Packing a rectangle with $m \times (m + 1)$ rectangles

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1. *Introduction*

We consider the packing of rectangles of dimension $m \times (m + 1)$ — where m is a natural number — into a larger rectangle. More specifically, we consider the following problem: What is the smallest area of a rectangle into which rectangles of dimensions 1×2, 2×3, 3×4,… , $n \times (n + 1)$ will fit without overlap?

Unlike the corresponding problem for squares of areas 1^2, 2^2, 3^2, … , n^2 (see [1]), where there is no known non-trivial example of an exact fit into a rectangle, in many cases we can achieve an exact fit for our set of $m \times (m + 1)$ rectangles. Intuitively, this is because each $m \times (m + 1)$ rectangle has two possible orientations, which considerably increases the chances of an exact fit. As in [1], we make the (possibly unnecessary) assumption that the sides of each $m \times (m + 1)$ rectangle are parallel to the sides of the bounding rectangle, whose dimensions are integral. For any given n, we consider two solutions to our problem to be distinct only if the bounding rectangles have different dimensions (but equal area).

2. *Solutions for small n*

For small values of n, the problem is easy to solve by hand and an exact fit can readily be obtained. We give the area of the bounding rectangle in parentheses:

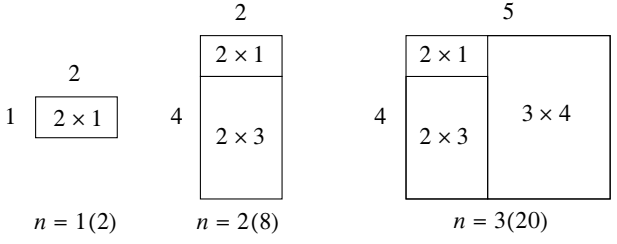
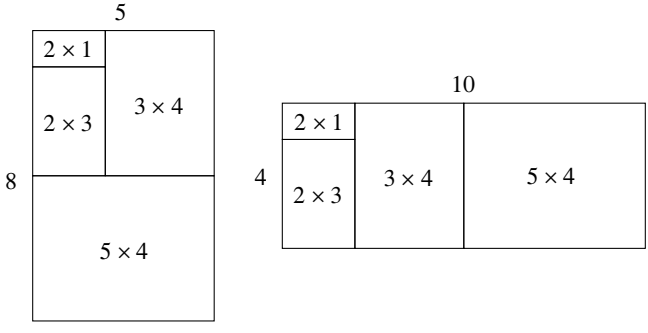


FIGURE 1

It is clear that these solutions are essentially unique. For $n = 4$, we see that there are exactly two different solutions:

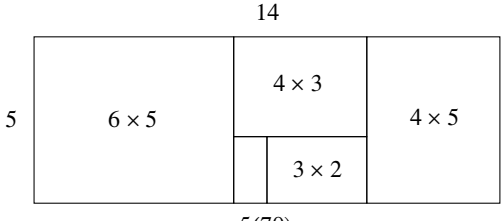
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$n = 4(40)$ $n = 4(40)$ (Alternative)

FIGURE 2

The first solution has the advantage that it can be augmented to give the solution to the next case, $n = 5$:



$n = 5(70)$

FIGURE 3

This solution turns out to be unique.

Note that as the diagrams become larger, we may sometimes omit the labels on the 1×2 and 2×3 rectangles to save space, and, to avoid confusion, if a rectangle is orientated horizontally, we will label it as $(m + 1) \times m$, i.e. width × height and if it is orientated vertically, we label it $m \times (m + 1)$.

So far so good, but in the next case, $n = 6$, an exact fit cannot be achieved. The total area of $m \times (m + 1)$ rectangles is

$$1 \times 2 + 2 \times 3 + \ldots + 6 \times 7 = 112$$

and so possible bounding rectangles have dimensions 1×112, 2×56, 4×28, 7×16 or 8×14. We can discount the first three possibilities.

In the 7×16 case, clearly the 5×6 rectangle (in either orientation) leaves a space that cannot be filled by the remaining $m \times (m + 1)$ rectangles.

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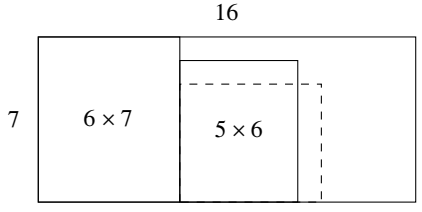


FIGURE 4

Similarly, in the 8×14 case, the 6×7 rectangle (in either orientation) leaves a space that cannot be filled by the remaining $m \times (m + 1)$ rectangles.

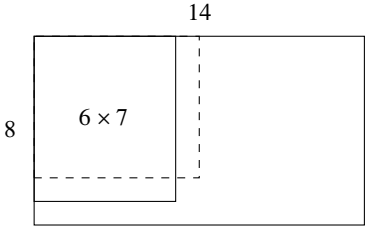
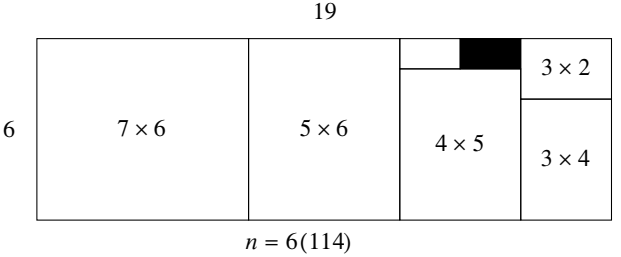


FIGURE 5

The next possibility is 113, which is prime, so this case can be ruled out also. This brings us to 114, with the only possible dimensions for the bounding rectangle being 6×19:



$n = 6(114)$

FIGURE 6

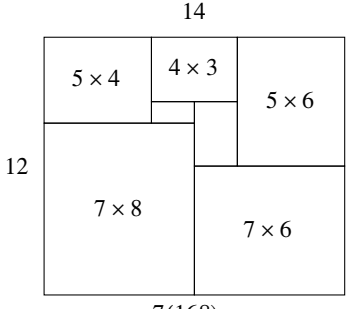
Here we can achieve a fit with a *remnant* (shaded area) of two square units. We say that the *occupancy* of this solution is $112/114 \approx 98.24\%$. This is optimal for $n = 6$.

Unfortunately, 'proofs' in this area of mathematics involve tedious and time-consuming case-by-case discussions, very prone to error and omission. In further examples, we will suppress these details, but the reader is invited to carry them out and not take our word for it. Fortunately, computer verifications are now possible (see [2, 3]), but one can have a lot of fun solving these mathematical jigsaws experimentally by hand.

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Proceeding in this way, we get the following solutions as n increases:

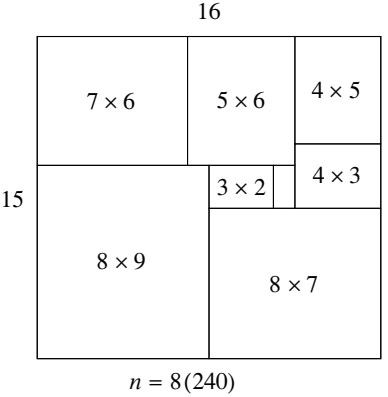
For $n = 7$ we achieve an exact fit and the solution is unique.



$n = 7(168)$

FIGURE 7

Also for $n = 8$ we achieve an exact fit and the solution is unique. We note that the bounding rectangle is itself the $m \times (m + 1)$ rectangle for $m = 15$ and ask when this situation arises again (it also happens in the case $n = 3$).

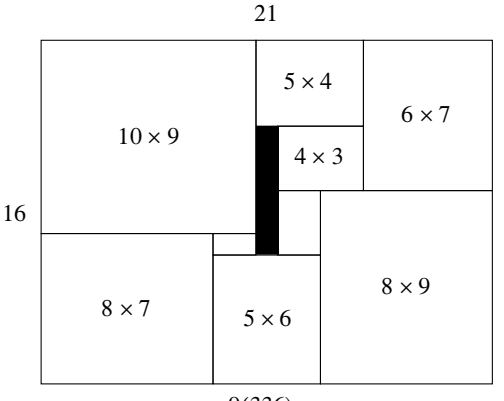


$n = 8(240)$

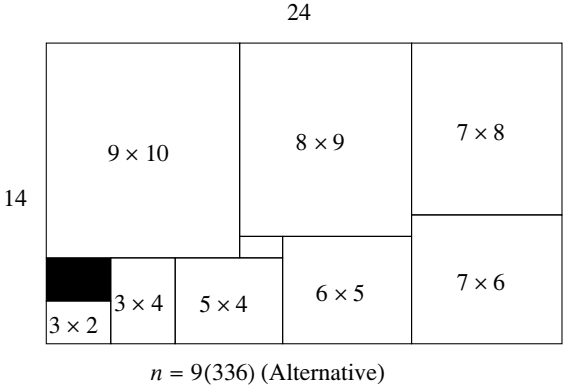
FIGURE 8

For $n = 9$ we get exactly two solutions arising from a lot of calculation and elimination. In each case, the remnant is quite large — six square units — and the occupancy is 98.21%.

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$n = 9(336)$

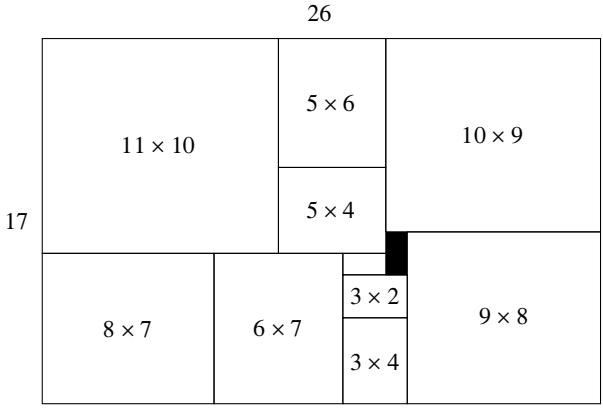


$n = 9(336)$ (Alternative)

FIGURE 9

As an extra constraint, one can add the condition that the remnant, when it exists, be a single connected piece. This has been achieved in each of the above solutions.

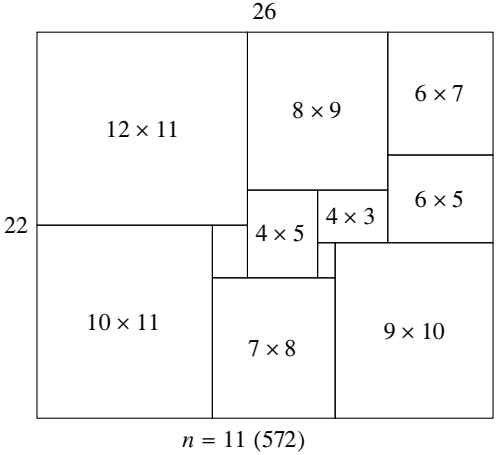
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$n = 10$ (442)

FIGURE 10

For $n = 10$ the remnant is 2 and the occupancy is $440/442 \approx 99.54\%$. The solution is unique. The remnant is connected; in fact it is clear that a connected remnant solution always exists, but perhaps at the cost of a bigger bounding rectangle.

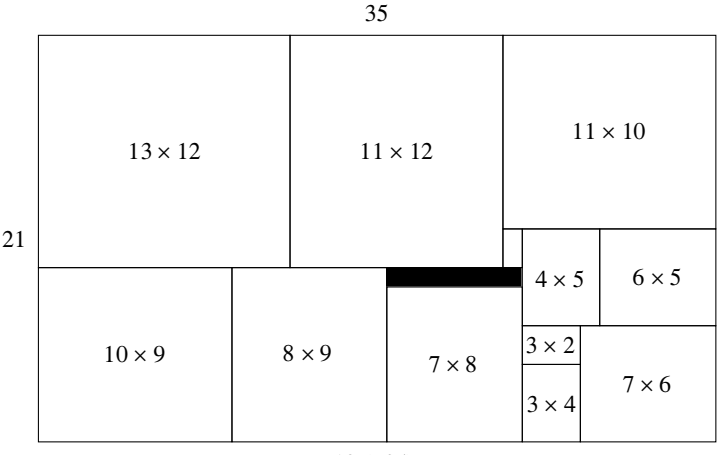


$n = 11$ (572)

FIGURE 11

In the case $n = 11$ we have a perfect fit and the solution is unique. These cases are of course much easier to solve than those with a positive remnant.

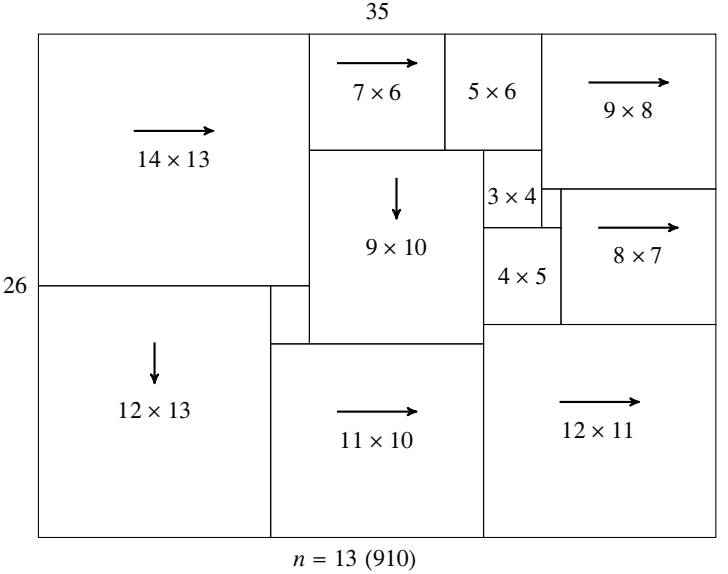
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$n = 12$ (735)

FIGURE 12

Here the occupancy is $728/735 \approx 99.05\%$. The remnant is seven square units and, with considerable effort, we have managed to make it connected, even a linear strip! Again, the solution is unique.



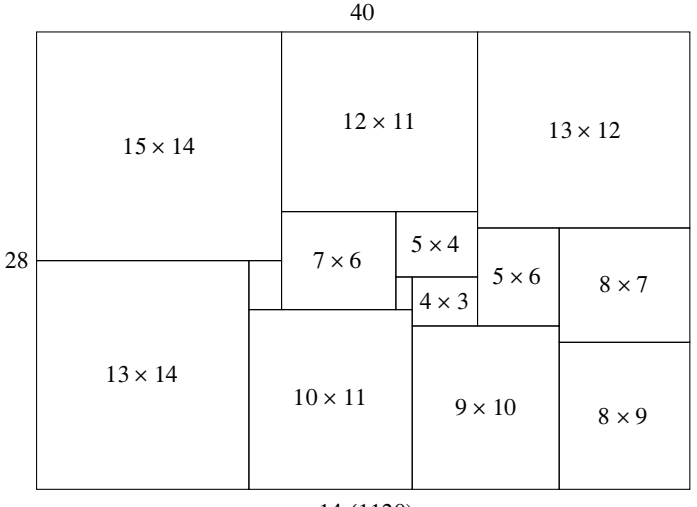
$n = 13$ (910)

FIGURE 13

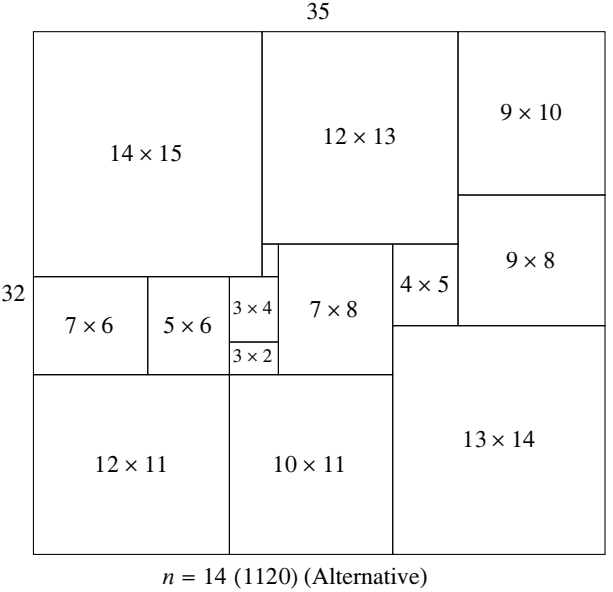
For $n = 13$ we get a perfect fit and the solution is unique. We mention that the numbers of the $\rightarrow$ and $\downarrow$ orientations of the pieces are approximately equal in this and several other cases and we wonder if any significance can be read into this fact.

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For $n = 14$ we get a perfect fit and exactly two different solutions.



$n = 14$ (1120)



$n = 14$ (1120) (Alternative)

FIGURE 14

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We have now very probably reached the limits of hand computation for this problem and the authors would be embarrassed to admit the number of hours they have spent testing and eliminating various configurations. Fortunately, the problem has been investigated by the marvellous computer work of Korf, Huang, Moffitt and Pollack [2, 3] which the interested reader is strongly recommended to consult.

3. *Solutions for large n*

Using a computer, Huang and Korf [3] have solved the problem as far as $n = 29$ and the reader may like to have fun solving the 'jigsaw puzzles' for the perfect fits they list of the following dimensions:

n	Bounding Rectangle	Area	
$n = 15$	34×40	1360	(unique solution)
$n = 16$	32×51	1632	(unique solution)
$n = 17$	34×57	1938	(unique solution)
$n = 18$	30×76	2280	(unique solution)
$n = 19$	35×76 38×70	2660	
$n = 20$	35×88 44×70 55×56	3080	

We ask if there is any significance in the repetition of the numbers 34, 35, 70 and 76 in the above list.

For the next value, $n = 21$, we see the reappearance of a remnant: the unique optimal solution has a bounding rectangle measuring 39×91 (area 3549), leaving a remnant of 7 square units. For the remaining values of n, all known solutions are a perfect fit. We also note the tendency towards multiple solutions from here on; in fact for $n = 26$ there are six optimal solutions!

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n	Bounding Rectangle	Area	
$n = 22$	44×92	4048	(unique solution)
$n = 23$	40×115 46×100	4600	
$n = 24$	40×130 52×100 65×80	5200	
$n = 25$	45×130 65×90 75×78	5850	
$n = 26$	42×156 52×126 56×117 63×104 72×91 78×84	6552	
$n = 27$	63×116	7308	(unique solution)
$n = 28$	56×145 70×116	8120	
$n = 29$	62×145	8990	(unique solution)

We recall that for $n = 3$ and $n = 8$, we had a perfect fit and in each case the bounding rectangle is itself an $m \times (m + 1)$ rectangle. This happens again for one of the configurations for $n = 20$; there is a perfect fit and the bounding rectangle is the 55×56 rectangle. We cannot resist displaying this very beautiful configuration (kindly supplied by Richard Korf) which would have been very difficult, if not impossible, to achieve by hand in Figure 15.

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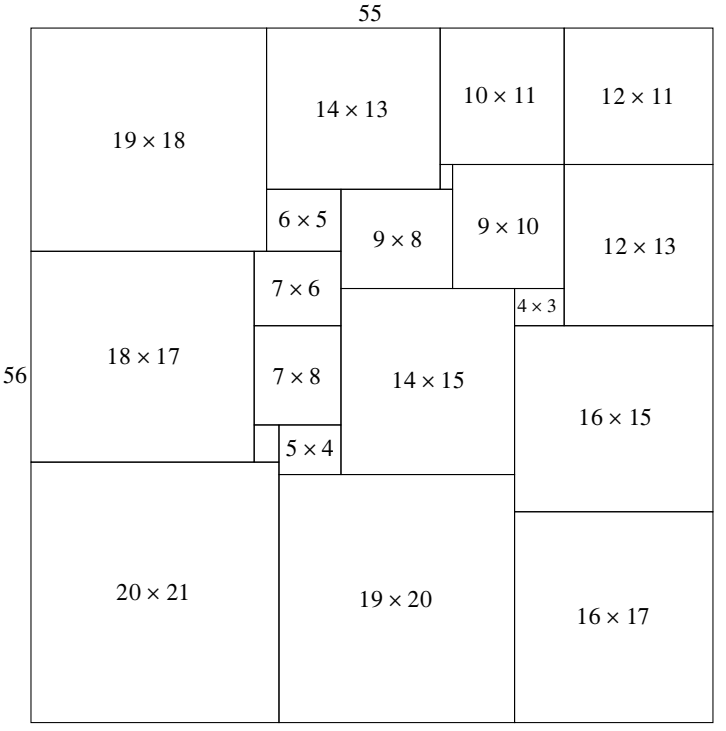


FIGURE 15

4. *Tiling a square*

In the corresponding problem for packing squares of areas $1^2, 2^2, \ldots, n^2$ into a rectangle, it is not known if a perfect fit can ever be achieved for $n > 1$. In the case of fitting those squares into a square, for $n \neq 24$, this is not even arithmetically possible as the only solutions to the diophantine equation

$$\sum_{r=1}^{n} r^2 = y^2$$

are $(n, y) = (1, 1)$ and $(n, y) = (24, 70)$. This was shown by Anglin in [5]. In addition, Korf [4] has shown that the 70×70 square cannot be tiled by squares of area $1^2, 2^2, \ldots, 24^2$; in fact, these squares cannot tile even a rectangle.

So what about tiling a square with distinct $m \times (m + 1)$ rectangles?

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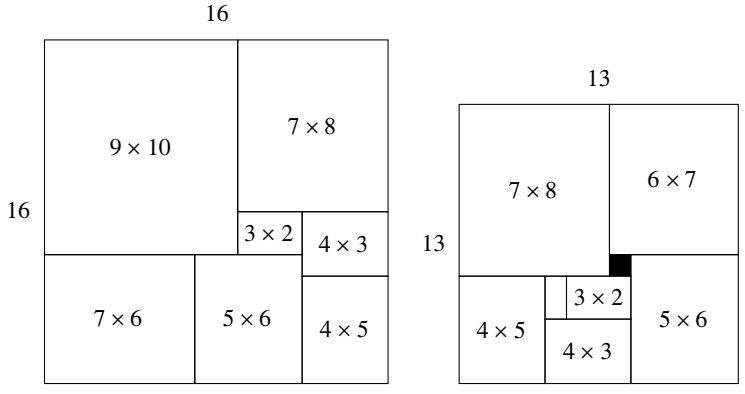


FIGURE 16

Well, the 16×16 square can be tiled with $m \times (m + 1)$ rectangles (leaving out 1×2 and 8×9) as above. Seven appears to be the smallest number of distinct $m \times (m + 1)$ rectangles with which this can be achieved. Furthermore, the 13×13 square can be covered with *all* of the rectangles 1×2, 2×3, … , 7×8 with remnant 1. We will now show that a remnant of 1 is as good as we can get.

Theorem 1: For each n, all of the rectangles $1 \times 2, 2 \times 3, \ldots, n \times (n+1)$ cannot tile a square exactly without overlap.

Proof: It is easy to show, by induction or otherwise, that the sum of the areas of the $m \times (m + 1)$ rectangles is

$$1 \times 2 + 2 \times 3 + \ \ldots \ + n(n + 1) = \tfrac{1}{3}n(n + 1)(n + 2).$$

If this is a square, then we have $n(n + 1)(n + 2) = 3y^2$ for some positive integer y. Putting $x = n + 1$, we get $(x - 1)x(x + 1) = 3y^2$ or $x^3 - x = 3y^2$. Finally, set $X = 3x$ and $Y = 9y$ and the equation becomes

$$Y^2 = X^3 - 9X.$$

Now this is the equation of an elliptic curve $Y^2 = f(X)$, where $f(X)$ is a monic cubic expression. In general, the determination of integral points on elliptic curves is a nontrivial computational exercise, but fortunately for us, thanks to the work of Professor John Cremona and others, implementations of such algorithms may be found in mathematical software packages such as Sage [6]. Consulting Sage, we find that all such solutions are $(X, Y) = (0, 0)$, $(-3, 0)$ or $(3, 0)$. Thus $Y = y = 0$ and our original equation has no positive integer solutions. Hence, even arithmetically, the total areas of the full set of $m \times (m + 1)$ rectangles cannot sum to a square, so the geometric configuration is also impossible.

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5. *Bounds for the minimal area*

Finally, we give bounds for the area of the bounding rectangle. A lower bound is clearly given by $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$ and this is achieved in cases of a perfect fit, including

$$n = 1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 11,\ 13,\ 14,\ \ldots,\ 20,\ 22,\ 23,\ \ldots,\ 29.$$

We conjecture that this happens for infinitely many values of n and maybe for all n from a certain point onward, but a proof of either of these conjectures is likely to be very difficult. For an upper bound, we treat the cases of n even and n odd separately.

If $n = 2k$, then we may arrange the $m \times (m+1)$ rectangles as follows:

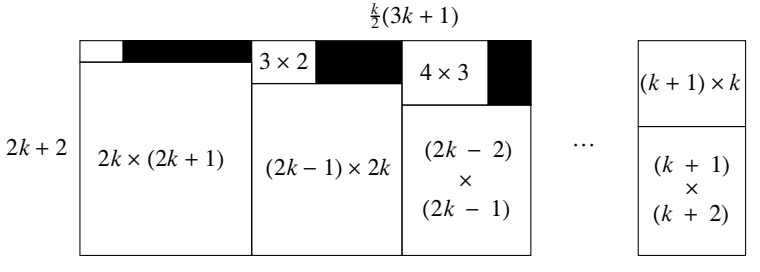


FIGURE 17

The height of the bounding rectangle is $2k + 2$ and the width is the sum of the arithmetic progression

$$2k + (2k-1) + (2k-2) + \ldots + (k+1) = \tfrac{1}{2}k(3k+1).$$

The area of the bounding rectangle is therefore

$$k(k+1)(3k+1) = \tfrac{3}{8}n^3 + n^2 + \tfrac{1}{2}n.$$

If $n = 2k + 1$, then we may place the $(2k+1) \times (2k+2)$ rectangle to the left of the above configuration:

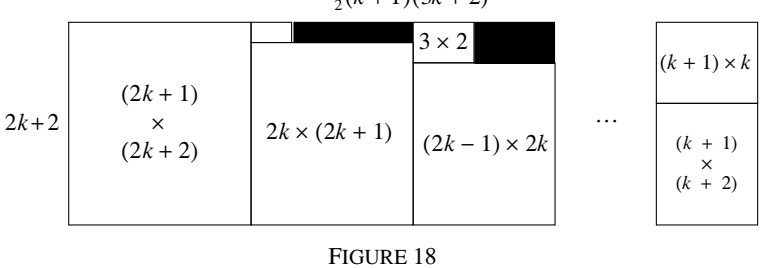


FIGURE 18

and the area of the bounding rectangle is then

$$(k+1)^2(3k+2) = \tfrac{3}{8}n^3 + \tfrac{7}{8}n^2 + \tfrac{5}{8}n + \tfrac{1}{8}.$$

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In particular, if $A(n)$ is the minimal area of the bounding rectangle, then for n even or n odd, we have

$$\tfrac{1}{3}n^3 < A(n) < \tfrac{3}{8}(n + 1)^3.$$

Acknowledgement

The authors would like to thank Professor John Cremona. His insights into elliptic functions proved pivotal in proving Theorem 1.

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