Proc. Camb. Phil. Soc. (1973), **73**, 57
PCPS 73–8
Printed in Great Britain

Fatou–Zygmund sets

BY KENNETH A. ROSS†
University of Oregon

(*Received* 10 *April* 1972)

1. Let G be a compact Abelian group with character group X. Let $\mathscr{S} = (\mathsf{X}_n)_{n=1}^{\infty}$ be an increasing sequence of finite symmetric subsets of X, and consider a symmetric subset P of $\bigcup_{n=1}^{\infty} \mathsf{X}_n$. For any Hermitian complex-valued function u on P, we write $s_n u$ for the real-valued trigonometric polynomial $\sum_{\chi \in \mathsf{P} \cap \mathsf{X}_n} u(\chi)\chi$. Edwards, Hewitt and Ross(4) investigated the following property for a non-void measurable subset W of G satisfying $W \subset (\operatorname{int} W)^-$:

$$(*) \quad \text{if} \quad \sup_n [\sup_{x \in W} s_n u(x)] < \infty, \quad \text{then} \quad u \in l^1(\mathsf{P}).$$

The validity of this implication was shown to be independent of the choice of $\mathscr{S}$. Accordingly, if (*) holds, P is called an $FZ(W)$-set. If P is an $FZ(W)$-set for all W, then P is termed a full FZ-set or full Fatou–Zygmund set. In this paper, we characterize the full FZ-sets as $FZ(G)$-sets satisfying a certain algebraic condition. In particular, we show that if G is connected, then a symmetric subset of X is an $FZ(G)$-set if and only if it is a full FZ-set. Some of the techniques are adaptations of those of Mme Déchamps-Gondim(1), (2). The class of full FZ-sets is not always closed under the operation of finite unions; this contrasts with the situation for Sidon sets and for $FZ(G)$-sets.

(1·1) DEFINITION. Let X_0 denote a subgroup of an Abelian group X. We will call a subset P of X X_0-*subtransversal* provided each coset of X_0 intersects P in at most one point. We will say that P is *almost* X_0-*subtransversal* if P is the union of a finite set and an X_0-subtransversal set.

(1·2) THEOREM. *Let G be a compact Abelian group with character group X. A symmetric subset P of X is a full FZ-set if and only if*

(i) *P is an $FZ(G)$-set; and*

(ii) *P is almost X_0-subtransversal for all finite subgroups X_0 of X.*

If G is connected, then X is torsion-free by (24·25)(5), and {so 1} is its only finite subgroup. Hence (ii) holds vacuously in this case and we have the following corollary.

(1·3) COROLLARY. *Let G be a compact connected Abelian group with character group X. A symmetric subset of X is a full FZ-set if and only if it is an $FZ(G)$-set.*

We prove Theorem (1·2) in section 2 and give some applications in section 3.

† Research partially supported by Grant GP-28250 of the National Science Foundation of the U.S.A. AMS 1970 Subject Classification: 42A44, 43A25, 43A40, 43A50.

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

2. *Proof of Theorem* (1·2). Throughout this section G will denote a compact Abelian group and X will denote its character group.

(2·1) LEMMA. *If* P *is a (symmetric) full FZ-set in* X, *then* P *is almost* X_0*-subtransversal for all finite subgroups* X_0 *of* X.

Proof. Assume that P is not almost X_0-subtransversal for some finite subgroup X_0 of X. Then an infinite number of cosets of X_0 intersect P in at least two points. Thus there are distinct characters $\chi_1, \chi_2, \ldots, \psi_1, \psi_2, \ldots$ in P such that $\chi_k \psi_k^{-1} \in \mathsf{X}_0$ for all k. Since X_0 is finite, we may suppose that all $\chi_k \psi_k^{-1}$ are equal to the same character ϕ in X_0. Clearly ϕ cannot be the character 1, so we can select x_0 in G so that $\phi(x_0) \neq 1$; let $\delta = |1 - \phi(x_0)|$. The annihilator G_0 of X_0 is an open subgroup of G, and so P is an $FZ((x_0 + G_0)^{-1})$-set. Theorem (8·7) in (4) (with $\beta = 1$) shows that there is a measure $\mu \in \mathbf{M}_+(x_0 + G_0)$ such that $|\hat{\mu}(\chi) - 1| < \frac{1}{4}\delta$ for all but finitely many χ in P. It follows that

$$|\hat{\mu}(\chi_k) - \hat{\mu}(\psi_k)| < \tfrac{1}{2}\delta \quad \text{and} \quad |\hat{\mu}(\chi_k)| > \tfrac{1}{2} \tag{1}$$

for all sufficiently large k. A direct computation shows that $\hat{\mu}(\psi_k) = \phi(x_0)\hat{\mu}(\chi_k)$ for all k. Hence

$$|\hat{\mu}(\chi_k) - \hat{\mu}(\psi_k)| = |\hat{\mu}(\chi_k) - \phi(x_0)\hat{\mu}(\chi_k)| \geqslant \tfrac{1}{2}|1 - \phi(x_0)| = \tfrac{1}{2}\delta$$

for all sufficiently large k, a contradiction to (1). ∎

The next lemma is a symmetric analogue to Lemme 1 (β) (1) (or 6·2 (2)).

(2·2) LEMMA. *Let* P *be a symmetric Sidon set in* X. *There is an integer* m *such that given a finite subset* Δ *of* X, P *can be written as a disjoint union*

$$\mathsf{P} = \Sigma \cup \left(\bigcup_{i \in I} \mathsf{P}_i \right)$$

where

(i) Σ *and all* P_i *are finite and symmetric,*

(ii) *each* P_i *has at most* m *elements,*

(iii) $\mathsf{P}_i \mathsf{P}_j \cap \Delta = \varnothing$ *for* $i \neq j$, $i, j \in I$.

Proof. We may suppose that Δ is symmetric and $1 \in \Delta$. By Lemme 1 (α) (1) (or 6·1 (2)), there is an integer n with the following property. If Φ is a finite subset of X, then P contains a finite subset Σ such that $(\chi\Phi) \cap (\mathsf{P} \setminus \Sigma)$ has at most n elements for every $\chi \in \mathsf{X}$. We apply this property to $\Phi = \Delta^{2n}$, so that

$$(\chi\Delta^{2n}) \cap (\mathsf{P} \setminus \Sigma) \quad \text{has at most } n \text{ elements, for each} \quad \chi \in \mathsf{X}. \tag{1}$$

We may suppose that Σ is symmetric.

Let us call a sequence $\chi_1, \chi_2, \ldots, \chi_s$ of distinct elements in X a Δ-chain provided

$$\chi_{j+1} \in \chi_j \Delta \cup \chi_j^{-1} \Delta \quad \text{for} \quad j = 1, 2, \ldots, s-1.$$

We claim that Δ-chains in $\mathsf{P} \setminus \Sigma$ have at most $2n$ elements. For suppose that

$$\chi_1, \chi_2, \cdots, \chi_{2n+1}$$

is a Δ-chain in $\mathsf{P} \setminus \Sigma$. A simple induction on j shows that for $j = 1, 2, \ldots, 2n+1$, there exist δ_j in $\{-1, 1\}$ such that $\chi_j^{\delta_j} \in \chi_1 \Delta^{j-1}$. Thus we have

$$\chi_j^{\delta_j} \in \chi_1 \Delta^{2n} \quad \text{for} \quad j = 1, 2, \ldots, 2n+1.$$

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

The $\chi_j^{\delta j}$ need not be distinct; however, no three of them can be equal. So the *set* $\{\chi_j^{\delta j}: j = 1, 2, \ldots, 2n+1\}$ has at least $n+1$ elements. Since $\mathsf{P}\backslash\Sigma$ is symmetric, this set is a subset of $(\chi_1 \Delta^{2n}) \cap (\mathsf{P}\backslash\Sigma)$, contrary to (1). This establishes our claim.

For χ, ψ in $\mathsf{P}\backslash\Sigma$, we define $\chi \sim \psi$ if a Δ-chain in $\mathsf{P}\backslash\Sigma$ contains them both. It is easy to verify that $\sim$ is an equivalence relation. It is also easy to see that the equivalence classes are symmetric sets having at most $2n$ elements. The lemma is completed by taking the P_i to be these equivalence classes and setting $m = 2n$. ∎

For a trigonometric polynomial f on G, we will write length (f) for the cardinal number of $\{\chi \in \mathsf{X}: \hat{f}(\chi) \neq 0\}$.

(2·3) Lemma. *Let m be a positive integer, and let U be a non-void open subset of G. There is a finite subgroup X_0 of X such that*

(i) *if f is a trigonometric polynomial that vanishes on U, if length $(f) \leqslant m$, and if* supp $(\hat{f})$ *is X_0-subtransversal, then f is identically zero on G.*

Here and in what follows X_0 might be $\{1\}$. In this case, the requirement that supp $(\hat{f})$ be X_0-subtransversal is a vacuous restriction.

Proof. Let x_0 be any element in U and let V be a symmetric neighbourhood of 0 such that $x_0 + mV \subset U$. Let G_0 be the (open-and-closed) subgroup of G generated by V, and let X_0 denote the annihilator of G_0. The subgroup X_0 is finite, since G/G_0 is finite. We prove (i) by proving the following:

if f vanishes on $x_0 + kV$, if length $(f) \leqslant k$, and

if supp $(\hat{f})$ is X_0-subtransversal, then f is identically zero on G.

This assertion is obvious for $k = 1$. We assume its validity for $k-1$ and consider f satisfying the hypotheses for k. We assume that f is not identically zero, so that length $(f) = k$. We can write f as $a\chi_0 + g$ where length $(g) = k-1$, and we may suppose that $a = 1$. Consider $v \in V$ and $x \in x_0 + (k-1)V$. Then x and $x+v$ both belong to $x_0 + kV$, and so

$$0 = f(x+v) - \chi_0(v) f(x) = \sum_{\chi \in \mathsf{X}} \hat{g}(\chi)\,[\chi(v) - \chi_0(v)]\,\chi(x).$$

The assertion for $k-1$ applies to $\sum_{\chi \in \mathsf{X}} \hat{g}(\chi)\,[\chi(v) - \chi_0(v)]\,\chi$, and so this trigonometric polynomial is identically zero. Thus if χ_1 is any character such that $\hat{g}(\chi_1) \neq 0$, then $\chi_1(v) - \chi_0(v) = 0$; this equality holds for all $v \in V$. It follows that $\chi_1 \chi_0^{-1}$ is identically one on V and hence on G_0. Thus $\chi_1 \chi_0^{-1}$ belongs to X_0, and χ_1 and χ_0 belong to the same coset of X_0. Since χ_1 and χ_0 are distinct elements of supp $(\hat{f})$ and supp $(\hat{f})$ is X_0-subtransversal, we have a contradiction. That is, f is identically zero. ∎

The next lemma is related to Lemme 6·3 in (2), which is a generalization to connected groups of a theorem of Zygmund.

(2·4) Lemma. *Let m be a positive integer, and let g be a non-zero function in $\mathfrak{C}^+(G)$. There is a finite subgroup X_0 of X and a positive number ϵ such that*

(i) *if f is in $\mathfrak{T}(G)$, if length $(f) \leqslant m$, and if* supp $(\hat{f})$ *is X_0-subtransversal, then*

$$\int_G |f|^2 g\, d\lambda \geqslant \epsilon \|f\|_2^2.$$

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

Proof. Let $U = \{x \in G : g(x) > 0\}$ and select X_0 as in Lemma (2·3). If (i) fails, there is a sequence (f_j) in $\mathfrak{T}(G)$ such that $\sup_j \text{length}(f_j) \leqslant m$, each $\text{supp}(\hat{f}_j)$ is X_0-subtransversal,

$$\lim_j \int_G |f_j|^2 g \, d\lambda = 0, \tag{1}$$

and

$$\|f_j\|_2 = 1 \quad \text{for all } j. \tag{2}$$

By passing to a subsequence we may assume that for some k in $\{1, 2, \ldots, m\}$,

$$\text{length}(f_j) = k \quad \text{for all } j.$$

We may also assume that k is *minimal*: no such sequence exists for smaller k. It is easy to see that k must be greater than 1. Each f_j has the form

$$f_j = \sum_{i=1}^{k} \hat{f}_j(\chi_{ij}) \chi_{ij}.$$

By replacing each f_j by $\overline{\chi_{1j}} f_j$, we may suppose that $\chi_{1j} = 1$ for all j. By re-ordering $\{1, 2, \ldots, k\}$ if necessary, we may suppose that there is an integer r such that $1 \leqslant r \leqslant k$ and $\{\chi_{ij} : j = 1, 2, \ldots\}$ is a finite set if and only if $1 \leqslant i \leqslant r$. By passing to further subsequences, we can also arrange for the following:

$$\text{for} \quad 1 \leqslant i \leqslant r, \quad \text{all } \chi_{ij} \text{ are equal (to } \chi_i, \text{ say)}; \tag{3}$$

$$\text{for} \quad r < i \leqslant k, \quad \text{all } \chi_{ij} \text{ are distinct}; \tag{4}$$

$$\text{for} \quad 1 \leqslant i \leqslant k, \quad \text{the limit } \lim_j \hat{f}_j(\chi_{ij}) \text{ exists (and equals } c_i, \text{ say)}. \tag{5}$$

Note that the χ_i's are distinct and that $\{\chi_1, \chi_2, \ldots, \chi_r\}$ is X_0-subtransversal. Let

$$f = \sum_{i=1}^{r} c_i \chi_i.$$

Case 1. If $r = k$, then $\lim_j f_j = f$ uniformly by (5) and (3), and hence (1) yields

$$\int_G |f|^2 g \, d\lambda = \lim_j \int_G |f_j|^2 g \, d\lambda = 0.$$

It follows that f vanishes on U, and by Lemma (2·3) f vanishes identically. On the other hand,

$$\|f\|_2^2 = \sum_{i=1}^{k} |c_i|^2 = \lim_j \sum_{i=1}^{k} |\hat{f}_j(\chi_{ij})|^2 = \lim_j \|f_j\|_2^2 = 1.$$

Case 2. Suppose that $r < k$ and $f = 0$. For each j, let

$$F_j = f_j - \sum_{i=1}^{r} \hat{f}_j(\chi_i) \chi_i = \sum_{i=r+1}^{k} \hat{f}_j(\chi_{ij}) \chi_{ij}.$$

Since $\lim_j \hat{f}_j(\chi_i) = c_i = 0$ for $i \leqslant r$, we have $\lim_j (F_j - f_j) = 0$ uniformly. It follows that

$$\lim_j \int_G |F_j|^2 g \, d\lambda = \lim_j \int_G |f_j|^2 g \, d\lambda = 0$$

and

$$\lim_j \|F_j\|_2 = \lim_j \|f_j\|_2 = 1.$$

The existence of the sequence (F_j) with these properties violates the minimality of k.

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

Case 3. Suppose that $r < k$ and f does not vanish identically. By Lemma (2·3), f does not vanish on U. Thus $fg \neq 0$ and there is a compact subset E of U such that $\int_E fg\,d\lambda \neq 0$. Let F_j be as in Case 2. By Hölder's inequality, we have

$$\left|\int_E f_j g\,d\lambda\right|^2 \leqslant \left(\int_G |f_j|^2 g\,d\lambda\right)\left(\int_G g\,d\lambda\right)$$

and so by (1), we have $\lim_j \int_E f_j g\,d\lambda = 0$. Since $\lim_j (f_j - F_j) = f$ uniformly, we conclude that

$$\lim_j \int_E F_j g\,d\lambda = \lim_j \int_E f_j g\,d\lambda - \int_E fg\,d\lambda \neq 0. \tag{6}$$

We can also write

$$\lim_j \int_E F_j g\,d\lambda = \lim_j \sum_{i=r+1}^{k} \hat{f}_j(\chi_{ij}) \int_E \chi_{ij} g\,d\lambda. \tag{7}$$

Now $\xi_E g$ belongs to $\mathfrak{L}^1(G)$ and so its Fourier transform belongs to $\mathfrak{C}_0(\mathsf{X})$. Property (4) hence implies

$$\lim_j \int_E \chi_{ij} g\,d\lambda = \lim_j (\xi_E g)\hat{}\,(\chi_{ij}^{-1}) = 0$$

for $i > r$ and thus $\lim_j \int_G F_j g\,d\lambda = 0$ by (7). This clearly contradicts (6). All three cases lead to contradictions and so (i) must hold. ∎

The next lemma is our analogue to Lemme 2 in (1); see also Lemme 6·4 (2).

(2·5) Lemma. *Let m be a positive integer, and let g be a non-zero function in $\mathfrak{C}^+(G)$. There is a finite subgroup X_0 of X and a positive number δ such that*

(i) *if $\{\chi_1, \chi_2, \ldots, \chi_k\}$ is an X_0-subtransversal subset of X with $2 \leqslant k \leqslant m$, then the determinant of the matrix $A = [\hat{g}(\chi_i \chi_j^{-1})]_{i,j=1}^k$ is greater than or equal to δ.*

Proof. Let X_0 and ϵ be as in Lemma (2·4). Let $\delta = \min(\epsilon^2, \epsilon^m)$. Since $\hat{g}$ is a positive-definite function, the matrix A is positive-definite and its determinant is a product of eigenvalues of A. It suffices then to show that $\alpha \geqslant \epsilon$ for each eigenvalue α of A. Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be an eigenvector for α: $A\mathbf{c} = \alpha\mathbf{c}$. Lemma (2·4) applies to the function $f = \sum_{i=1}^{k} c_i \chi_i$, and therefore

$$\alpha \|f\|_2^2 = \langle \alpha\mathbf{c}, \mathbf{c}\rangle = \langle A\mathbf{c}, \mathbf{c}\rangle = \sum_{i,j} \bar{c}_i c_j \hat{g}(\chi_i \chi_j^{-1})$$

$$= \int_G \sum_{i,j} \bar{c}_i c_j \bar{\chi}_i \chi_j g\,d\lambda = \int_G |f|^2 g\,d\lambda \geqslant \epsilon \|f\|_2^2. \quad ∎$$

We need one more lemma.

(2·6) Lemma. *Let P be an $FZ(G)$-set not containing 1, and let $\epsilon > 0$. There is a constant M (depending only on P and ϵ) with the following property. If $K \geqslant 1$ and if α is a bounded Hermitian function on P satisfying $\|\alpha\|_\infty \leqslant K$, then there is a measure $\nu \in \mathbf{M}_+(G)$ such that $\hat{\nu}(\chi) = \alpha(\chi)$ for $\chi \in \mathsf{P}$, $\hat{\nu}(1) = \|\nu\| \leqslant MK^2$, and $|\hat{\nu}(\chi)| \leqslant \epsilon$ for all $\chi \in \mathsf{X} \setminus (\mathsf{P} \cup \{1\})$.*

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

Proof. By (8·4) (4), there is a constant $\kappa > \epsilon$ such that: given a bounded Hermitian function β on P, there exist $g \in \mathfrak{L}_r^\infty(G)$ and $\mu_0 \in \mathbf{M}_+(G)$ such that

$$\beta(\chi) = \hat{g}(\chi) + \hat{\mu}_0(\chi) \quad \text{for} \quad \chi \in \mathsf{P},$$

and

$$\|g\|_\infty + \|\mu_0\| \leqslant \kappa \|\beta\|_\infty.$$

Setting $\mu = (\|g\|_\infty + g)\lambda + \mu_0$ as in (8·10) (4), we obtain μ in $\mathbf{M}_+(G)$ such that

$$\beta(\chi) = \hat{\mu}(\chi) \quad \text{for} \quad \chi \in \mathsf{P}, \tag{1}$$

and

$$\|\mu\| \leqslant 2\kappa \|\beta\|_\infty. \tag{2}$$

Let $\mathsf{P}_1 = \{\chi \in \mathsf{P} : \chi^2 = 1\}$ and $\mathsf{P}_2 = \mathsf{P} \setminus \mathsf{P}_1$. Let K and α be as in the lemma. By the last paragraph, there are measures ν_1 and ν_2 in $\mathbf{M}_+(G)$ such that $\hat{\nu}_j(\chi) = \alpha(\chi)$ for $\chi \in \mathsf{P}_j$, $\hat{\nu}_j(\chi) = 0$ for $\chi \in \mathsf{P} \setminus \mathsf{P}_j$, and $\|\nu_j\| \leqslant 2\kappa \|\alpha\|_\infty$, $j = 1, 2$. By (9·4) (4), there is a constant $c \geqslant 1$ (depending only on P) and measures μ_1 and μ_2 in $\mathbf{M}_+(G)$ such that $\hat{\mu}_j(\chi) = 1$ for $\chi \in \mathsf{P}_j$, $|\hat{\mu}_j(\chi)| \leqslant \epsilon/(4\kappa K)$ for $\chi \in \mathsf{X} \setminus (\mathsf{P}_j \cup \{1\})$, and $\|\mu_j\| \leqslant 4c^4 \cdot 4\kappa K/\epsilon$, $j = 1, 2$. Let $\nu = \nu_1 * \mu_1 + \nu_2 * \mu_2$ and $M = 64\kappa^2 c^4/\epsilon$. ∎

PROOF OF THEOREM (1·2). Lemma (2·1) shows that (1·2ii) holds for full FZ-sets, and (1·2i) obviously holds for full FZ-sets. Suppose now that P is a (symmetric) $FZ(G)$-set and that P is almost X_0-subtransversal for all finite subgroups X_0 of X. Let W be a non-void measurable subset of G such that $W \subset (\operatorname{int} W)^-$. We need to show that P is an $FZ(W)$-set. Let β be an Hermitian function on P such that $|\beta(\chi)| = 1$ for all $\chi \in \mathsf{P}$. By Theorem (8·7) (4), it suffices for us to find $\mu \in \mathbf{M}_+((W^-)^{-1})$ such that

$$|\hat{\mu}(\chi) - \beta(\chi)| \leqslant \tfrac{1}{2} \tag{1}$$

for all but finitely many χ in P. We may suppose that $1 \notin \mathsf{P}$.

Let g be a function in $\mathfrak{C}^+(G)$ such that $\hat{g}$ is in $l^1(\mathsf{X})$, $\operatorname{supp}(g) \subset (W^-)^{-1}$, and

$$\hat{g}(1) = \int_G g \, d\lambda = 1;$$

the existence of g follows from the regularity of the algebra of functions on G with absolutely convergent Fourier series. Let $\epsilon > 0$ be so that $\epsilon < 1$ and

$$4\epsilon \sum_{\chi \in \mathsf{X}} |\hat{g}(\chi)| \leqslant 1. \tag{2}$$

Next let $M \geqslant 1$ be the constant given by Lemma (2·6) corresponding to this ϵ. By (5·7) in (4), P is a Sidon set. Accordingly, we let m be as given by Lemma (2·2). For this m and g, let X_0 and $\delta > 0$ be as in Lemma (2·5). Let $K = \max(1, m!/\delta)$ and let Δ be a finite symmetric subset of X containing 1 and satisfying

$$4MK^2 \sum_{\chi \in \mathsf{X} \setminus \Delta} |\hat{g}(\chi)| \leqslant 1. \tag{3}$$

Now for *this* Δ, decompose P as in Lemma (2·2). Since we only need to establish (1) for all but finitely many χ in P and since P is almost X_0-subtransversal, we may suppose: Σ in (2·2) is void, P is X_0-subtransversal, and $\mathsf{P} \cap \Delta = \varnothing$.

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

We now define α on $\mathsf{P} = \bigcup_{i\in I} \mathsf{P}_i$ so that $\|\alpha\|_\infty \leqslant K$ as follows. If P_i has only one element χ_i, we put $\alpha(\chi_i) = \beta(\chi_i)$. In this case,

$$\chi_i^2 = 1 \quad \text{and} \quad \alpha(\chi_i) = \beta(\chi_i) \quad \text{is a real number.} \tag{4}$$

Otherwise, $\mathsf{P}_i = \{\chi_{i1}, \dots, \chi_{ik}\}$ where $2 \leqslant k \leqslant m$ by (2·2ii), and we consider the linear system

$$\sum_{s=1}^{k} x_s \hat{g}(\chi_{ir}\chi_{is}^{-1}) = \beta(\chi_{ir}) \quad (1 \leqslant r \leqslant k). \tag{5}$$

By Lemma (2·5), the determinant of this system is $\geqslant \delta$, and so its unique solution $(x_1, \dots, x_k)$ satisfies $|x_s| \leqslant m!/\delta$ for $1 \leqslant s \leqslant k$. We define $\alpha(\chi_{is}) = x_s$ for $s = 1, 2, \dots, k$, so that

$$\sum_{s=1}^{k} \alpha(\chi_{is}) \hat{g}(\chi_{ir}\chi_{is}^{-1}) = \beta(\chi_{ir}) \quad (1 \leqslant r \leqslant k). \tag{6}$$

Since P_i is symmetric (2·2i), there is a permutation π of $\{1, 2, \dots, k\}$ such that

$$\chi_{i,\pi(s)} = \chi_{is}^{-1}$$

for all s in $\{1, 2, \dots, k\}$. Next we replace all s and r in (6) by $\pi(s)$ and $\pi(r)$, respectively, and use the fact that $\hat{g}$ and β are Hermitian; we find that $x_s = \overline{\alpha(\chi_{is}^{-1})}$ $(s = 1, 2, \dots, k)$ is also a solution of (5). Since the solution of (5) is unique, we infer that

$$\alpha(\chi) = \overline{\alpha(\chi^{-1})} \quad \text{for all} \quad \chi \in \mathsf{P}_i. \tag{7}$$

The function α just defined on P is bounded by K, and α is Hermitian by (4) and (7).

By Lemma (2·6), there is a measure $\nu \in \mathbf{M}_+(G)$ such that $\hat{\nu}(\chi) = \alpha(\chi)$ for $\chi \in \mathsf{P}$, $\hat{\nu}(1) \leqslant MK^2$, and $|\hat{\nu}(\chi)| \leqslant \epsilon$ for $\chi \in \mathsf{X}\setminus(\mathsf{P} \cup \{1\})$. Finally, let $\mu = g\nu$; clearly μ is in $\mathbf{M}_+((W^-)^{-1})$. It remains for us to show that (1) holds for all $\chi \in \mathsf{P}$; note that we removed the 'bad' finite part of P from P after formula (3). Consider a fixed $\chi_0 \in \mathsf{P}_i \subset \mathsf{P}$. It is easy to see that

$$\hat{\mu}(\chi_0) = \sum_{\chi\in\mathsf{X}} \hat{\nu}(\chi)\hat{g}(\chi_0\chi^{-1}).$$

We break this sum into three pieces: S_1 is the sum over P_i; S_2 is the sum over $\chi_0\Delta\setminus\mathsf{P}_i$; S_3 is the sum over the remainder of X. An application of (6) or (4) shows that $S_1 = \beta(\chi_0)$. An application of (2), together with (2·2iii), yields $|S_2| \leqslant \frac{1}{4}$ and the inequality $|S_3| \leqslant \frac{1}{4}$ follows from (3). Hence we have $|\hat{\mu}(\chi_0) - \beta(\chi_0)| = |S_2 + S_3| \leqslant \frac{1}{2}$, and so (1) holds. ∎

3. *Some applications.* We show that infinite full FZ-sets exist in abundance.

(3·1) THEOREM. *If Δ is a countably infinite dissociate set in X, then $\mathsf{P} = \Delta \cup \Delta^{-1}$ is a full FZ-set.*

Proof. We first show

$$\text{for} \quad \phi \in \mathsf{X}\setminus\{1\}, \quad \chi^2 = \phi \quad \text{for at most two} \quad \chi \in \mathsf{P}. \tag{1}$$

If (1) fails, there exist χ_1 and χ_2 in P so that $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_2^{-1}$, and $\chi_1^2 = \chi_2^2 \neq 1$. There are ϕ_1, ϕ_2 in Δ and δ_1, δ_2 in $\{-1, 1\}$ such that $\phi_j^{\delta_j} = \chi_j$ for $j = 1, 2$. Then we have $\phi_1 \neq \phi_2$ and $\phi_1^{2\delta_1}\phi_2^{-2\delta_2} = 1$. Since $\phi_1^{2\delta_1} \neq 1$, this contradicts the fact that Δ is a dissociate set; see Definition (37·12) (5).

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

Now we prove that P is a full FZ-set. The set P is an $FZ(G)$-set by (10·4) (4). By Theorem (1·2), we need only show that P is almost X_0-subtransversal for all finite subgroups X_0 of X. Assume the contrary. Then infinitely many cosets of X_0 intersect P in at least two points. Thus there exist distinct $\chi_1, \chi_2, \ldots, \psi_1, \psi_2, \ldots$ in P so that $\chi_k \psi_k^{-1} \in X_0$ for all k. Since X_0 is finite, we may suppose that

$$\chi_k \psi_k^{-1} = \chi_1 \psi_1^{-1} \quad \text{for all} \quad k. \tag{2}$$

Because of (1), we may also suppose that

$$\chi_k^2 \quad \text{is not in} \quad X_0 \backslash \{1\} \quad \text{for all} \quad k. \tag{3}$$

Select $k > 1$ so that neither χ_k nor ψ_k belongs to $\{\chi_1, \psi_1, \chi_1^{-1}, \psi_1^{-1}\}$. Since neither χ_k^2 nor χ_1^2 belongs to $X_0 \backslash \{1\}$ by (3), it follows that *no two elements in* $\{\chi_1, \psi_1, \chi_k, \psi_k\}$ *are inverses to each other*. If, for example, we were to have $\chi_1 = \psi_1^{-1}$, then $\chi_1^2 = \chi_1 \psi_1^{-1}$ would be in X_0, so that $\chi_1^2 = 1$ and hence $\chi_1 = \psi_1$, a contradiction. There exist ϕ_j in Δ and δ_j in $\{-1, 1\}$ for $j = 1, 2, 3, 4$, such that

$$\phi_1^{\delta_1} = \chi_1, \quad \phi_2^{\delta_2} = \psi_1, \quad \phi_3^{\delta_3} = \chi_k, \quad \text{and} \quad \phi_4^{\delta_4} = \psi_k.$$

The italicized statement above shows that the ϕ_j's are distinct. From (2) we obtain

$$1 = \chi_k \psi_k^{-1} \chi_1^{-1} \psi_1 = \phi_3^{\delta_3} \phi_4^{-\delta_4} \phi_1^{-\delta_1} \phi_2^{\delta_2}.$$

Again this contradicts the fact that Δ is dissociate. ∎

(3·2) COROLLARY. *If G is an infinite compact Abelian group, then* X *contains an infinite full FZ-set* P.

Proof. It is proved, though not stated, in (37·18) (5) that X contains a countably infinite dissociate set Δ. Let $P = \Delta \cup \Delta^{-1}$. ∎

(3·3) EXAMPLE. We show that X might contain an infinite symmetric subset P such that every full FZ-subset is finite.

Select G so that X has a character ψ of order 3 and an infinite sequence (χ_k) of characters of order 2. For example, G can be taken to be $Z(3) \times Z(2)^{\aleph_0}$. Let

$$\Delta = \{\psi \chi_k : k = 1, 2, \ldots\},$$

so that $\Delta^{-1} = \{\psi^2 \chi_k : k = 1, 2, \ldots\}$, and let $P = \Delta \cup \Delta^{-1}$. Let $X_0 = \{1, \psi, \psi^2\}$. No infinite symmetric subset of P is almost X_0-subtransversal. Hence by (1·2), P contains no infinite full FZ-subsets.

Drury (3) has shown that the union of two Sidon sets is again a Sidon set. The same technique shows that the union of two $FZ(G)$-sets is again an $FZ(G)$-set; see (9·1) (4). By (1·3), the same property obviously holds for full FZ-sets if G is connected. We now show that this result does not generalize to all G.

(3·4) EXAMPLE. Suppose that $\{\chi \in X : \chi^2 = 1\}$ is infinite. Then X contains full FZ-sets P_1 and P_2 such that $P_1 \cup P_2$ is not a full FZ-set.

Let $Y = \{\chi \in X : \chi^2 = 1\}$. Since Y is a direct sum of two-element groups by (A. 25) (5), there exist a two-element subgroup $X_0 = \{1, \chi_0\}$ and an infinite subgroup Y_0 of Y

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press

such that $X_0 \cap Y_0 = \{1\}$. As shown in (37·18) (5), Y_0 contains an infinite dissociate set P_1. It is easy to see that $P_2 = \chi_0 P_1$ is also dissociate. Since P_1 and P_2 are symmetric, Theorem (3·1) shows that they are full FZ-sets. The union $P_1 \cup P_2 = X_0 P_1$ is not almost X_0-subtransversal and so it is not a full FZ-set by (1·2).

REFERENCES

(1) Déchamps-Gondim, M. Compacts associés à un ensemble de Sidon. *C.R. Acad. Sci. Paris, Sér. A*, **271** (1970), 590–592.

(2) Déchamps-Gondim, M. Ensembles de Sidon topologiques. *Ann. Inst. Fourier (Grenoble)* 22, **3** (1972).

(3) Drury, S. W. Sur les ensembles de Sidon. *C.R. Acad. Sci. Paris, Sér. A*, **271** (1970), 162–163.

(4) Edwards, R. E., Hewitt, E. and Ross, K. A. Lacunarity for compact groups. III. *Studia Math.* **44** (1972), 19–66.

(5) Hewitt, E. and Ross, K. A. *Abstract harmonic analysis*, 2 vols. (Berlin–Heidelberg–New York: Springer-Verlag, 1963, 1970.)

https://doi.org/10.1017/S0305004100047460 Published online by Cambridge University Press