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An Approximation by Lacunary Sequence of Vectors

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Let $(t_k)_{k=0}^{\infty}$ be a sequence of real numbers satisfying $t_0 \neq 0$ and $|t_{k+1}| \geqslant (1+1/M)|t_k|$ for each $k \geqslant 0$, where $M \geqslant 1$ is a fixed number. We prove that, for any sequence of real numbers $(\xi_k)_{k=0}^{\infty}$, there is a real number ξ such that $\|t_k\xi - \xi_k\| > 1/(80M\log(28M))$ for each $k \geqslant 0$. Here, $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer. This is a corollary derived from our main theorem, which is a more general matrix version of this statement with explicit constants.

1. Introduction

Throughout, let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of real numbers, integers and positive integers, respectively. For a vector $\mathbf{x} = (x_1, \dots, x_m)$ in $\mathbb{R}^m$, we denote its length by $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_m^2}$. Also, let $\|\mathbf{x}\|$ be the minimum of $|\mathbf{x} - \mathbf{n}|$, where $\mathbf{n} = (n_1, \dots, n_m)$ runs over every vector of $\mathbb{Z}^m$. In other words, $\|\mathbf{x}\|$ denotes the distance from $\mathbf{x}$ to the nearest point of the lattice $\mathbb{Z}^m$.

In this note we shall prove the following.

Theorem 1.1. *Given $m \in \mathbb{N}$, let $(\boldsymbol{\xi}_k)_{k=0}^{\infty}$ be a sequence of vectors in $\mathbb{R}^m$, and let $(S_k)_{k=0}^{\infty}$ be a sequence of $m \times m$ orthogonal matrices with real entries. Suppose that $h \in \mathbb{N}$, $c(h)$ is a positive real number greater than $8(eh/2)^{1/m}$, and $(t_k)_{k=0}^{\infty}$ is a sequence of positive numbers satisfying*

$$t_{k+h} \geqslant c(h)t_k \tag{1.1}$$

for each integer $k \geqslant 0$. Then there is a vector $\boldsymbol{\xi}$ in $\mathbb{R}^m$ such that

$$\|t_k\boldsymbol{\xi}S_k - \boldsymbol{\xi}_k\| > \delta \tag{1.2}$$

for every $k \geqslant 0$, where

$$\delta = \frac{1}{16}\left(\frac{2}{eh} - \frac{8^m}{c(h)^m}\right)^{1/m}. \tag{1.3}$$

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Inequality (1.2) of the theorem is equivalent to the fact that the balls centred at $(\boldsymbol{\xi}_k + \mathbf{n})S_k^{-1}/t_k$, where $\mathbf{n} \in \mathbb{Z}^m$, with radii δ/t_k, $k = 0, 1, 2, \ldots$, do not cover the space $\mathbb{R}^m$.

An important class of sequences for which (1.1) holds consists of all *lacunary* sequences, namely, those $(t_k)_{k=0}^{\infty}$ with $t_0 > 0$ for which there is a real number $\theta > 1$ such that $t_{k+1} \geqslant \theta t_k$ for every $k \geqslant 0$. Below, we shall write θ in the form $\theta = 1 + 1/M$, where $M > 0$. By choosing an appropriate $h \in \mathbb{N}$ we shall derive the following corollary from Theorem 1.1.

Corollary 1.2. *Given $m \in \mathbb{N}$, let $(\boldsymbol{\xi}_k)_{k=0}^{\infty}$ be a sequence of vectors in $\mathbb{R}^m$, and let $(S_k)_{k=0}^{\infty}$ be a sequence of $m \times m$ orthogonal matrices with real entries. Suppose that $(t_k)_{k=0}^{\infty}$ is a sequence of positive numbers satisfying*

$$t_{k+1} \geqslant (1 + 1/M)t_k \tag{1.4}$$

for some real number $M \geqslant 1$ and each integer $k \geqslant 0$. Then there is a vector $\boldsymbol{\xi}$ in $\mathbb{R}^m$ such that

$$\|t_k \boldsymbol{\xi} S_k - \boldsymbol{\xi}_k\| > \frac{1}{16(5M \log(28M))^{1/m}} \tag{1.5}$$

for every $k \geqslant 0$.

A special case of Corollary 1.2, corresponding to $m = 1$ and $S_k = \pm 1$, is the following statement.

Corollary 1.3. *Let $(\xi_k)_{k=0}^{\infty}$ be a sequence of real numbers, and let $M \geqslant 1$. Suppose that $(t_k)_{k=0}^{\infty}$ is a sequence of real numbers satisfying $t_0 \neq 0$ and $|t_{k+1}| \geqslant (1 + 1/M)|t_k|$ for each integer $k \geqslant 0$. Then there is a real number ξ such that*

$$\|t_k \xi - \xi_k\| > \frac{1}{80M \log(28M)} \tag{1.6}$$

for every $k \geqslant 0$.

Suppose that $a + ib$ is a complex number of modulus $\theta = \sqrt{a^2 + b^2} > 1$. Set $\theta = 1 + 1/M$ and take (in Corollary 1.2) $m = 2$, $t_k = \theta^k$,

$$S_k = \begin{pmatrix} \cos(k\phi) & \sin(k\phi) \\ -\sin(k\phi) & \cos(k\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}^k$$

for every $k \geqslant 0$, $\boldsymbol{\xi} = (u, v)$ and $\boldsymbol{\xi}_k = (u_k, v_k)$ for $k \geqslant 0$. It follows that $\boldsymbol{\xi} S_k = (u\cos(k\phi) - v\sin(k\phi), v\cos(k\phi) + u\sin(k\phi))$. Put $\xi = u + iv \in \mathbb{C}$ and $z_k = u_k + iv_k \in \mathbb{C}$. Note that $\|t_k \boldsymbol{\xi} S_k - \boldsymbol{\xi}_k\|$ is equal to the distance from $(a + ib)^k \xi - z_k$ to the nearest Gaussian integer in $\mathbb{Z}[i]$. Hence Corollary 1.2 implies the following.

Corollary 1.4. *Let $(z_k)_{k=0}^{\infty}$ be a sequence of complex numbers, and let $M \geqslant 1$. Suppose that $a + ib$ is a complex number of modulus $\sqrt{a^2 + b^2} = 1 + 1/M$, where $M \geqslant 1$. Then there is a complex number ξ such that, for each $k \geqslant 0$, the distance from $(a + ib)^k \xi - z_k$ to the nearest Gaussian integer in $\mathbb{Z}[i]$ is at least $1/(36\sqrt{M \log(28M)})$.*

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In the next section we shall explain the origin of the problem and recall some earlier results in this direction. The proof of Theorem 1.1 is based on the Lovász local lemma from probabilistic combinatorics. This beautiful relation between the Lovász local lemma and the distribution of intervals with denominators t_k was discovered in a recent paper of Peres and Schlag [14]. We follow their idea in our proof.

The proof of Theorem 1.1 is given in Section 3. It is completely self-contained. The proof of Corollary 1.2 is given in Section 4. All parameters and constants featuring in Corollary 1.2 were found by calculating the minimum point of the involved function.

2. Earlier results

The problem of whether, for every lacunary sequence of positive numbers $(t_k)_{k=0}^{\infty}$, there is a number $\xi \in \mathbb{R}$ for which the sequence of fractional parts $\{t_k\xi\}_{k=0}^{\infty}$ is not dense in $[0,1)$ was posed by Erdős [8]. It was solved by de Mathan [12] and Pollington [15] independently. Their results imply that if $t_0 > 0$ and $t_{k+1} \geqslant (1+1/M)t_k$ for some $M > 0$ and each $k \geqslant 0$, then there are $\xi \in \mathbb{R}$ and $\tau = \tau(M) > 0$ such that $\|t_k\xi\| > \tau$ for every $k \geqslant 0$. Curiously, the problem of Erdős was already solved by Khintchine many years before it was posed: see Hilfssatz III in [11]. It seems likely that Khintchine's result was simply forgotten.

The relation of this kind of result to another problem of Erdős on the chromatic number of some graphs was found by Katznelson [10]. In this context, the dependence of τ on M becomes an important problem. (Following Khintchine's proof [11], one gets $\tau(M) = c/(M\log M)^2$ with an absolute positive constant c.) Subsequent improvements of the function $\tau(M)$ were obtained in [10], [1] and [5]. Recently, Peres and Schlag [14] proved that $\tau(M)$ can be taken as $1/(cM\log(M))$ for $M \geqslant 2$, where c is some large but explicit constant. This estimate is sharp up to a logarithmic factor. Moshchevitin [13] proved that one can take $\tau(M) = 1/(2^{11}M\log M)$ if $t_0 \geqslant 2$ and $M \geqslant 8$. Our estimate (1.6) of Corollary 1.3 strengthens this bound to $\tau(M) = 1/(80M\log(28M))$ under the relaxed condition $M \geqslant 1$. In addition, it proves this bound for $\|t_k\xi - \xi_k\|$, where $(\xi_k)_{k=0}^{\infty}$ is an arbitrary sequence of real numbers, which for some applications can be quite useful. See [5] for a result concerning $\tau(M)$ in the range $M < 1$. (See also [2] and [6] for some related work.)

An additional motivation for the study of $\tau(M)$ is the problem of distribution of the fractional parts $\{\xi\alpha^k\}_{k=0}^{\infty}$, where $\alpha > 1$ and $\xi \neq 0$. This is an old problem initiated by Weyl [16]. We remark that even the distribution of the simplest sequences, such as $\{(3/2)^k\}_{k=0}^{\infty}$, is far from understood. See, for instance, [9] for an estimate between two extreme limit points of the sequence $\{\xi\alpha^k\}_{k=0}^{\infty}$ when $\alpha > 1$ is a rational number, and [4] for a similar estimate when α is an algebraic number. (There is almost nothing known if α is a transcendental number.) Inequality (1.6) applied to the sequence $t_k = \alpha^k$ thus indicates the limits beyond which one cannot go in such estimates.

Similarly, one can study the distribution of a sequence of complex numbers, for instance, the sequence of powers of a complex number $a + ib$, where $a^2 + b^2 > 1$, with respect to the Gaussian integers $\mathbb{Z}[i]$. This problem was studied, for example, in [7]. Corollary 1.4 improves upon corresponding estimates obtained earlier in [7] and [13].

A version of Corollary 1.2 with the denominator in (1.5) being $2^{13}m(M\log M)^{1/m}$, under the somewhat stronger restriction $M \geqslant 8$ instead of $16(5M\log(28M))^{1/m}$ for $M \geqslant 1$ (as in (1.5)),

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was recently obtained by Moshchevitin [13]. Another useful observation is that we remove some unnecessary restrictions on t_0.

Finally, we remark that Theorem 1.1 can be applied not only to lacunary sequences $(t_k)_{k=0}^{\infty}$, but also to their unions and similar sequences. Also, usually, for a given lacunary sequence, one gets stronger result by applying Theorem 1.1 directly rather than by Corollary 1.3. For example, let us consider the sequence $t_k = (11/10)^k$, where $k \geqslant 0$. Corollary 1.3 with $M = 10$ implies the existence of $\xi \in \mathbb{R}$ for which $\|\xi(11/10)^k - \xi_k\| > 0.0002218$ for each $k \geqslant 0$. On the other hand, by Theorem 1.1 with $m = 1$, we find that $c(h) = (11/10)^h$. Hence, by (1.3), we have $\delta = (1/(eh) - 4(10/11)^h)/8$. A simple calculation shows that the maximal value of $\delta = 0.0004258\ldots$ is attained at $h = 96$. Thus, applying Theorem 1.1 directly, we derive that for any sequence of real numbers $(\xi_k)_{k=0}^{\infty}$ there is a real number ξ such that $\|\xi(11/10)^k - \xi_k\| > 0.0004258$ for each $k \geqslant 0$.

3. Proof of Theorem 1.1

The following probabilistic lemma is given in [14]. It is a version of the Lovász local lemma (see, *e.g.*, [3, Chapter 5]). Moshchevitin [13] showed that one can avoid the use of this probabilistic lemma and give a direct argument instead. However, the proof based on such a lemma is simpler. For the convenience of the reader we shall give its proof.

Lemma 3.1. *Let A_j, $j = 1, 2, \ldots$, be events in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Put $B_j = \Omega$ for $j \leqslant 1$ and $B_j = A_1^c \cap \cdots \cap A_{j-1}^c$ for $j \geqslant 2$. If $\tau \in (0,1)$ and $h \in \mathbb{N}$ are such that $\mathbb{P}(A_i \mid B_{i-h+1}) \leqslant \tau(1-\tau)^{h-1}$ for each $i \in \mathbb{N}$, then $\mathbb{P}(B_j) \geqslant (1-\tau)^{j-1}$ for each $j \in \mathbb{N}$.*

Proof. We have $B_j \subseteq B_{j-1}$. Note that $\mathbb{P}(B_j) = \mathbb{P}(B_{j-1})(1 - \mathbb{P}(A_{j-1} \mid B_{j-1}))$, so $\mathbb{P}(B_j) = \prod_{i=1}^{j-1}(1 - \mathbb{P}(A_i \mid B_i))$. The claim would follow from the inequality $\mathbb{P}(A_i \mid B_i) \leqslant \tau$. We shall prove this inequality by induction on i.

For $i \leqslant h$, we have $B_{i-h+1} = \Omega$, hence $\mathbb{P}(A_i) = \mathbb{P}(A_i \mid B_{i-h+1}) \leqslant \tau(1-\tau)^{h-1}$. This implies that $\mathbb{P}(A_i \mid B_i) \leqslant \mathbb{P}(A_i) \leqslant \tau(1-\tau)^{h-1} \leqslant \tau$. Next, suppose that $i > h$. Then

$$\mathbb{P}(A_i \mid B_i)\,\mathbb{P}(B_i \mid B_{i-h+1}) = \mathbb{P}(A_i \cap B_i \mid B_{i-h+1}) \leqslant \mathbb{P}(A_i \mid B_{i-h+1}) \leqslant \tau(1-\tau)^{h-1}. \tag{3.1}$$

Note that $\mathbb{P}(B_i \mid B_{i-h+1}) = \prod_{k=i-h+1}^{i-1}(1 - \mathbb{P}(A_k \mid B_k))$. By induction on i, we have $\mathbb{P}(A_k \mid B_k) \leqslant \tau$ for every k satisfying $i - h + 1 \leqslant k \leqslant i - 1$. Hence $\mathbb{P}(B_i \mid B_{i-h+1}) \geqslant (1-\tau)^{h-1}$. Inserting this lower bound into (3.1) yields $\mathbb{P}(A_i \mid B_i) \leqslant \tau$, as claimed. □

Proof of Theorem 1.1. Fix a positive number ε. On replacing $\boldsymbol{\xi}$ by $N^{-1}\boldsymbol{\xi}$ and each t_k by Nt_k, we have $t_k\boldsymbol{\xi} = Nt_kN^{-1}\boldsymbol{\xi}$. So, without loss of generality every t_k can be assumed to be arbitrarily large, say, $t_k > 1/\varepsilon$.

Fix some $\sigma > \delta$. Fix $k \geqslant 0$. Note that the inequality $\|t_k\mathbf{x}S_k - \boldsymbol{\xi}_k\| < \sigma$, where $\mathbf{x} = (x_1, \ldots, x_m)$, $S_k = (a_{ij})_{1 \leqslant i,j \leqslant m}$ and $\boldsymbol{\xi}_k = (\xi_1, \ldots, \xi_m)$, is equivalent to

$$(a_{11}x_1 + \cdots + a_{m1}x_m - (\xi_1 + n_1)/t_k)^2 + \cdots + (a_{m1}x_1 + \cdots + a_{mm}x_m - (\xi_m + n_m)/t_k)^2 < (\sigma/t_k)^2.$$

Here, n_j is the nearest integer to $t_k(a_{j1}x_1 + \cdots + a_{jm}x_m) - \xi_j$ for every $j = 1, \ldots, m$. Since the matrix S_k is orthogonal, this inequality defines a union E_k of at most $(\lceil t_k \rceil + 1)^m$ open balls of

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radii σ/t_k in $[0,1)^m$ with centres at $((\xi_1+n_1)/t_k,\dots,(\xi_m+n_m)/t_k)$, where each x_j and each $(\xi_j+n_j)/t_k$ is understood as a number modulo 1.

Put $\ell_k=\lfloor \log_2(t_k/(2\sigma))\rfloor$. Then $2^{-\ell_k-1}<2\sigma/t_k\leqslant 2^{-\ell_k}$. Divide the unit cube $[0,1]^m$ into $2^{m\ell_k}$ closed cubes with side $2^{-\ell_k}$. The number $2^{-\ell_k}$ is greater than or equal to the diameter $2\sigma/t_k$ of a ball in E_k, so at most 2^m of those cubes have a non-empty intersection with one ball of E_k. By the inequality $2^{-\ell_k}<4\sigma/t_k$, the Lebesgue measure of E_k is at most

$$(\lceil t_k\rceil+1)^m 2^m 2^{-m\ell_k}<(\lceil t_k\rceil+1)^m 2^m 4^m\sigma^m/t_k^m<(1+2/t_k)^m 8^m\sigma^m.$$

Since $1/t_k<\varepsilon$, the above inequality can be written in the form

$$\mu(A_k)<(1+2\varepsilon)^m 8^m\sigma^m, \tag{3.2}$$

where A_k is a union of cubes with side $2^{-\ell_k}$ having a common point with E_k and $A_k^c=[0,1]^m\setminus A_k$ is a union of all remaining closed cubes with side $2^{-\ell_k}$.

Set $B_j=A_1^c\cap\cdots\cap A_{j-1}^c$. Since the sets B_j are all closed and $B_j\subseteq B_{j-1}$, for the proof of (1.2) it suffices to show that the set $\cap_{i=1}^\infty B_i$ is non-empty. Then every $\boldsymbol{\xi}\in\cap_{i=1}^\infty B_i$ satisfies $\|t_k\boldsymbol{\xi}S_k-\boldsymbol{\xi}_k\|\geqslant\sigma>\delta$ for each $k\geqslant 0$. We shall use (3.2) and Lemma 3.1.

It suffices to show that the sets B_j are non-empty for every $j\in\mathbb{N}$. Let us estimate $\mu(A_i\cap B_{i-h+1})$ for $i>h$. Note that B_{i-h+1} is a union of closed cubes C_s with side $2^{-\ell_{i-h}}$, *i.e.*, $B_{i-h+1}=\cup_s C_s$. Clearly, $\mu(B_{i-h+1})=\sum_s\mu(C_s)$. By induction on i, we can assume that B_{i-h+1} is non-empty. We have $\mu(A_i\cap B_{i-h+1})=\sum_s\mu(A_i\cap C_s)$. Each connected part of A_i is contained into a cube with side $2(2\sigma/t_i)\leqslant 2^{1-\ell_i}$. Each cube C_s contains at most $(1+(1+\lceil t_i\rceil)2^{-\ell_{i-h}})^m$ of those connected parts. Hence

$$\mu(A_i\cap C_s)\leqslant(1+(1+\lceil t_i\rceil)2^{-\ell_{i-h}})^m 2^{m(1-\ell_i)}<(1+(2+t_i)\mu(C_s)^{1/m})^m 2^{m(1-\ell_i)}.$$

Using the inequalities $(1+y)^m\leqslant 2^{m-1}(1+y^m)$ and $1/t_i<\varepsilon$, we further bound

$$\begin{aligned}\mu(A_i\cap C_s)&<2^{m-1}(1+(2+t_i)^m\mu(C_s))2^{m(1-\ell_i)}\\&<2^{2m-1}\big(2^{-m\ell_i}+(1+2\varepsilon)^m t_i^m 2^{-m\ell_i}\mu(C_s)\big)\\&=2^{2m-1}\mu(C_s)\big(2^{m(\ell_{i-h}-\ell_i)}+(1+2\varepsilon)^m t_i^m 2^{-m\ell_i}\big).\end{aligned}$$

Next, from $2^{\ell_{i-h}-\ell_i}<2t_{i-h}/t_i$ and $t_i2^{-\ell_i}<4\sigma$, we obtain that

$$\begin{aligned}\mu(A_i\cap C_s)&<\mu(C_s)2^{2m-1}(2^m(t_{i-h}/t_i)^m+(1+2\varepsilon)^m 2^{2m}\sigma^m)\\&=\mu(C_s)(2^{3m-1}(t_{i-h}/t_i)^m+2^{4m-1}(1+2\varepsilon)^m\sigma^m).\end{aligned}$$

Summing over all s and using $\sum_s\mu(C_s)=\mu(B_{i-h+1})$, we obtain

$$\mu(A_i\cap B_{i-h+1})=\sum_s\mu(A_i\cap C_s)<\mu(B_{i-h+1})(2^{3m-1}(t_{i-h}/t_i)^m+2^{4m-1}(1+2\varepsilon)^m\sigma^m).$$

By (1.1), $t_{i-h}/t_i\leqslant 1/c(h)$, so

$$\mu(A_i\cap B_{i-h+1})/\mu(B_{i-h+1})<2^{3m-1}c(h)^{-m}+2^{4m-1}(1+2\varepsilon)^m\sigma^m.$$

Defining $\mathbb{P}(A)=\mu(A)$, we deduce that

$$\mathbb{P}(A_i\mid B_{i-h+1})=\mu(A_i\cap B_{i-h+1})/\mu(B_{i-h+1})<2^{3m-1}c(h)^{-m}+2^{4m-1}(1+2\varepsilon)^m\sigma^m. \tag{3.3}$$

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Taking into account (3.2), we see that (3.3) holds not only for $i > h$ but for each non-negative integer i.

Let us apply Lemma 3.1 with $\tau = 1/h$. Since $\tau(1-\tau)^{h-1} = (1-1/h)^{h-1}/h$, by (3.3), it suffices to check that the right-hand side of (3.3) is at most $(1-1/h)^{h-1}/h$, *i.e.*, that

$$h(1+1/(h-1))^{h-1}(2^{3m-1}c(h)^{-m} + 2^{4m-1}(1+2\varepsilon)^m\sigma^m) < 1.$$

Note that, by (1.3), $(16\delta)^m + (8/c(h))^m = 2/(eh)$. Therefore, it remains to show that

$$h(1+1/(h-1))^{h-1}(2^{3m-1}c(h)^{-m} + 2^{4m-1}(1+2\varepsilon)^m\sigma^m) < he(2^{3m-1}c(h)^{-m} + 2^{4m-1}\delta^m) = 1.$$

By taking ε small enough and σ close enough to δ, we see that this inequality holds, because of $(1+1/(h-1))^{h-1} < e$. This completes the proof of the theorem. □

4. Proof of Corollary 1.2

Obviously (1.4) implies that (1.1) holds with $c(h) = (1+1/M)^h$. From (1.3) we find that

$$(16\delta)^m + 8^m(1+1/M)^{-hm} = 2/(eh).$$

Comparing (1.2) and (1.5), we see that it suffices to show that $(16\delta)^m > 1/(5M\log(28M))$. Clearly, this inequality holds and implies (1.5) provided that, for some $h \in \mathbb{N}$, we have

$$h/(5M\log(28M)) < 2/e - 8^m h(1+1/M)^{-hm} \tag{4.1}$$

for every $m \in \mathbb{N}$.

Put $h = \lfloor cM\log(28M)\rfloor$ for some constant $c > 1$ to be chosen later. Using $h \leqslant cM\log(28M)$, we find that the left-hand side of (4.1) is at most $c/5$. It is easy to see that the function $G(h) = h(1+1/M)^{-hm}$ is decreasing in h if $mh\log(1+1/M) > 1$. This obviously holds because $c > 1$ and $M \geqslant 1$. Hence, using $h > cM\log(28M) - 1$, we find that the right hand side of (4.1) is greater than

$$2/e - 8^m(cM\log(28M) - 1)(1+1/M)^{m(1-cM\log(28M))}.$$

It remains to show that there is a $c > 1$ for which

$$c/5 < 2/e - 8^m(cM\log(28M) - 1)(1+1/M)^{m(1-cM\log(28M))}.$$

Equivalently, setting

$$F(c,m,M) = c/5 + 8^m(cM\log(28M) - 1)(1+1/M)^{m(1-cM\log(28M))},$$

we need to show that there is a $c > 1$ such that $F(c,m,M) < 2/e$ for every $m \in \mathbb{N}$ and every $M \geqslant 1$.

Take $c = 3.2$. It is easy to see that the function $F(3.2,m,M)$ is decreasing in $m \geqslant 1$ and in $M \geqslant 1$. Hence, by calculating $F(3.2,1,1)$, we find that

$$F(3.2,m,M) \leqslant F(3.2,1,1) = 0.7353\cdots < 0.7357\cdots = 2/e.$$

This completes the proof of the corollary.

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