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An elastostatic circle theorem

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Abstract. An isotropic infinite plane containing a circular inhomogeneity is subjected to an arbitrary loading condition. Assuming that the unperturbed elastic field is known, it is proved that the disturbed Papkovich potentials are expressible in terms of the potentials for the homogeneous plane. This knowledge can save considerable labour in the actual construction of the elastic field for the composite plane. The results are applied to some particular problems which include that of an arc displacement discontinuity in the composite plane. This example has as yet not been solved explicity by other methods and reveals that the displacement field at the cavity surface is independent of the elastic constants.

Introduction. In a recent paper (3), two-dimensional classical elasticity theory was employed by the author to show that the elastic field in a plane containing a circular inhomogeneity is expressible in terms of the corresponding results for the homogeneous plane. The analysis was restricted to the case in which the homogeneous plane solution is harmonic.

Here we treat the general case when the composite plane is subjected to an arbitrary body force distribution or local transformation. The analysis is based upon the stress–function approach of Papkovich and Neuber. It is shown that within the class of elastic fields distributed according to prescribed cosine- and sine-laws, the potentials generating the composite plane solution are readily deducible from those for the homogeneous plane. This knowledge can pave the way for the avoidance of a great deal of mathematical analysis where perhaps rigour rather than potential application is often the first consideration. Full appreciation of such a knowledge will also prevent duplication of analysis.

The results are applied to a dislocation problem in which a uniform radial displacement discontinuity occurs across an arc in the composite plane. It is shown that the displacement field at the cavity surface is independent of the elastic constants. We also briefly cite the fundamental case in which the composite body is in a plane state of pure shear at infinity, the case of an arbitrary uniform plane field of stress at infinity being obtainable by the superposition of the trivial solution corresponding to that of an all-round tension at infinity. We also reconsider the problem of a transforming circular inclusion in a plane containing a circular inhomogeneity. Using complex variable methods Bhargava and Kapoor (7) recently solved this problem in a somewhat

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less direct method than the one adopted here. To avoid duplication, we do not record the solution but show how it may be deduced from the homogeneous plane solution.

The general conclusion drawn in this paper is fundamentally similar to one by Milne-Thomson (13) in connexion with the flow of fluid over circular obstacles. An electrostatic form of the theorem of Milne-Thomson is also available. This gives the complex potential of a field in the presence of a earthed conducting circular cylinder in terms of the original potential. A general sphere theorem in potential theory is also available (12).

Perhaps it will prove possible to extend the present investigation to simple domains which may be mapped onto a circular disc by means of rational functions.

Governing equations. Let (x_1, x_2) be the rectangular Cartesian coordinates and (r, θ) the plane polar coordinates. These coordinates are connected through

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

Let the entire two-dimensional plane be occupied by an isotropic material with elastic constants $\mu^{(1)}$ and $\nu^{(1)}$ except for a circular region $r < a$ which is occupied by a different material with elastic constants $\mu^{(2)}$ and $\nu^{(2)}$. The materials are assumed to be perfectly bonded at $r = a$.

As proved in courses on the mathematical theory of elasticity, the cartesian tensor form of the field equations associated with two-dimensional elastostatics is given by

$$(q-1)\,U_{i,jj} + U_{j,ji} = 0, \tag{1}$$

$$\sigma_{ij} = \mu\left(U_{i,j} + U_{j,i} + \frac{2-q}{q-1}U_{k,k}\delta_{ij}\right) \quad (i,j = 1,2), \tag{2}$$

where

$$q = \begin{cases} 2/(1-\nu) & \text{for plane strain,} \\ 2/(1+\nu) & \text{for plane stress.} \end{cases} \tag{3}$$

In (1)–(3), (U_1, U_2) stands for the elastic displacement vector and σ_{ij} for the components of the stress tensor. The constants μ and ν stand for the shear modulus and Poisson's ratio, respectively, while δ_{ij} stands for the Kronecker delta. The customary convention of summation over repeated indices is also implied.

Equations (1) and (2) are to be accompanied by the perfect bond continuity conditions

$$\left.\begin{aligned} U_r^{(1)} &= U_r^{(2)}, \quad U_\theta^{(1)} = U_\theta^{(2)}, \\ \sigma_{rr}^{(1)} &= \sigma_{rr}^{(2)}, \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} \end{aligned}\right\} \tag{4}$$

at $r = a$.

When referred to the polar coordinates, the Laplacian operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{5}$$

while the complete solution of the field equations is expressible in terms of the Papkovich potentials ψ_0, ψ_1 and ψ_2 in the form

$$U_r = \frac{\partial\psi}{\partial r} - 2q(\psi_1\cos\theta + \psi_2\sin\theta),$$

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$$U_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} + 2q(\psi_1 \sin\theta - \psi_2 \cos\theta),$$

$$\frac{\sigma_{rr}}{2\mu} = \frac{\partial^2 \psi}{\partial r^2} + (2-q)\left(\frac{\partial \psi_1}{\partial \theta}\sin\theta - \frac{\partial \psi_2}{\partial \theta}\cos\theta\right)\frac{1}{r} - (2+q)\left(\frac{\partial \psi_1}{\partial r}\cos\theta + \frac{\partial \psi_2}{\partial r}\sin\theta\right), \tag{6}$$

$$\frac{\sigma_{\theta\theta}}{2\mu} = \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - (2-q)\left(\frac{\partial \psi_1}{\partial r}\cos\theta + \frac{\partial \psi_2}{\partial r}\sin\theta\right) + (2+q)\left(\frac{\partial \psi_1}{\partial \theta}\sin\theta - \frac{\partial \psi_2}{\partial \theta}\cos\theta\right)\frac{1}{r},$$

$$\frac{\sigma_{r\theta}}{2\mu} = \frac{1}{r}\frac{\partial^2 \psi}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} + q\left(\frac{\partial \psi_1}{\partial r}\sin\theta - \frac{\partial \psi_2}{\partial r}\cos\theta\right) - q\left(\frac{\partial \psi_1}{\partial \theta}\cos\theta + \frac{\partial \psi_2}{\partial \theta}\sin\theta\right)\frac{1}{r},$$

$$\psi = \psi_0 + r(\psi_1 \cos\theta + \psi_2 \sin\theta) \tag{7}$$

provided

$$\nabla^2 \psi_k = 0 \quad (k = 0, 1, 2). \tag{8}$$

Our main objective is to compare the response of the composite plane with that of the homogeneous plane while both are subjected to exactly the same loading condition. It will be shown that the following proposition holds:

Any set of Papkovich potentials for the plane containing a circular inhomogeneity is related to the corresponding set for the homogeneous plane. If the normal stresses are distributed according to $\cos n\theta$ ($\sin n\theta$) *and the shear stress according to* $\sin n\theta$ ($\cos n\theta$) $n \geqslant 1$, *the relation remains unchanged irrespective of the nature of the loading condition.*

Proof in the case of exterior loading. We suppose that the unperturbed elastic field $(S) \equiv (U_i, \sigma_{ij})$ for the homogeneous plane is known. Here we assume that this solution has been obtained by subjecting the matrix to any body force distribution or local transformation. No restriction is placed on the components U_i which need not vanish at infinity. If they contain logarithmic terms the analysis developed here may still be applied as such terms can be expanded in terms of Fourier series.

To prove the proposition, we must assume that the displacement components U_i of the given solution (S) take the general form, that is, they are biharmonic. Since the Papkovich stress functions are complete, (11), harmonic functions ψ_0, ψ_1 and ψ_2 then exist such that their substitution into (6) gives expressions for the components of (S). Let these functions admit the representations

$$\left.\begin{aligned} \psi_0 &= a^2 \sum_{n=1}^{\infty} n^{-1} F_n r^n \cos n\theta, \\ \psi_1 &= \sum_{n=0}^{\infty} G_n r^{n+1} \cos(n+1)\theta, \\ \psi_2 &= \sum_{n=0}^{\infty} G_n r^{n+1} \sin(n+1)\theta \end{aligned}\right\} \tag{9}$$

in the neighbourhood of the inhomogeneity, where F_n and G_n are known superposition coefficients. Equations (9) imply that the normal and shear stresses are assumed distributed according to cosine- and sine-laws, respectively. The case of elastic field which is unsymmetrical with respect to the coordinate line $\theta = 0$ does not require any new methods of solution. The homogeneous plane solution (U_i, σ_{ij}) may now be obtained from the insertion of (9) into (6).

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To obtain the solution of the residual problem, we first set

$$(\mathbf{U}^{(1)}, \boldsymbol{\sigma}^{(1)}) = (\mathbf{U} + \mathbf{U}^*, \boldsymbol{\sigma} + \boldsymbol{\sigma}^*) \tag{10}$$

for the total field in $r > a$ and write $(\mathbf{U}^{(2)}, \boldsymbol{\sigma}^{(2)})$ for the total field in $r < a$. We then construct the additional Papkovich functions in the following form:

For $r > a$

$$\left.\begin{aligned} \psi_0^* &= a^2 \Sigma A_n n^{-1} r^{-n} \cos n\theta, \\ \psi_1^* &= \Sigma B_n r^{1-n} \cos(1-n)\theta, \\ \psi_2^* &= \Sigma B_n r^{1-n} \sin(1-n)\theta; \end{aligned}\right\} \tag{11}$$

and for $r < a$

$$\left.\begin{aligned} \psi_0^{(2)} &= a^2 \Sigma C_n n^{-1} r^{-n} \cos n\theta, \\ \psi_1^{(2)} &= \Sigma D_n r^{n+1} \cos(n+1)\theta, \\ \psi_2^{(2)} &= \Sigma D_n r^{n+1} \sin(n+1)\theta, \end{aligned}\right\} \tag{12}$$

where the summation sign is here and hereforth, unless otherwise stated, held to mean from n equals one to infinity. The trivial solution associated with the term $n = 0$ is readily deducible from the formulae for the Lame problem and actually corresponds to a plane isotropic state of stress at infinity.

The coefficients A_n, B_n, C_n and D_n are to be determined from the interface conditions (4), using formulae (6) for the components of elastic field. The satisfaction of these conditions implies a system of four linear equations whose solution is given by

$$\left.\begin{aligned} A_n &= (\alpha(n-1)F_n + (\alpha(n-1)(n+1) + 1 - \beta)G_n)a^{2n}, \\ B_n &= -\alpha(F_n + (n+1)G_n)a^{2n}, \\ C_n &= g^{-1}((1-\alpha)F_n + (1-\alpha-\beta)(n+1)G_n), \\ D_n &= g^{-1}\beta G_n, \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} \alpha &= (1-g)(1+g(2q^{(1)}-1))^{-1}, \\ \beta &= 2gq^{(1)}(g+2q^{(2)}-1)^{-1}, \\ g &= \mu^{(2)}/\mu^{(1)}. \end{aligned}\right\} \tag{14}$$

Having found the coefficients A_n, B_n, C_n and D_n in terms of F_n and G_n, we are now in a position to compare the series representations of the stress functions for the composite plane with the representations of the stress functions for the homogeneous plane. To do this, we merely insert (13) into (11) and (12) and compare the resulting expressions with (9). On performing straightforward algebraic manipulations, one concludes easily that the potentials for the composite plane are connected with the potentials for the homogeneous plane according to

$$\left.\begin{aligned} \psi_0^{(1)}(r,\theta) &= \psi_0(r,\theta) - \alpha\left(1 + r\frac{\partial}{\partial r}\right)\psi_0(\rho,\theta) - \alpha a^2 r \frac{\partial}{\partial r}(\psi_1(\rho,\theta)) - a^2(1-\alpha-\beta)\frac{\partial}{\partial\theta}(\psi_2(\rho,\theta)), \\ \psi_1^{(1)}(r,\theta) &= \psi_1(r,\theta) + \alpha r \frac{\partial}{\partial r}(\psi_0(\rho,\theta)) - \alpha a^2\left(2 - r\frac{\partial}{\partial r}\right)\psi_1(\rho,\theta), \\ \psi_2^{(1)}(r,\theta) &= \psi_2(r,\theta) - \alpha\frac{\partial}{\partial\theta}(\psi_0(\rho,\theta)) + \alpha a^2\left(2 - r\frac{\partial}{\partial r}\right)\psi_2(\rho,\theta); \end{aligned}\right\} \tag{15}$$

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$$\left.\begin{aligned}\psi_0^{(2)}(r,\theta) &= (1-\alpha)g^{-1}\psi_0(r,\theta)+(1-\alpha-\beta)g^{-1}a^2\big(\psi_1(r,\theta)-\int\psi_2(r,\theta)\,d\theta\big),\\ \psi_1^{(2)}(r,\theta) &= \beta g^{-1}\psi_1(r,\theta),\\ \psi_2^{(2)}(r,\theta) &= \beta g^{-1}\psi_2(r,\theta),\end{aligned}\right\} \tag{16}$$

where α, β and g are once again given by (14) while

$$\rho = a^2/r. \tag{17}$$

Therefore, if we have previously determined the potentials for the unperturbed elastic field, the potentials generating the composite plane solution are readily deducible by means of (15) and (16). It is worthwhile noting that these equations simplify considerably in the special case when the unperturbed elastic field is harmonic. For then the potentials $\psi_k^{(2)}$, $k = 1, 2$ vanish identically, while the only non-vanishing potential $\psi_0^{(2)}$ in the inhomogeneity is a constant multiple of the undisturbed potential ψ_0.

By a procedure similar to that used in obtaining the foregoing results, we can now treat the internal problem, in which the loading is applied to the embedded disc only. However, there is no need for this, since the analysis for this case follows from above by replacing n by $-n$ and reversing the roles of the superscripts 1 and 2. For a detailed discussion of a special case of this problem, reference may be made to the author's doctoral thesis where a correspondence relation was established.

Some applications.

1. *Displacement discontinuity across an arc in a plane containing a circular inhomogeneity.* The two-dimensional problem of a vertical displacement discontinuity in a semi-infinite isotropic plane has been thoroughly considered by Berry (4) who showed that the displacement field at the stress-free surface is independent of the elastic constants. In a second paper, Berry and Sales (5) considered the same physical problem for an orthotropic half-plane. The method of solution was essentially the same as before. The three-dimensional counterpart of the line problem, corresponding to a vertical displacement discontinuity over a bounded rectangular region, was first considered by Rongved (14) for the infinite isotropic solid and by Rongved and Frasier (15) for the semi-infinite solid. The solution of the problem of such a displacement discontinuity in a transversely isotropic half-space was discussed in a third paper by Berry and Sales (6). The space problems were reduced to standard problems in potential theory.

Here we consider an infinite isotropic plane which contains a circular inhomogeneity of radius a and possesses a uniform discontinuity ϵ in its radial displacement component across the arc $|\theta| < \omega$ of a circle of radius $b(> a)$. Our main objective is to use our previous analysis to show how the composite plane solution may be deduced from a knowledge of the homogeneous plane solution. If such a deduction is possible, then it is reasonable to expect that the solutions to the half-space (or half-plane) problems alluded to above are directly deducible from the solutions for the infinite solid in both cases of complete and transversal isotropy. That this is indeed the case has been confirmed by the present author in a paper to follow the present publication, the more general problem of bonded transversely isotropic half-spaces being considered.

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Returning now to the arc problem, a straightforward analysis leads to the conclusion that the potentials generating the homogeneous plane solution admit the representations

$$\left.\begin{aligned}\psi_0 &= bk\Sigma(n+2)\,n^{-2}(r/b)^n\cos n\theta\sin n\omega,\\ \psi_1 &= -k\Sigma n^{-1}(r/b)^{n+1}\cos(n+1)\,\theta\sin n\omega,\\ \psi_2 &= -k\Sigma n^{-1}(r/b)^{n+1}\sin(n+1)\,\theta\sin n\omega,\end{aligned}\right\} \tag{18}$$

where

$$k = \epsilon/2\pi q \tag{19}$$

the expressions in (18) being valid for $r < b$.

Now, let

$$\left.\begin{aligned} r_{1,2}^2 &= (x-x_0)^2+(y\mp y_0)^2,\\ \theta_{1,2} &= \arctan\left((y\mp y_0)/(x-x_0)\right),\\ x_0 &= b\cos\omega,\quad y_0 = b\sin\omega.\end{aligned}\right\} \tag{20}$$

Then with the aid of known trigonometric relations and the following formulae which hold when $r < b$

$$\left.\begin{aligned}\theta_1 &= \omega-\Sigma n^{-1}(r/b)^n\sin n(\theta-\omega),\\ \log r_1 &= \log b-\Sigma n^{-1}(r/b)^n\cos n(\theta-\omega),\end{aligned}\right\} \tag{21}$$

one concludes easily that the Papkovich potentials defined by (18) are expressible in terms of elementary functions in the form

$$\left.\begin{aligned}\psi_0 &= \tfrac{1}{2}kba^{-2}\left(\theta_1-\theta_2-2\omega-2\int\log(r_1/r_2)\,d\theta\right),\\ \psi_1 &= \tfrac{1}{2}krb^{-1}((\theta_2-\theta_1-2\omega)\cos\theta+\sin\theta\log(r_1/r_2)),\\ \psi_2 &= \tfrac{1}{2}krb^{-1}((\theta_2-\theta_1-2\omega)\sin\theta+\cos\theta\log(r_2/r_1)).\end{aligned}\right\} \tag{22}$$

The stress functions for the composite plane are now computable from the insertion of (22) into (15) and (16). Equations (6) then permit one to write down the components of elastic field in the composite plane. A remarkable feature of this solution concerns the manner in which the elastic constants $\mu^{(m)}$ and $\nu^{(m)}$ $(m = 1, 2)$ enter the expressions for the displacements and stresses. It is easily shown that the stress field depends on $\mu^{(1)}/q^{(1)}$ merely proportionally and on the four elastic constants by way of the composite parameters α and β only. This is in agreement with a recent theorem of Dundurs(9). As for the displacement field, there is no reduction in its dependence on the elastic constants for any arbitrary combination of the materials. However, in the special case of a circular hole, the displacements on $r = a$ are given by

$$\left.\begin{aligned}U_r^{(1)} &= \epsilon b(T_1-T_2)/\pi a+\epsilon(b^2-a^2)\,(R_2^{-2}\sin(\theta+\omega)-R_1^{-2}\sin(\theta-\omega))/2\pi,\\ U_\theta^{(1)} &= \epsilon(b^2-a^2)\,(\log(R_1/R_2))/\pi ab+\epsilon(b^2-a^2)\,(a-b\cos(\theta-\omega))/2\pi bR_1^2\\ &\quad -\epsilon(b^2-a^2)\,(a-b\cos(\theta+\omega))/2\pi bR_2^2,\end{aligned}\right\} \tag{23}$$

where

$$\left.\begin{aligned}T_{1,2} &= \tan^{-1}\left((a\sin\theta\mp b\sin\omega)/(a\cos\theta-b\cos\omega)\right),\\ R_{1,2}^2 &= a^2+b^2-2ab\cos(\theta\mp\omega).\end{aligned}\right\} \tag{24}$$

Equations (23) reveal that the displacement field at the cavity surface is independent of the elastic constants. A consideration of the method of the general analysis shows that this conclusion is valid for the class of problems whose potentials are representable

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in the form (15) and (16), provided that the original potentials ψ_0, ψ_1 and ψ_2 are all directly proportional to q^{-1}. The next example further illustrates this point.

2. *The transforming circular inclusion in a plane containing a circular inhomogeneity.* Let a circular subregion of an infinite isotropic plane containing a central circular inhomogeneity of radius a be undergoing a general transformation. For convenience the inclusion is centred at $(b, 0)$, $b \geqslant a+1$, and its radius is taken as unity. Suppose that the inclusion tends to undergo the arbitrary uniform uniaxial expansions

$$e_{xx} = \epsilon_1, \quad e_{yy} = \epsilon_2$$

so that any point within it tends to undergo displacement components

$$U_1 = (x-b)\,\epsilon_1, \quad U_2 = y\epsilon_2.$$

Then, guided by the boundary conditions at the inclusion interface, the Papkovich potentials generating the homogeneous plane solution turn out to be

$$\left.\begin{aligned} 2q\psi_0 &= (\epsilon_1+\epsilon_2)\log R + \tfrac{1}{4}(\epsilon_1-\epsilon_2)\,((x-b)^2-y^2)\,R^{-4}, \\ 2q\psi_1 &= -\tfrac{1}{2}(\epsilon_1-\epsilon_2)\,(x-b)\,R^{-2}, \\ 2q\psi_2 &= \tfrac{1}{2}(\epsilon_1-\epsilon_2)\,yR^{-2}, \end{aligned}\right\} \tag{25}$$

for $R > 1$; and

$$\left.\begin{aligned} 2q\psi_0 &= -\tfrac{1}{4}(\epsilon_1-\epsilon_2)\,((x-b)^2-y^2), \\ 2q\psi_1 &= \tfrac{1}{2}(\epsilon_1+\epsilon_2)\,(x-b), \\ 2q\psi_2 &= \tfrac{1}{2}(\epsilon_1+\epsilon_2)\,y, \end{aligned}\right\} \tag{26}$$

for $R < 1$, where R is measured from the inclusion centre. When $r < b$, the expressions in (25) have the Fourier series repesentations

$$\left.\begin{aligned} 2q\psi_0 &= -(\epsilon_1+\epsilon_2)\,\Sigma n^{-1}(r/b)^n \cos n\theta + \tfrac{1}{4}(\epsilon_1-\epsilon_2)\,b^{-2}\Sigma(n+1)\,(r/b)^n \cos n\theta, \\ 2q\psi_1 &= \tfrac{1}{2}(\epsilon_1-\epsilon_2)\,b^{-1}\Sigma(r/b)^{n+1}\cos(n+1)\,\theta, \\ 2q\psi_2 &= \tfrac{1}{2}(\epsilon_1-\epsilon_2)\,b^{-1}\Sigma(r/b)^{n+1}\sin(n+1)\,\theta. \end{aligned}\right\} \tag{27}$$

A straightforward comparison of (27) with equations (9) in the general analysis shows that

$$\left.\begin{aligned} F_n &= \tfrac{1}{2}a^{-2}q^{-1}(\tfrac{1}{4}b^{-2}(\epsilon_1-\epsilon_2)\,(n+1)\,n-(\epsilon_1+\epsilon_2))\,b^{-n}, \\ G_n &= \tfrac{1}{4}q^{-1}(\epsilon_1-\epsilon_2)\,b^{-n-2}. \end{aligned}\right\} \tag{28}$$

Therefore the general equations (15) and (16) which permit one to write down immediately the potentials generating the composite plane solution remain in force for the present problem. We merely substitute (25) into (15) and (16) to obtain the desired potentials. To avoid duplication, and because of their excessive length, the final expressions for the components of elastic field will not be recorded here, since Bhargava and Kapoor (7) recently considered this problem with the aid of complex variable methods. However, they did not remark upon the possibility of expressing the complex potentials for the composite plane in terms of those for the homogeneous plane, not even in the special case $\epsilon_1 = \epsilon_2$ when the inclusion is undergoing a purely dilatational transformation. It is interesting to point out that the *explicit* relations connecting the

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elastic fields before and after the introduction of the inhomogeneity are available in this case:

$$\left.\begin{aligned}
U_r^{(1)}(r,\theta) &= U_r(r,\theta) + ((1-\alpha)\,g^{-1}-1)\,U_r(\rho,\theta) + \alpha r^{-1}(a^2-r^2)\frac{\partial}{\partial r}(U_r(\rho,\theta)),\\
U_\theta^{(1)}(r,\theta) &= U_\theta(r,\theta) + ((1-\alpha)\,g^{-1}-1)\,U_\theta(\rho,\theta) - \alpha r^{-1}(a^2-r^2)\frac{\partial}{\partial r}(U_\theta(\rho,\theta)),\\
\sigma_{rr}^{(1)}(r,\theta) &= \sigma_{rr}(r,\theta) + \alpha a^2 r^{-4}(3a^2-4r^2)\,\sigma_{rr}(\rho,\theta) + \alpha a^2 r^{-3}(r^2-a^2)\frac{\partial}{\partial r}(\sigma_{rr}(\rho,\theta)),\\
\sigma_{\theta\theta}^{(1)}(r,\theta) &= \sigma_{\theta\theta}(r,\theta) + 3\alpha(a/r)^4\,\sigma_{\theta\theta}(\rho,\theta) + \alpha a^2 r^{-3}(r^2-a^2)\frac{\partial}{\partial r}(\sigma_{\theta\theta}(\rho,\theta)),\\
\sigma_{r\theta}^{(1)}(r,\theta) &= \sigma_{r\theta}(r,\theta) - \alpha(a/r)^4\,\sigma_{r\theta}(\rho,\theta) + \alpha a^2 r^{-4}(r^2-a^2)\frac{\partial}{\partial \theta}(\sigma_{rr}(\rho,\theta)),
\end{aligned}\right\} \tag{29}$$

$$(U_i^{(2)}, \sigma_{ij}^{(2)}) = (1-\alpha)\,(g^{-1}U_i, \sigma_{ij}). \tag{30}$$

It follows from equations (29) and (30) that the displacements and stresses on $r = a$ are constant multiples of the corresponding results in the homogeneous plane, which might be of interest to the stress analysts. Specifically,

$$\left.\begin{aligned}
U_r^{(1)} &= U_r^{(2)} = (1-\alpha)\,(\epsilon_1+\epsilon_2)\,(a-b\cos\theta)/2gq^{(1)}(a^2+b^2-2ab\cos\theta),\\
U_\theta^{(1)} &= U_\theta^{(2)} = (1-\alpha)\,(\epsilon_1+\epsilon_2)\,b\sin\theta/2gq^{(1)}(a^2+b^2-2ab\cos\theta)
\end{aligned}\right\} \tag{31}$$

when $r = a$. On recalling the definition of α in (14) equations (31) reveal that the displacement field on $r = a$ is independent of the elastic constants when the region $r < a$ is occupied by the empty space. This conclusion is easily shown to be equally valid in the general case $\epsilon_1 \neq \epsilon_2$. Bhargava and Kapoor did not give the explicit expressions for the displacement field and so overlooked this point.

Other problems for which equations (15) and (16) may be applied are those appropriate to the following loading conditions:

(a) Pressure (shear) discontinuity across an arc in the composite plane.

(b) Concentrated couple at an arbitrary location.

(c) A plane state of pure shear at infinity.

The limiting case of case (a) does not yield the solution of the problem of an isolated force in the composite plane, which requires a separate treatment and has actually been thoroughly considered by Dundurs and Hetenyi (**10**). Similar uniqueness questions arise in three-dimensional elasticity (**16**, **17**). Therefore, it appears that the Kelvin's definition for a concentrated force applied at an internal point requires a reconsideration.

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