between "unit jumps" $Z_1, Z_2, \ldots$; that is, $B(t) = k \Leftrightarrow \sum_{i=1}^{k} Z_i \leq t$ and $\sum_{i=1}^{k+1} Z_i > t$, $k \geq 0$, $t \geq 0$ (here, the Z_i's can be equal to zero with positive probability).
- The times of the "unit jumps" $T_1, T_2, \ldots$; that is, $T_k = \sum_{i=1}^{k} Z_i$, $k \geq 1$ (here consecutive T_k's may be equal to each other with positive probability).
- The jump amounts $J_0, J_1, \ldots$; that is, $J_0 = B(0)$, and $J_t = B(t) - B(t-1)$, $t \geq 1$ (here, of course, the J_t's can be equal to zero with positive probability and they can also be larger than one).

A process $\{B(t), t = 0,1,\ldots\}$, as above, will be said to have the UOSP $(\leq)$ if $T_1, T_2, \ldots$ have this property.

A process $\{B(t), t = 0,1,\ldots\}$, as above (i.e., which can have jumps larger than unity), can be a discrete-time discrete-state mixed uniform sample process. In order to define it formally, let $\tilde{t} < \infty$ be a positive integer and let $U_1, U_2, \ldots$ be a sequence of independent random variables distributed uniformly on $\{0,1,\ldots,\tilde{t}\}$. Let L be a positive integer-valued random variable, independent of the U_i's. Given $L = l$, let $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ be the empirical distribution function based on $U_1, U_2, \ldots, U_l$. The unconditional process $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ will be called a discrete-time discrete-state mixed uniform sample process. Such processes are useful in studies where the discrete uniform distribution can model some real phenomenon, and a sample, of a size that cannot be predetermined in advance of the experiment, is taken from that distribution.

For processes, as above, that can have general (rather than unit) jumps, we have the following equivalent conditions.

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THEOREM 4.7: *Let $\{B(t), t = 0,1,\ldots\}$ be a nondecreasing discrete-time discrete-state process. The following statements are equivalent:*

(i) The process $\{B(t), t = 0,1,\ldots\}$ has the UOSP ($\leq$).
(ii) The process $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ is a mixed uniform sample process for every $\tilde{t} \geq 0$ such that $P\{B(s) > B(\tilde{t})$ for some $s > \tilde{t}\} = 1$.
(iii) The interepoch intervals $Z_1, Z_2, \ldots, Z_k$ have a Schur-constant density on $(\mathbb{N} \cup \{0\})^k$ for every $k \geq 1$ such that $P\{B(t) \geq k$ for some $t \geq 1\} = 1$.
(iv) The epoch times $T_1, T_2, \ldots, T_k$ have a joint $\ell_\infty^{\leq}$-spherical density on $(\mathbb{N} \cup \{0\})^k$ for every $k \geq 1$ such that $P\{B(t) \geq k$ for some $t \geq 1\} = 1$.
(v) The jump amounts $J_0, J_1, \ldots, J_t$ have a Schur-constant density on $(\mathbb{N} \cup \{0\})^{t+1}$ for every $t \geq 0$ such that $P\{B(s) > B(t)$ for some $s > t\} = 1$.

PROOF: The equivalence (i) $\Leftrightarrow$ (iv) follows from Proposition 4.2. The equivalence (iii) $\Leftrightarrow$ (iv) follows from the comment after (4.4).

(i) $\Rightarrow$ (ii): The proof of this statement is a modification of the proof of Theorem 4 of Puri [15]. Fix an integer $\tilde{t} \geq 0$ and set $L = B(\tilde{t})$. From the assumption that $\{B(t), t = 0,1,\ldots\}$ has the UOSP ($\leq$) it follows that, given $L = l$, the jump times $T_1 \leq T_2 \leq \cdots \leq T_l$ of $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ are distributed as the order statistics of a sample of size l from a uniform distribution on $\{0,1,\ldots,\tilde{t}\}$; that is, $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ is an empirical distribution function corresponding to a sample of size l from a uniform distribution on $\{0,1,\ldots,\tilde{t}\}$.

(ii) $\Rightarrow$ (i): Fix a $\tilde{t} \geq 0$ and let $L = B(\tilde{t})$. Given $L = l$, we have that $\{B(t), t = 0,1,\ldots,\tilde{t}\}$ is an empirical distribution function corresponding to a sample of size l from a uniform distribution on $\{0,1,\ldots,\tilde{t}\}$. Let $T_1 \leq T_2 \leq \cdots \leq T_l$ be the jump points of this empirical distribution (some of the T_i's might be equal to each other if a jump is larger than unity). These T_i's obviously have the UOSP ($\leq$).

(i) $\Rightarrow$ (v): Fix a $t \geq 0$. Let $(j_0, j_1, \ldots, j_t) \in (\mathbb{N} \cup \{0\})^{t+1}$ be a possible realization of $(J_0, J_1, \ldots, J_t)$. Set

$$\begin{aligned}
\alpha_1 &= \min\{l \,|\, 0 \leq l \leq t, j_l \geq 1\},\\
\alpha_2 &= \min\{l \,|\, \alpha_1 < l \leq t, j_l \geq 1\},\\
&\vdots\\
\alpha_k &= \min\{l \,|\, \alpha_{k-1} < l \leq t, j_l \geq 1\};
\end{aligned}$$

these are the corresponding realized epochs of actual jumps—obviously, there is a finite number k of them. Now we describe the corresponding realization of the epoch times t_m's of the zero and nonzero jumps.

The first t_m ($\tilde{t}_1$, say) is equal to α_1.
The next $j_{\alpha_1} - 1$ of the t_m's are all equal to $\tilde{t}_1$.
The next t_m ($\tilde{t}_2$, say) is equal to α_2.
The next $j_{\alpha_2} - 1$ of the t_m's are all equal to $\tilde{t}_2$.
$\vdots$

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The next t_m ($\tilde{t}_k$, say) is equal to α_k.

The last $j_{\alpha_k} - 1$ of the t_m's are all equal to $\tilde{t}_k$.

Note that above we have described

$$p \equiv k + \sum_{u=1}^{k} (j_{\alpha_u} - 1) = \sum_{v=0}^{t} j_v$$

epoch times t_m's; the second equality here follows from the fact that j_v's with indices different than α_u are all zeros. Let us order these t_m's as $0 \le t_1 \le t_2 \le \cdots \le t_p$ (note that $t_p \le t$). Then,

$$\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\} = \{T_1 = t_1, T_2 = t_2, \ldots, T_p = t_p, T_{p+1} > t\}. \qquad \textbf{(4.7)}$$

So,

$$\begin{aligned} &P\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\} \\ &\quad = P\{T_1 = t_1, T_2 = t_2, \ldots, T_p = t_p | T_p \le t, T_{p+1} > t\} P\{T_p \le t, T_{p+1} > t\} \\ &\quad = \frac{P\{T_p \le t, T_{p+1} > t\}}{\binom{t+p}{p}}, \end{aligned}$$

where the second equality follows from the assumption that $T_1, T_2, \ldots, T_p$ have the UOSP ($\le$); that is, $P\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\}$ does not depend on the actual values $j_0, j_1, \ldots, j_t$, but only on $p = \sum_{v=0}^{t} j_v$. Thus, $J_0, J_1, \ldots, J_t$ have a Schur-constant density.

(v) $\Rightarrow$ (iv): Fix a $k \ge 1$ and $0 \le t_1 \le t_2 \le \cdots \le t_k$. Let

$$\begin{aligned} \varepsilon_1 &= 1, \\ \varepsilon_i &= \min\{l | 1 \le l \le k, t_l > t_{\varepsilon_{i-1}}\}, \qquad i = 2,3,\ldots,h; \end{aligned}$$

that is, $t_{\varepsilon_1} < t_{\varepsilon_2} < \cdots < t_{\varepsilon_h}$ are the different values in the sequence $t_1 \le t_2 \le \cdots \le t_k$. Obviously, h above is finite (it satisfies $h \le k$). Now, define t_k values $j_0, j_1, \ldots, j_{t_k - 1}$ as follows: Let

$$j_{t_{\varepsilon_i}} = \varepsilon_{i+1} - \varepsilon_i, \qquad i = 1,2,\ldots,h-1,$$

and let the other $t_k - h + 1$ values j_m's be zero. Also, let

$$a = \max\{b \ge 0 | t_k = t_{k-b}\} + 1;$$

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that is, a is the number of the last t_n's in the sequence $t_1 \le t_2 \le \cdots \le t_k$ that are equal to each other. Obviously, $1 \le a \le k$. Then,

$$\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\} = \{J_0 = j_0, J_1 = j_1, \ldots, J_{t_k - 1} = j_{t_k - 1}, J_{t_k} \ge a\}. \quad \textbf{(4.8)}$$

Note that

$$\sum_{m=0}^{t_k - 1} j_m + a = j_{t_{\varepsilon_1}} + j_{t_{\varepsilon_2}} + \cdots + j_{t_{\varepsilon_{h-1}}} + a = \varepsilon_h - 1 + a = k;$$

the last equality follows by noting that ε_h is the index of the (first) largest value in the sequence $t_1 \le t_2 \le \cdots \le t_k$ and the number of t_n's that are equal to t_{ε_h} is a. Therefore,

$$P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\} = \sum_{j_{t_k} = a}^{\infty} P\{J_0 = j_0, J_1 = j_1, \ldots, J_{t_k - 1} = j_{t_k - 1}, J_{t_k} = j_{t_k}\}.$$

Since $(J_0, J_1, \ldots, J_{t_k})$ has a Schur-constant density on $(\mathbb{N} \cup \{0\})^{t_k + 1}$, it follows that there exists a function φ_{t_k} such that

$$\begin{aligned} P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\} &= \sum_{j_{t_k} = a}^{\infty} \varphi_{t_k}\left(\sum_{i=0}^{t_k - 1} j_i + j_{t_k}\right) \\ &= \sum_{j_{t_k} = 0}^{\infty} \varphi_{t_k}\left(\sum_{i=0}^{t_k - 1} j_i + a + j_{t_k}\right) \\ &= \sum_{j=0}^{\infty} \varphi_{t_k}(k + j). \end{aligned}$$

Thus, for a fixed $k \ge 1$, when $0 \le t_1 \le t_2 \le \cdots \le t_k$, the probability $P\{T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k\}$ depends only on t_k; that is, $T_1, T_2, \ldots, T_k$ have a joint $\ell_\infty^{\le}$-spherical density on $(\mathbb{N} \cup \{0\})^k$. ■

It is of interest to compare the statement in Theorem 4.5(v) with the one in Theorem 4.7(v). The condition on the jump variables in the former is exchangeability, whereas in the latter, the condition on the jump variables is that they have a Schur-constant density. However, in the setup of Theorem 4.5, these two conditions are the same because for binary random variables, Schur-constant density and exchangeability are equivalent conditions.

It is also of interest to compare the statement in Theorem 4.5(ii) with the one in Theorem 4.7(ii). It is seen that in the discrete-time case, the mixed geometric process is a natural tool for characterizing processes that have unit step jumps and are unbounded, whereas the mixed uniform sample process plays a similar role for characterizing processes that can be bounded and that can have jumps larger than unity.

Remark 4.8: An interesting relationship exists between processes $\{A(t), t = 0, 1, \ldots\}$ such that $P\{\lim_{t \to \infty} A(t) = \infty\} = 1$ with the UOSP $(<)$ and processes $\{B(t),$

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$t = 0,1,\ldots\}$ such that $P\{\lim_{t\to\infty} B(t) = \infty\} = 1$ with the UOSP $(\leq)$. Let $Z_1, Z_2, \ldots$ be the interepoch intervals of $\{A(t),\ t = 0,1,\ldots\}$ as in Theorem 4.5(iii). Define

$$Z_i' = Z_i - 1, \qquad i = 1,2,\ldots, \tag{4.9}$$

and let the Z_i''s be the interepoch intervals of a process $\{B(t),\ t = 0,1,\ldots\}$ as in Theorem 4.7(iii). If the $Z_1, Z_2, \ldots, Z_k$ have a Schur-constant density on $\mathbb{N}^k$ for each $k \geq 1$, then so do $Z_1', Z_2', \ldots, Z_k'$ on $(\mathbb{N} \cup \{0\})^k$, and vice versa. We thus see that the transformation $Z_i \to Z_i'$ described above, and its inverse, enable us to go back and forth between processes that have the UOSP $(<)$ and the UOSP $(\leq)$, provided they satisfy $P\{\lim_{t\to\infty} A(t) = \infty\} = 1$ and $P\{\lim_{t\to\infty} B(t) = \infty\} = 1$.

The transformation (4.9) indicates that unbounded processes with unit step jumps (as in Theorem 4.5) that satisfy any of the equivalent conditions in Theorem 4.5 also satisfy a version of the statement in Theorem 4.7(ii). A continuous-time analog of this observation is Theorem 6.6 in Grandell [7].

We will now obtain a relationship among discrete Schur-constant densities that is a discrete analog of (2.1). This will be used in the sequel. Let $W_1, W_2, \ldots$ be a sequence of random variables taking on values in $\mathbb{N} \cup \{0\}$, such that the density function of $(W_1, W_2, \ldots, W_n)$ is of the form (4.2) where the function φ there is denoted here by φ_n, $n = 1,2,\ldots$. Then, for any $p \geq 1$ and $(w_1, w_2, \ldots, w_p) \in (\mathbb{N} \cup \{0\})^p$, we have

$$\begin{aligned}
\varphi_p\left(\sum_{j=1}^{p} w_j\right) &= P\{W_1 = w_1, W_2 = w_2, \ldots, W_p = w_p\} \\
&= \sum_{w=0}^{\infty} P\{W_1 = w_1, W_2 = w_2, \ldots, W_p = w_p, W_{p+1} = w\} \\
&= \sum_{w=0}^{\infty} \varphi_{p+1}\left(\sum_{j=1}^{p} w_j + w\right);
\end{aligned}$$

that is,

$$\varphi_p(v) = \sum_{w=0}^{\infty} \varphi_{p+1}(v + w), \qquad p \geq 1,\ v \geq 0. \tag{4.10}$$

Next, we express the joint density of the jump amounts $J_0, J_1, \ldots, J_t$ by means of the densities of the interepoch intervals Z_i's. For this purpose, let us assume that the density of $(Z_1, Z_2, \ldots, Z_n)$ is of the form (4.2) where the function φ there is denoted here by φ_n, $n = 1,2,\ldots$. For $t \geq 0$, fix a $(j_0, j_1, \ldots, j_t) \in (\mathbb{N} \cup \{0\})^{t+1}$ and let $\sum_{v=0}^{t} j_v = p$. If $p = 0$, then we have

$$P\{J_0 = 0, J_1 = 0, \ldots, J_t = 0\} = P\{Z_1 > t\}, \qquad t \geq 0. \tag{4.11}$$

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On the other hand, if $p \geq 1$, then

$$\begin{aligned} P\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\} &= P\{J_0 = p, J_1 = 0, \ldots, J_t = 0\} \\ &= P\{Z_1 = 0, Z_2 = 0, \ldots, Z_p = 0, Z_{p+1} > t\} \\ &= \sum_{k=t+1}^{\infty} \varphi_{p+1}(k) \\ &= \sum_{k=0}^{\infty} \varphi_{p+1}(t+1+k) \\ &= \varphi_p(t+1) \\ &= P\{Z_1 = 0, Z_2 = 0, \ldots, Z_{p-1} = 0, Z_p = t+1\}, \end{aligned} \tag{4.12}$$

where the first equality follows from the fact that $(J_0, J_1, \ldots, J_t)$ has a Schur-constant density, the second equality essentially follows from (4.7), and the next to last equality follows from (4.10).

In a similar manner, we can express the joint density of the interepoch intervals $Z_1, Z_2, \ldots, Z_k$ by means of the densities of the jump amounts J_i's. For $k \geq 1$, fix a $(z_1, z_2, \ldots, z_k) \in (\mathbb{N} \cup \{0\})^k$ and let $\sum_{v=0}^{k} z_v = w$. If $w = 0$, then we have

$$P\{Z_1 = 0, Z_2 = 0, \ldots, Z_k = 0\} = P\{J_0 \geq k\}, \qquad k \geq 1.$$

On the other hand, if $w \geq 1$, then it can be shown, in a manner similar to the proof of (4.12) (but using (4.8) rather than (4.7)), that

$$P\{Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k\} = P\{J_0 = 0, J_1 = 0, \ldots, J_{w-2} = 0, J_{w-1} = k\}.$$

Theorem 4.7 and the transformation (4.9), together with the above observations, enable us to obtain interesting Schur-constant densities on $(\mathbb{N} \cup \{0\})^k$. The following example illustrates this.

Example 4.9: Let Θ and the Z_i's be as in Example 4.6 and let Z_i''s be defined by (4.9). Then, it is a simple matter to see that the Schur-constant density of the first k Z_i''s is given by

$$P\{Z_1' = z_1, Z_2' = z_2, \ldots, Z_k' = z_k\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+k)\Gamma\left(\beta + \sum_{i=1}^{k} z_i\right)}{\Gamma\left(\alpha+\beta+k+\sum_{i=1}^{k} z_i\right)}, \qquad z_i \geq 0,\ i = 1, 2, \ldots, k. \tag{4.13}$$

This density is a special case of the beta-compound negative multinomial density (see Johnson et al. [11, p. 104]).

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In order to find the joint density of the first $t+1 \geq 1$ jumps amounts $J_0, J_1, \ldots, J_t$, we use (4.12). Fix a $t \geq 0$ and $(j_0, j_1, \ldots, j_t) \in (\mathbb{N} \cup \{0\})^{t+1}$ and denote $\sum_{v=0}^{t} j_v = p$. If $p \geq 1$, then, by (4.12) and (4.13), we have

$$\begin{aligned} P\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\} \\ = P\{Z_1' = 0, Z_2' = 0, \ldots, Z_{p-1}' = 0, Z_p' = t+1\} \\ = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma\left(\alpha + \sum_{v=0}^{t} j_v\right)\Gamma(\beta+t+1)}{\Gamma\left(\alpha+\beta+t+\sum_{v=0}^{t} j_v + 1\right)}. \end{aligned}$$

If $p = 0$, then, by (4.11), the corresponding probability $P\{J_0 = 0, J_1 = 0, \ldots, J_t = 0\}$ is a sum of expressions of the form (4.6).

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