Canad. Math. Bull. 2026, pp. 1–16
http://dx.doi.org/10.4153/S0008439526102008
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

Estimates for strongly singular operators along curves

Magali Folch-Gabayet and Ricardo A. Sáenz

Abstract. For an appropriate function f on the plane, we study the operator

$$Tf(x,y) = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} f(x-t, y-\phi(t))\frac{e^{2\pi i\gamma(t)}}{\psi(t)}dt,$$

where ϕ is convex and ψ and γ are functions defined near the origin such that $\psi(t)\to 0$ and $|\gamma(t)|\to\infty$ as $t\to 0$. We give sufficient regularity and growth conditions on ψ and γ for its multiplier to be a bounded function, and thus for the operator to be bounded on $L^2(\mathbb{R}^2)$. We also consider its boundedness on $L^p(\mathbb{R}^2), 1<p<\infty$.

1 Preliminaries

In trying to generalize the Calderón–Zygmund method of rotations to kernels with nonstandard homogeneity, we are led to the study of the Hilbert transform along curves. This work was started by Fabes and Rivière in [FR66]. See [Ste86] for a very nice explanation of the difficulties on dealing with such operators.

It is known that

$$H_\phi f(x) = \text{P. V.}\int_{-1}^{1} f(x-\phi(t))\frac{dt}{t}$$

is bounded on L^p for certain well-curved ϕ in $\mathbb{R}^d$ (see [CNSW99, NRW74, NRW76, SW78]). However, when the singularity is given by a power greater than one, say $t|t|^\alpha$ with $\alpha>0$, it can be proved that the operator is bounded as long as a sufficiently rapid oscillatory term is involved. Indeed, consider the operator

$$T_{\alpha,\beta}f(x,y) = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} f(x-t, y-\phi(t))\frac{e^{2\pi i t^{-\beta}}}{t^{1+\alpha}}dt \tag{1.1}$$

for the curve $t\mapsto(t,\phi(t))$ with $\phi(t)=t^k$, $k\geq 2$. Zielinski [Zi85] ($k=2$) and Chandarana [Cha96] ($k>2$) proved that $T_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^2)$ if and only if

$$\beta\geq 3\alpha.$$

Received by the editors May 6, 2025; revised March 3, 2026; accepted April 21, 2026.
Published online on Cambridge Core April 30, 2026.
AMS subject classification: **42B20, 44A15.**
Keywords: Singular integrals along curves, oscillatory integrals, strongly singular operators.

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

Chandarana also proved that the operator $T_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^2)$ as long as $\beta > 3\alpha$ and

$$\frac{3\alpha(\beta+1)}{\beta(3\alpha+\beta+2)} < \frac{1}{p} < \frac{\beta(\beta+2)-3\alpha}{\beta(3\alpha+\beta+2)}.$$

This range of exponents was improved by Chen et al. in [CFWZ08], who proved that the operator (1.1) is bounded on $L^p(\mathbb{R}^2)$ if $\beta > 3\alpha$ and

$$\frac{3\alpha}{2\beta} < \frac{1}{p} < \frac{2\beta-3\alpha}{2\beta}. \tag{1.2}$$

They also extended this result to operators

$$T^{\Phi}_{\alpha,\beta} f(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} f(x-\Phi(t)) \frac{e^{2\pi i t^{-\beta}}}{t^{1+\alpha}} dt,$$

where $\Phi(t)$ is a curve in $\mathbb{R}^n$ of the form $\Phi(t) = (t^{k_1}, \dots, t^{k_n})$, and concluded the analogous result that $T^{\Phi}_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^n)$ if $\beta \geq (n+1)\alpha$ and is bounded on L^p if

$$\frac{(n+1)\alpha}{2\beta} < \frac{1}{p} < \frac{2\beta-(n+1)\alpha}{2\beta}.$$

Other authors have studied the boundedness of the operators $T^{\Phi}_{\alpha,\beta}$ on diverse function spaces, as Sobolev, Wiener, Besov spaces, and α-modulation spaces (see, for instance, [LG14, SXX21, WY17]).

Chen et al. considered in [CDZ17] curves $t \mapsto (t, \phi(t))$, where ϕ is a convex function, with $\phi(0) = \phi'(0) = 0$, that satisfies the condition

(C1) there exists $\delta > 0$ such that $\delta < \beta + 2$ and $t^{\delta}\phi''(t)$ is increasing in $(0,1)$.

They proved the boundedness on $L^p(\mathbb{R}^2)$ of the operator (1.1) for p satisfying (1.2). In [LY18], Li and Yu proved that the operator (1.1) is bounded on $L^p(\mathbb{R}^2)$, for the same p, if the curve satisfies the following conditions:

(C2) for each $j \in \mathbb{N}$, $\phi''(t) > 0$ and $\phi'''(t) \geq 0$ on $(2^{-j-1}, 2^{-j})$, and

(C3) there exist constants $C_2 \geq C_1 > 0$ such that, for each $j \in \mathbb{N}$,

$$C_1 h'(t) \leq \frac{h(t)}{t} \leq C_2 h'(t)$$

on $(2^{-j-1}, 2^{-j})$, where $h(t) = t\phi'(t) - \phi(t)$.

They also proved the existence of a function ϕ that satisfies conditions (C2) and (C3), but not (C1).

In this work, we discuss the boundedness of the operator

$$Tf(x,y) = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} f(x-t, y-\phi(t)) \frac{e^{2\pi i \gamma(t)}}{\psi(t)} dt, \tag{1.3}$$

along a convex curve $t \mapsto (t, \phi(t))$, where ϕ satisfies the condition

(C) $\phi''(t) > 0$ and $\phi'''(t) \geq 0$ on $(0,1)$.

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

Note that we no longer require $\phi(0)$ nor $\phi'(0)$ to be 0, nor any of the conditions (C1) or (C3) above. We observe that any function $\phi(t) = t^\kappa$ satisfies condition (C) if $\kappa \geq 2$, but not if $1 < \kappa < 2$. In this latter case, however, ϕ satisfies condition (C3) on the whole interval.

We allow the singularity $\psi(t)$ to be much worse than a power, say of the order of $t \to e^{-1/t}$ or worse. The authors have studied oscillatory integrals with such kernels previously in [FG99, FGS19]. We assume that the functions $\gamma \in C^3((0,1])$ and $\psi \in C^2((0,1])$ satisfy the following:

(a.1) γ, ψ, and their derivatives $\gamma', \gamma'', \gamma'''$, and ψ' are all monotone. For convenience, we also assume that $\gamma''(t) > 0$ for $t \in (0,1]$, γ'' is strictly decreasing, $\lim_{t\to 0} \gamma''(t) = \infty$ and that $\gamma''(1) = 1$, as well as $\psi(t) > 0$.

(a.2) There exists $C > 0$ such that, for $s \geq 1$,

$$|\gamma'''(\gamma''^{-1}(2s))| \leq C|\gamma'''(\gamma''^{-1}(s))|.$$

(a.3) For small $\rho > 0$, there exists $C > 0$ such that

$$|\gamma'''(t)| \leq C\gamma''(t)^{3/2-\rho}.$$

(a.4) There exists $C > 0$ such that

$$\frac{1}{\psi(t)} \leq C|\gamma'''(t)|^{1/3}.$$

These assumptions are satisfied by $\gamma(t) = t^{-\beta}$ and $\psi(t) = t^{\alpha+1}$ if α and β satisfy, as in [Cha96], $\beta \geq 3\alpha$. Assumption (a.4) is not satisfied by our example $\gamma(t) = e^{1/t}$ and $\psi(t) = e^{-1/2t}$ in [FGS19], but it is satisfied by the function $\psi(t) = e^{-1/3t}$. Alternatively, it is satisfied by the pair of functions $\gamma(t) = e^{3/t}$ and $\psi(t) = e^{-1/t}$.

1.1 Existence

It is not hard to see that the limit in (1.3) exists for, say, $f \in C_c^\infty(\mathbb{R}^2)$. Indeed, if we write

$$\Gamma(s) = \int_0^s e^{2\pi i\gamma(t)} dt,$$

for $0 < \varepsilon' < \varepsilon$, we can integrate by parts to obtain

$$\begin{aligned}\int_{\varepsilon'}^{\varepsilon} f(x-t, y-\phi(t))\frac{\Gamma'(t)}{\psi(t)}dt \\ = f(x-t, y-\phi(t))\frac{\Gamma(t)}{\psi(t)}\Big|_{\varepsilon'}^{\varepsilon} - \int_{\varepsilon'}^{\varepsilon} \Gamma(t)\frac{d}{dt}\frac{f(x-t, y-\phi(t))}{\psi(t)}dt. \end{aligned} \tag{1.4}$$

The first term in (1.4) goes to zero as $\varepsilon', \varepsilon \to 0$ because, by the van der Corput's lemma [Ste93, Proposition 2 in section 1.2 of chapter VIII],

$$|\Gamma(t)| \lesssim \frac{1}{\gamma''(t)^{1/2}},$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

and therefore, using assumptions (a.3) and (a.4) and the fact that f is bounded,

$$\left| f(x-t, y-\phi(t)) \frac{\Gamma(t)}{\psi(t)} \right| \lesssim \frac{1}{\gamma''(t)^{1/2}} |\gamma'''(t)|^{1/3} \lesssim \frac{1}{\gamma''(t)^{\rho/3}} \to 0$$

as $t \to 0$.

For the second integral in (1.4), we note that

$$\begin{aligned} &\left| \int_{\varepsilon'}^{\varepsilon} \Gamma(t) \frac{d}{dt} \frac{f(x-t, y-\phi(t))}{\psi(t)} dt \right| \\ &\qquad \leq \int_{\varepsilon'}^{\varepsilon} |\Gamma(t)| \left| \frac{df(x-t, y-\phi(t))}{dt} \right| \frac{1}{|\psi(t)|} dt \\ &\qquad\quad + \int_{\varepsilon'}^{\varepsilon} |\Gamma(t)| |f(x-t, y-\phi(t))| \left| \frac{\psi'(t)}{\psi(t)^2} \right| dt, \end{aligned} \tag{1.5}$$

and the first integral in (1.5) goes to 0 as above, because the derivatives of f and ϕ are also bounded. For the second integral, choose $l \in \mathbb{N}$ such that $\gamma''^{-1}(2^{l+1}) < \varepsilon \leq \gamma''^{-1}(2^l)$. As $0 < \varepsilon' < \varepsilon \leq \gamma''^{-1}(2^l)$, we have, using the fact that f is bounded,

$$\begin{aligned} &\int_{\varepsilon'}^{\varepsilon} |\Gamma(t)| |f(x-t, y-\phi(t))| \left| \frac{\psi'(t)}{\psi(t)^2} \right| dt \\ &\qquad \lesssim \int_0^{\gamma''^{-1}(2^l)} |\Gamma(t)| \left| \frac{\psi'(t)}{\psi(t)^2} \right| dt \lesssim \sum_{j=l}^{\infty} \frac{1}{2^{j/2}} \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \left| \frac{\psi'(t)}{\psi(t)^2} \right| dt, \end{aligned}$$

where we have again used the van der Corput's lemma so

$$|\Gamma(\gamma''^{-1}(2^l))| \lesssim \frac{1}{\gamma''(\gamma''^{-1}(2^l))^{1/2}} = \frac{1}{2^{j/2}}.$$

Now, as ψ is monotone, we have

$$\begin{aligned} \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \left| \frac{\psi'(t)}{\psi(t)^2} \right| dt &= \left| \frac{1}{\psi(\gamma''^{-1}(2^{j+1}))} - \frac{1}{\psi(\gamma''^{-1}(2^j))} \right| \\ &\lesssim |\gamma'''(\gamma''^{-1}(2^{j+1}))|^{1/3}, \end{aligned}$$

where we have also used (a.4). Using (a.3), we see that the integral is then bounded by

$$\sum_{j=l}^{\infty} \frac{1}{2^{j/2}} \left(2^{j/3}\right)^{3/2-\rho} = \sum_{j=l}^{\infty} \frac{1}{2^{\rho/3}} \lesssim \frac{1}{2^l},$$

which goes to 0 as $l \to \infty$ (and thus as $\varepsilon', \varepsilon \to 0$).

2 L^2 boundedness

We will prove the following theorem.

Theorem 2.1 *The operator T given by (1.3) extends to a bounded operator on $L^2(\mathbb{R}^2)$.*

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

In order to prove the L^2 boundedness of operator (1.3), we need to prove that its multiplier

$$m(\xi,\eta)=\int_0^1 e^{2\pi i(\gamma(t)-\xi t-\eta\phi(t))}\frac{dt}{\psi(t)},\qquad \xi,\eta\in\mathbb{R}, \tag{2.1}$$

is bounded. We write (2.1) as the integral

$$\int_0^1 e^{2\pi i g(t)}\frac{dt}{\psi(t)}, \tag{2.2}$$

where $g(t)=\gamma(t)-\xi t-\eta\phi(t)$. If we let

$$G(s)=\int_0^s e^{2\pi i g(t)}dt,$$

then we can also write (2.2) as

$$\int_0^1 \frac{G'(t)}{\psi(t)}dt.$$

The boundedness of (2.2) will follow from estimates to G, obtained by van der Corput's lemma provided we can prove that the derivatives of g can be bounded from below, uniformly for ξ and η in proper subsets of $\mathbb{R}^2$. The derivatives of the function g are given by

$$g'(t)=\gamma'(t)-\xi-\eta\phi'(t),$$

and

$$g^{(p)}(t)=\gamma^{(p)}(t)-\eta\phi^{(p)}(t),\qquad p=2,3.$$

We thus observe that bounds from below for g', g'', or g''' will depend on the signs of ξ and η. We prove then the boundedness of (2.2) by splitting in cases.

2.1 Case $\eta\le 0$

In this case, we consider the second derivative g''. For $\eta\le 0$,

$$g''(t)=\gamma''(t)-\eta\phi''(t)\ge\gamma''(t),$$

because $\phi''(t)>0$. Hence, by van der Corput's lemma,

$$|G(t)|\lesssim\frac{1}{\gamma''(t)^{1/2}}.$$

We integrate by parts to obtain

$$\int_0^1\frac{G'(t)}{\psi(t)}dt=\lim_{t\to 0}\Big(\frac{G(1)}{\psi(1)}-\frac{G(t)}{\psi(t)}\Big)+\int_0^1\frac{G(t)\psi'(t)}{\psi(t)^2}dt.$$

The limit is bounded because, using assumptions (a.3) and (a.4),

$$\Big|\frac{G(t)}{\psi(t)}\Big|\lesssim\frac{1}{\gamma''(t)^{1/2}}|\gamma'''(t)|^{1/3}\lesssim\frac{1}{\gamma''(t)^{\rho/3}}\le 1.$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

For the integral, we partition the interval $(0,1]$ in the intervals

$$\gamma''^{-1}\big((2^j, 2^{j+1}]\big),$$

$j = 0, 1, 2\ldots$, and we use the monotonicity of ψ, so ψ' doesn't change sign. Thus,

$$\begin{aligned}
\Big|\int_0^1 \frac{G(t)\psi'(t)}{\psi(t)^2}dt\Big| &\le \sum_{j=0}^{\infty}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} |G(t)|\Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\
&\lesssim \sum_{j=0}^{\infty}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \frac{1}{\gamma''(t)^{1/2}}\Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\
&\le \sum_{j=0}^{\infty}\frac{1}{2^{j/2}}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\
&\le \sum_{j=0}^{\infty}\frac{1}{2^{j/2}}\Big|\frac{1}{\psi(\gamma''^{-1}(2^j))} - \frac{1}{\psi(\gamma''^{-1}(2^{j+1}))}\Big| \\
&\lesssim \sum_{j=0}^{\infty}\frac{1}{2^{j/2}}|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{1/3} \\
&\lesssim \sum_{j=0}^{\infty}\frac{1}{2^{j/2}}\big(2^{j(3/2-\rho)}\big)^{1/3} = \sum_{j=0}^{\infty}\frac{1}{2^{j\rho/3}} < \infty.
\end{aligned}$$

2.2 Case $\eta > 0$

In this case, both g' and g'' might be zero. Since both $\gamma'' > 0$ and $\phi'' > 0$, γ'' is decreasing and ϕ'' is increasing, for $\eta > 0$, we will have that

$$g''(t) = \gamma''(t) - \eta\phi''(t)$$

will have a unique zero in $(0,1)$ if $\eta\phi''(1) > \gamma''(1) = 1$.

Let t_0 be the zero of $g''(t)$ if $\eta\phi''(1) > 1$; otherwise, let $t_0 = 1$. Hence, t_0 satisfies

$$\gamma''(t_0) = \eta\phi''(t_0)$$

if $t_0 < 1$, or $1 \ge \eta\phi''(1)$ if $t_0 = 1$. Either way we have the estimate

$$\gamma''(t_0) \ge \eta\phi''(t_0). \tag{2.3}$$

Let $l \in \mathbb{N}$ be such that

$$\gamma''^{-1}(2^{l+1}) < t_0 \le \gamma''^{-1}(2^l). \tag{2.4}$$

We thus split integral (2.2) as

$$\int_0^{\gamma''^{-1}(2^{l+2})}\frac{G'(t)}{\psi(t)}dt + \int_{\gamma''^{-1}(2^{l+2})}^{\gamma''^{-1}(2^{l-1})}\frac{G'(t)}{\psi(t)}dt + \int_{\gamma''^{-1}(2^{l-1})}^{1}\frac{G'(t)}{\psi(t)}dt,$$

where the third integral is nonzero only if $l > 1$ (if $l = 0$, the second integral is only over the interval $[\gamma''^{-1}(4), 1]$, of course). We estimate each of these integrals separately.

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

2.2.1 The first integral

Here, we integrate over t such that $0 < t \le \gamma''^{-1}(2^{l+2})$. For such t, we have, using the monotonicity of γ'' and $\phi''(t)$, (2.3) and (2.4),

$$\begin{aligned}\gamma''(t) &\ge 2^{l+2} = 2 \cdot 2^{l+1} = 2\gamma''(\gamma''^{-1}(2^{l+1}))\\ &\ge 2\gamma''(t_0) \ge 2\eta\phi''(t_0) \ge 2\eta\phi''(t).\end{aligned}$$

Hence

$$g''(t) \ge \frac{1}{2}\gamma''(t)$$

over $0 < t \le \gamma''^{-1}(2^{l+2})$, and by van der Corput's lemma,

$$|G(t)| \lesssim \frac{1}{\gamma''(t)^{1/2}}$$

over this interval. We can therefore estimate the first integral as in Section 2.1.

2.2.2 The third integral

If $l > 1$, we integrate over $\gamma''^{-1}(2^{l-1}) \le t \le 1$. This implies that $t_0 < 1$ so, in particular, $\gamma''(t_0) = \eta\phi''(t_0)$. Using the monotonicity of γ'' and $\phi''(t)$, and (2.4),

$$\eta\phi''(t) \ge \eta\phi''(t_0) = \gamma''(t_0) \ge 2^l = 2\gamma''(\gamma''^{-1}(2^{l-1})) \ge \gamma''(t),$$

so we have again, by van der Corput's lemma,

$$|G(t)| \lesssim \frac{1}{\gamma''(t)^{1/2}}$$

and we proceed as above.

2.2.3 The middle integral

We finally integrate over the interval that contains t_0. Since $-\eta < 0$ and $\phi''' \ge 0$, we have

$$g'''(t) = \gamma'''(t) - \eta\phi'''(t) \le \gamma'''(t).$$

Thus,

$$|g'''(t)| \ge |\gamma'''(t)|.$$

for $\gamma''^{-1}(2^{l+2}) \le t \le \gamma''^{-1}(2^{l-1})$. By van der Corput's lemma (see the corollary to Proposition 2 in [Ste93, VIII.1.2]), we obtain

$$\Big|\int_{\gamma''^{-1}(2^{l+2})}^{\gamma''^{-1}(2^{l-1})} \frac{e^{2\pi i g(t)}}{\psi(t)}\,dt\Big| \lesssim \frac{1}{|\gamma'''(\gamma''^{-1}(2^{l-1}))|^{1/3}}\,\frac{1}{|\psi(\gamma''^{-1}(2^{l+2}))|}.$$

Thus, by hypothesis (a.4),

$$\Big|\int_{\gamma''^{-1}(2^{l+2})}^{\gamma''^{-1}(2^{l-1})} \frac{e^{2\pi i g(t)}}{\psi(t)}\,dt\Big| \lesssim \frac{1}{|\gamma'''(\gamma''^{-1}(2^{l-1}))|^{1/3}}|\gamma'''(\gamma''^{-1}(2^{l+2}))|^{1/3},$$

which is bounded by a constant by assumption (a.2).

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

3 A result on sharpness

Assume that γ satisfies assumptions (a.1)–(a.3). We prove that, under assumption

(b.4), there exists $C > 0$ and a small $\mu > 0$ such that, for $t \in (0,1]$,

$$\frac{1}{|\psi(t)|} \leq C\gamma''(t)^{1/2-\mu},$$

assumption (a.4) is sharp, in the sense that, if we assume

(b.5), the quotient

$$\frac{1}{|\psi(t)||\gamma'''(t)|^{1/3}}$$

is unbounded,

then the multiplier is not bounded for any curve $\phi(t)$ satisfying hypothesis (C) above. Indeed, it fails for the parabola.

Proposition 3.1 *Under hypotheses (a.1)–(a.3), (b.4), and (b.5), the multiplier $m(\xi,\eta)$ for the parabola $\phi(t) = t^2$ is unbounded.*

Proof Let $t_n \to 0$ such that

$$\frac{1}{|\psi(t_n)||\gamma'''(t_n)|^{1/3}} \to \infty.$$

Now, for each n, set

$$\eta_n = \frac{1}{2}\gamma''(t_n) \qquad \text{and} \qquad \xi_n = \gamma'(t_n) - \gamma''(t_n)t_n.$$

Note that this implies that $g'(t_n) = g''(t_n) = 0$. We claim that $|m(\xi_n,\eta_n)| \to \infty$.

Fix $n \in \mathbb{N}$ and choose $l = l(n) \in \mathbb{N}$ such that

$$\gamma''^{-1}(2^{l+1}) < t_n \leq \gamma''^{-1}(2^l).$$

Let $h \in C^\infty((0,1))$ be a cutoff function such that

(1) $\operatorname{supp} h \subset [\gamma''^{-1}(2^{l+3}), \gamma''^{-1}(2^{l-2})]$;
(2) $h(t) = 1$ on $[\gamma''^{-1}(2^{l+2}), \gamma''^{-1}(2^{l-1})]$;
(3) h is increasing on $[\gamma''^{-1}(2^{l+3}), \gamma''^{-1}(2^{l+2})]$ and decreasing on $[\gamma''^{-1}(2^{l-1}), \gamma''^{-1}(2^{l-2})]$.

Hence, with the notation used above,

$$\begin{aligned} m(\xi_n,\eta_n) = \int_0^1 \frac{G'(t)}{\psi(t)}dt = &\int_0^{\gamma''^{-1}(2^{l+2})} \frac{G'(t)}{\psi(t)}(1-h(t))dt \\ &+ \int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l-2})} \frac{G'(t)}{\psi(t)}h(t)dt \\ &+ \int_{\gamma''^{-1}(2^{l-1})}^{1} \frac{G'(t)}{\psi(t)}(1-h(t))dt. \end{aligned} \tag{3.1}$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

To estimate the first integral, we integrate by parts to obtain

$$\begin{aligned}\int_0^{\gamma''^{-1}(2^{l+2})} \frac{G'(t)}{\psi(t)}(1-h(t))dt &= -\lim_{t\to 0}\frac{G(t)}{\psi(t)} + \int_0^{\gamma''^{-1}(2^{l+2})} \frac{G(t)\psi'(t)}{\psi(t)^2}(1-h(t))dt \\ &\quad - \int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l+2})} \frac{G(t)}{\psi(t)}h'(t)dt. \end{aligned} \tag{3.2}$$

As in Section 2.2.1, if $0 < t < \gamma''^{-1}(2^{l+2})$,

$$\gamma''(t) \geq 2^{l+2} = 2\gamma''(\gamma''^{-1}(2^{l+1})) \geq 2\gamma''(t_n) = 4\eta_n,$$

so $g''(t) = \gamma''(t) - 2\eta_n \geq \frac{1}{2}\gamma''(t)$ and, by van der Corput's lemma,

$$|G(t)| \lesssim \frac{1}{\gamma''(t)^{1/2}}.$$

So, by hypothesis (b.4),

$$\Big|\frac{G(t)}{\psi(t)}\Big| \lesssim \frac{1}{\gamma''(t)^{\mu}}.$$

This implies that the limit in (3.2) is zero, as well as the third integral is bounded, as

$$\begin{aligned}&\Big|\int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l+2})} \frac{G(t)}{\psi(t)}h'(t)dt\Big| \\ &\quad\leq \int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l+2})} \Big|\frac{G(t)}{\psi(t)}\Big|h'(t)dt \lesssim \frac{1}{2^l}\int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l+2})} h'(t)dt = 2^{-l},\end{aligned}$$

because h is increasing from 0 to 1 in the interval $[\gamma''^{-1}(2^{l+3}), \gamma''^{-1}(2^{l+2})]$.

For the second integral in (3.2), we decompose the interval $(0, \gamma''^{-1}(2^{l+2})]$ and use the fact that ψ is monotone, as in Section 2.1, to obtain

$$\begin{aligned}\Big|\int_0^{\gamma''^{-1}(2^{l+2})} \frac{G(t)\psi'(t)}{\psi(t)^2}(1-h(t))dt\Big| &\leq \sum_{j=l+2}^{\infty}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} |G(t)|\Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\ &\lesssim \sum_{j=l+2}^{\infty}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \frac{1}{\gamma''(t)^{1/2}}\Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\ &\leq \sum_{j=l+2}^{\infty}\frac{1}{2^{j/2}}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \Big|\frac{\psi'(t)}{\psi(t)^2}\Big|dt \\ &\leq \sum_{j=l+2}^{\infty}\frac{1}{2^{j/2}}\Big|\frac{1}{\psi(\gamma''^{-1}(2^j))} - \frac{1}{\psi(\gamma''^{-1}(2^{j+1}))}\Big|\end{aligned}$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

$$\lesssim \sum_{j=l+2}^{\infty} \frac{1}{2^{j/2}} \gamma''(\gamma''^{-1}(2^{j+1}))^{1/2-\mu}$$
$$\lesssim \sum_{j=l+2}^{\infty} \frac{1}{2^{j/2}} (2^j)^{1/2-\mu} = \sum_{j=l+2}^{\infty} \frac{1}{2^{j\mu}} < \infty.$$

The third integral in (3.1) can be estimated similarly. We conclude then that both of these integrals are bounded.

Now, since $g'(t_n) = g''(t_n) = 0$, we have the asymptotic estimate

$$\int_{\gamma''^{-1}(2^{l+3})}^{\gamma''^{-1}(2^{l-2})} \frac{e^{2\pi i g(t)}}{\psi(t)} h(t)dt \sim C \frac{e^{2\pi i g(t_n)}}{\psi(t_n)(\gamma'''(t_n))^{1/3}}$$

for the second integral in (3.1), where C is independent of n (see, e.g., equations (4), (17), and (20) in [Erd56, Section 2.9]). By the construction of the t_n, we have that

$$|m(\xi_n, \eta_n)| \to \infty$$

as $n \to \infty$. ∎

4 An extension to L^p spaces

In this section, we consider the operator

$$Tf(x,y) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f(x-t, y-\phi(t)) \frac{e^{2\pi i \gamma(t)}}{t^{\theta}\, \psi(t)^{1-\theta}} dt, \tag{4.1}$$

for $0 < \theta < 1$, and γ and ψ satisfy hypotheses (a.1)–(a.4) above, as well as assuming that $\psi > 0$, $\phi'', \phi''' \geq 0$. We also make the following assumption on γ'':

(a.5) There exists a constant $\varepsilon > 0$, with $(1+\varepsilon)^2 < 2$, such that, for $0 < t < t_0$,

$$\gamma''(t) \geq 2\gamma''((1+\varepsilon)t).$$

We remark that assumption (a.5) is analogous to assumption (a.4) in [FGS19], which stated that

> "there exist $\varepsilon > 0$, $A > 1+\varepsilon$ such that $\gamma'(t) \geq A\gamma'((1+\varepsilon)t)$ for for $0 < t < t_0$."

It is expected, as we are now working along the curve $t \mapsto (t, \phi(t))$, to require a stronger condition to the corresponding one in [FGS19], so this time we need to assume such condition on γ''.

Using the fact that γ'' is decreasing, we can rewrite (a.5) as

$$t \leq \gamma''^{-1}(2\gamma''((1+\varepsilon)t))$$

and, by substituting $\gamma''((1+\varepsilon)t)$ by s, we obtain

$$\gamma''^{-1}(s) \leq (1+\varepsilon)\gamma''^{-1}(2s) \tag{4.2}$$

for large $s > 1$. Together with the fact that γ''^{-1} is decreasing, we obtain that

$$\gamma''^{-1}(2s) \leq \gamma''^{-1}(s) \lesssim \gamma''^{-1}(2s) \tag{4.3}$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

for large values of s, so $\gamma''^{-1}(s)$ and $\gamma''^{-1}(2s)$ have comparable sizes. This is analogous to the fact that $|\gamma'''(\gamma''^{-1}(s))|$ and $|\gamma'''(\gamma''^{-1}(2s))|$ also have comparable sizes, as, from assumption (a.2), we have that

$$|\gamma'''(\gamma''^{-1}(s))| \le |\gamma'''(\gamma''^{-1}(2s))| \lesssim |\gamma'''(\gamma''^{-1}(s))|. \tag{4.4}$$

Remark 4.1. The function $\gamma(t) = t^{-\beta}$, with $\beta > 0$, also satisfies assumption (a.5). Indeed, $\gamma''(t) = \beta(\beta+1)t^{-\beta-2}$, so

$$\gamma''(t) = (1+\varepsilon)^{\beta+2}\gamma''((1+\varepsilon)t) \ge 2\gamma''((1+\varepsilon)t)$$

if we choose $(1+\varepsilon)^{\beta+2} = 2$, which satisfies $(1+\varepsilon)^2 < 2$ because $\beta > 0$.

Remark 4.2. Functions γ such that γ'' is of the form $\gamma''(t) = ce^{\sigma/t}$, for $\sigma > 0$, also satisfy (a.5). Indeed,

$$\gamma''(t) = e^{\frac{\sigma}{t}\frac{\varepsilon}{1+\varepsilon}}\gamma''((1+\varepsilon)t) \ge 2\gamma''((1+\varepsilon)t)$$

for $\varepsilon = \log 2^{1/3} < 2^{1/2} - 1$ as long as $t < \sigma/\big(3(1+\varepsilon)\big)$.

Lemma 4.3 *There exist constants $C > 0$ and $\delta > 0$ such that, for small $t > 0$,*

$$|\gamma'''(t)| \ge C\frac{\gamma''(t)^\delta}{t^3}.$$

Proof We first proceed as in [FG99] to verify that

$$|\gamma'''(t)| \gtrsim \frac{\gamma''(t)}{t}. \tag{4.5}$$

Indeed, as $t \mapsto -\gamma'''(t)$ is decreasing, we have

$$-\gamma'''(t)\cdot\varepsilon t \ge \int_t^{(1+\varepsilon)t} -\gamma'''(s)ds = \gamma''(t) - \gamma''((1+\varepsilon)t) \ge \frac{1}{2}\gamma''(t)$$

by assumption (a.5). So we obtain

$$|\gamma'''(t)| \ge \frac{1}{2\varepsilon}\frac{\gamma''(t)}{t}.$$

The lemma will follow from (4.5) once we establish the existence of $\delta > 0$ such that

$$\gamma''(t) \gtrsim \frac{\gamma''(t)^\delta}{t^2}. \tag{4.6}$$

To verify this, observe that, by assumption (a.5),

$$\gamma''\Big(\frac{t_0}{(1+\varepsilon)^n}\Big) \ge 2^n\gamma''(t_0) = C\Big(\frac{2}{1+\varepsilon}\Big)^{n+1}\frac{1}{t_0/(1+\varepsilon)^{n+1}}$$

for every $n \in \mathbb{N}$, where $C = t_0\gamma''(t_0)/2$. For $t \in (0, t_0)$, choose n such that

$$\frac{t_0}{(1+\varepsilon)^{n+1}} \le t < \frac{t_0}{(1+\varepsilon)^n}.$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

Then $(1+\varepsilon)^{n+1} \geq t_0/t$ and hence $n+1 \geq \log(t_0/t)/\log(1+\varepsilon)$, which implies

$$\left(\frac{2}{1+\varepsilon}\right)^{n+1} \geq \left(\frac{t_0}{t}\right)^{\alpha},$$

where

$$\alpha = \frac{\log(2/(1+\varepsilon))}{\log(1+\varepsilon)} > 1$$

because $2 > (1+\varepsilon)^2$. Therefore, as γ'' is decreasing,

$$\gamma''(t) > \gamma''\left(\frac{t_0}{(1+\varepsilon)^n}\right) \geq C \cdot \left(\frac{t_0}{t}\right)^{\alpha} \cdot \frac{1}{t} = \frac{C'}{t^{1+\alpha}}. \tag{4.7}$$

If we choose $\delta = (\alpha-1)/(1+\alpha)$, we obtain

$$\gamma''(t) > \gamma''(t)^{\delta} \cdot \left(\frac{C'}{t^{1+\alpha}}\right)^{1-\delta} = C'' \frac{\gamma''(t)^{\delta}}{t^2}.$$

■

Remark 4.4. Substituting t by $\gamma''^{-1}(s)$ in Lemma 4.3, we obtain

$$\gamma''^{-1}(s)|\gamma'''(\gamma''^{-1}(s))|^{1/3} \gtrsim s^{\delta/3},$$

which guarantees, in particular, that for any $\varepsilon > 0$, the series

$$\sum_{j=0}^{\infty} \frac{1}{\left(\gamma''^{-1}(2^j)|\gamma'''(\gamma''^{-1}(2^j))|^{1/3}\right)^{\varepsilon}} < \infty.$$

The existence of the limit in (4.1) can be verified analogously to the existence of the limit in (1.3) above. From the proof of Theorem 2.1, in particular, the estimates for the multiplier (2.2), we obtain the following result.

Corollary 4.5 *Under assumptions (a.1)–(a.5), the operator (4.1) is bounded on* $L^p(\mathbb{R}^2)$ *if*

$$\frac{1}{2} \leq \frac{1}{p} < \frac{1+\theta}{2}. \tag{4.8}$$

Remark 4.6. We take a moment to note that our result in Corollary 4.5 is consistent with previous known results for the operator (1.1). If we write $t^{\alpha+1}$ as $t^{\theta}\psi(t)^{1-\theta}$, with

$$\psi(t) = t^{\frac{\alpha}{1-\theta}+1},$$

then the condition $\beta \geq 3\alpha/(1-\theta)$ implies

$$\theta < \frac{\beta - 3\alpha}{\beta},$$

so we see that the operator (1.1) is bounded on $L^p(\mathbb{R}^2)$ if $\beta > 3\alpha$ and

$$\frac{1}{p} < \frac{1}{2} + \frac{\beta - 3\alpha}{2\beta} = \frac{2\beta - 3\alpha}{2\beta},$$

which is of course (1.2) by duality.

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

Proof of Corollary 4.5. We follow the idea in [CDZ17] and formally write $Tf = \sum_{j=0}^{\infty} T_j f$, where

$$T_j f(x,y) = \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} f(x-t, y-\phi(t)) \frac{e^{2\pi i \gamma(t)}}{t^\theta \psi(t)^{1-\theta}} dt,$$

and we estimate the L^1 and L^2 norms of each T_j, $j \geq 0$.

For the L^1 norms, we start with the trivial estimate

$$\begin{aligned}\int |T_j f(x,y)| dx dy &\leq \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \frac{1}{t^\theta \psi(t)^{1-\theta}} dt \|f\|_{L^1} \\ &\lesssim \frac{\gamma''^{-1}(2^j)}{\gamma''^{-1}(2^{j+1})^\theta \psi(\gamma''^{-1}(2^{j+1}))^{1-\theta}} \|f\|_{L^1}.\end{aligned}$$

Using (a.4) and (4.4), we have

$$\frac{1}{\psi(\gamma''^{-1}(2^{j+1}))^{1-\theta}} \lesssim |\gamma'''(\gamma''^{-1}(2^{j+1}))|^{(1-\theta)/3} \lesssim |\gamma'''(\gamma''^{-1}(2^j))|^{(1-\theta)/3}$$

and, by (4.3), we have

$$\gamma''^{-1}(2^{j+1}) \gtrsim \gamma''^{-1}(2^j),$$

so we have the estimate

$$\|T_j\|_{L^1 \to L^1} \lesssim \gamma''^{-1}(2^j)^{1-\theta} |\gamma'''(\gamma''^{-1}(2^j))|^{(1-\theta)/3}. \tag{4.9}$$

For the L^2 norm, we estimate the supremum of its multiplier

$$m_j(\xi,\eta) = \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} \frac{G'(t)}{t^\theta \psi(t)^{1-\theta}} dt,$$

with the same notation as above. Integrating by parts,

$$\begin{aligned}m_j(\xi,\eta) = &\frac{G(\gamma''^{-1}(2^j))}{\gamma''^{-1}(2^j)^\theta \psi(\gamma''^{-1}(2^j))^{1-\theta}} - \frac{G(\gamma''^{-1}(2^{j+1})}{\gamma''^{-1}(2^{j+1})^\theta \psi(\gamma''^{-1}(2^{j+1})^{1-\theta}} \\ &+ \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^j)} G(t) \Big(\frac{\theta}{t^{\theta+1}\psi(t)^{1-\theta}} + \frac{(1-\theta)\psi'(t)}{t^\theta \psi(t)^{2-\theta}} \Big) dt.\end{aligned} \tag{4.10}$$

Again, by Van der Corput's lemma,

$$|G(t)| \lesssim \frac{1}{|\gamma'''(t)|^{1/3}}.$$

Using (a.4) again, each of the first two terms in (4.10) is estimated by

$$\begin{aligned}\Big| \frac{G(\gamma''^{-1}(2^j))}{\gamma''^{-1}(2^j)^\theta \psi(\gamma''^{-1}(2^j))^{1-\theta}} \Big| &\lesssim \frac{|\gamma'''(\gamma''^{-1}(2^j))|^{(1-\theta)/3}}{\gamma''^{-1}(2^j)^\theta |\gamma'''(\gamma''^{-1}(2^j))|^{1/3}} \\ &\lesssim \frac{1}{\gamma''^{-1}(2^j)^\theta |\gamma'''(\gamma''^{-1}(2^j))|^{\theta/3}}\end{aligned}$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

and, also using (4.3) and (4.4),

$$\Big|\frac{G(\gamma''^{-1}(2^{j+1}))}{\gamma''^{-1}(2^{j+1})^{\theta}\psi(\gamma''^{-1}(2^{j+1}))^{1-\theta}}\Big| \lesssim \frac{|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{(1-\theta)/3}}{\gamma''^{-1}(2^{j+1})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{1/3}} \lesssim \frac{1}{\gamma''^{-1}(2^{j})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j}))|^{\theta/3}}.$$

Similarly, we estimate the integral by

$$\begin{aligned}\Big|\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{G(t)\theta}{t^{\theta+1}\psi(t)^{1-\theta}}dt\Big| &\le \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{|G(t)|}{t^{\theta+1}\psi(t)^{1-\theta}}dt \\ &\lesssim \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{|\gamma'''(t)|^{(1-\theta)/3}}{t^{\theta+1}|\gamma'''(t)|^{1/3}}dt \\ &\lesssim \frac{1}{|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{\theta/3}}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{1}{t^{\theta+1}}dt \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j}))|^{\theta/3}},\end{aligned}$$

where we have again used (4.4), and, using the fact that ψ is monotone (and thus increasing),

$$\begin{aligned}\Big|\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{G(t)(1-\theta)\psi'(t)}{t^{\theta}\psi(t)^{2-\theta}}dt\Big| &\le \int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{|G(t)|}{t^{\theta}}\frac{\psi'(t)}{\psi(t)^{2-\theta}}dt \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j+1})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{1/3}}\int_{\gamma''^{-1}(2^{j+1})}^{\gamma''^{-1}(2^{j})} \frac{\psi'(t)}{\psi(t)^{2-\theta}}dt \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j+1})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{1/3}}\frac{1}{\psi(\gamma''^{-1}(2^{j+1}))^{1-\theta}} \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j+1})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j+1}))|^{\theta/3}} \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j}))|^{\theta/3}}.\end{aligned}$$

We therefore conclude that

$$\|T_j\|_{L^2\to L^2} \lesssim \frac{1}{\gamma''^{-1}(2^{j})^{\theta}|\gamma'''(\gamma''^{-1}(2^{j}))|^{\theta/3}}. \tag{4.11}$$

By interpolation, if $0 \le \tau \le 1$ and

$$\frac{1}{p} = \tau + \frac{1-\tau}{2},$$

then T_j is bounded on $L^p(\mathbb{R}^2)$ with norm

$$\begin{aligned}\|T_j\|_{L^p\to L^p} &\le \|T_j\|_{L^1\to L^1}^{\tau}\|T_j\|_{L^2\to L^2}^{1-\tau} \\ &\lesssim \frac{1}{\gamma''^{-1}(2^{j})^{\theta-\tau}|\gamma'''(\gamma''^{-1}(2^{j}))|^{(\theta-\tau)/3}}.\end{aligned} \tag{4.12}$$

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

By Remark 4.4, the series

$$\sum_{j=0}^{\infty} \|T_j\|_{L^p\to L^p} \lesssim \sum_{j=0}^{\infty} \frac{1}{\gamma''^{-1}(2^j)^{\theta-\tau}|\gamma'''(\gamma''^{-1}(2^j))|^{(\theta-\tau)/3}}$$

converges as long as $\theta > \tau$, that is,

$$\frac{1}{p} = \frac{1+\tau}{2} < \frac{1+\theta}{2}.$$

■

5 Discussion

In [FG99], the author proved that the operator

$$f \mapsto \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} f(x-t)\frac{e^{2\pi i\gamma(t)}}{t^{\theta}\,\psi(t)^{1-\theta}}dt \tag{5.1}$$

is bounded on $L^p(\mathbb{R})$ as long as γ, ψ satisfy assumptions that are analogous to our assumptions (a.1)–(a.5) (as discussed in this article) and p satisfies

$$\frac{1}{2} \le \frac{1}{p} \le \frac{1+\theta}{2},$$

for $0 \le \theta < 1$. Thus, a natural question is whether Corollary 4.5 is true for the endpoint case

$$\frac{1}{p} = \frac{1+\theta}{2}.$$

References

[CDZ17] J. Chen, B. M. Damtew, and X. Zhu, *Oscillatory hyper Hilbert transforms along general curves*. Front. Math. China **12**(2017), no. 2, 281–299. https://doi.org/10.1007/s11464-016-0574-3

[CFWZ08] J. Chen, D. Fan, M. Wang, and X. Zhu, *L^p bounds for oscillatory hyper-Hilbert transform along curves*. Proc. Am. Math. Soc. **136**(2008), no. 9, 3145–3153. https://doi.org/10.1090/s0002-9939-08-09325-8

[Cha96] S. Chandarana, *L^p-bounds for hypersingular integral operators along curves*. Pac. J. Math. **175**(1996), no. 2, 389–416. https://doi.org/10.2140/pjm.1996.175.389

[CNSW99] M. Christ, A. Nagel, E. M. Stein, and S. Wainger, *Singular and maximal radon transforms: analysis and geometry*. Ann. Math. **150**(1999), no. 2, 489–577. https://doi.org/10.2307/121088

[Erd56] A. Erdélyi, *Asymptotic expansions, VI*, Dover Publications, Inc., New York, NY, 1956, p. 108.

[FG99] M. Folch-Gabayet, *A family of strongly singular operators*. J. Austral. Math. Soc. Ser. A **67**(1999), no. 1, 58–84. https://doi.org/10.1017/s1446788700000872

[FGS19] M. Folch-Gabayet and R. A. Sáenz, *Weak-type $(1,1)$ estimates for strongly singular operators*. Anal. Math. **45**(2019), no. 3, 505–514. https://doi.org/10.1007/s10476-019-0925-x

[FR66] E. B. Fabes and N. M. Rivière, *Singular integrals with mixed homogeneity*. Stud. Math. **27**(1966), 19–38. https://doi.org/10.4064/sm-27-1-19-38

[LG14] J. Li and G. Gao, *Boundedness of oscillatory hyper-Hilbert transform along curves on Sobolev spaces*. J. Funct. Spaces **2014**(2014), no. 5, 489068. (English). https://doi.org/10.1155/2014/489068

[LY18] J. Li and Y. Haixia, *The oscillatory hyper-Hilbert transform associated with plane curves*. Can. Math. Bull. **61**(2018), no. 4, 802–811. https://doi.org/10.4153/cmb-2017-064-9

[NRW74] A. Nagel, N. Rivière, and S. Wainger, *On Hilbert transforms along curves*. Bull. Am. Math. Soc. **80**(1974), 106–108. https://doi.org/10.1090/s0002-9904-1974-13374-4

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press

[NRW76] A. Nagel, N. M. Rivière, and S. Wainger, *On Hilbert transforms along curves. II*. Am. J. Math. **98**(1976), 395–403. https://doi.org/10.2307/2373893

[Ste86] E. M. Stein, *Beijing lectures in harmonic analysis. (Summer School in analysis, Beijing, The People's Republic of China, September 1984)*, *Annals of Mathematics Studies, 112*, Princeton University Press, Princeton, NJ, 1986. (English).

[Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 95c:42002.

[SW78] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*. Bull. Am. Math. Soc. **84**(1978), 1239–1295. https://doi.org/10.1090/s0002-9904-1978-14554-6

[SXX21] W. Sun, R.-L. Xie, and X. Liang-Yu, *Oscillatory hyper-Hilbert transform on Wiener amalgam spaces*. Open Math. **19**(2021), 1579–1587. https://doi.org/10.1515/math-2021-0106

[WY17] X. Wu and Y. Xiao, *Strongly singular integrals along curves on α-modulation spaces*. J. Inequal. Appl. **2017**(2017), no. 13, 185. https://doi.org/10.1186/s13660-017-1458-0

[Zi85] M. F. Zielinski, *Highly oscillatory singular integrals along curves*. Ph.D. thesis, University of Wisconsin -Madison, 1985.

Universidad Nacional Autónoma de México, Mexico
e-mail: folchgab@matem.unam.mx

Universidad de Colima, Mexico
e-mail: rasaenz@ucol.mx

https://doi.org/10.4153/S0008439526102008 Published online by Cambridge University Press