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THE RESONANCE METHOD FOR LARGE CHARACTER SUMS

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Abstract. We consider the size of large character sums, proving new lower bounds for $\Delta(N, q) = \sup_{\chi \neq \chi_0 \bmod q} |\sum_{n<N} \chi(n)|$ in almost all ranges of N. The proofs use the resonance method and saddle point analysis.

§1. *Introduction.* Let q be a large modulus and let $N < q$. Thinking of the character sum $S(N, \chi) = \sum_{n \leqslant N} \chi(n)$ as a function of the non-principal Dirichlet character $\chi \bmod q$, set $\Delta(N, q)$ for its L^∞-norm,

$$\Delta(N, q) = \|S(N, \chi)\|_\infty = \sup_{\chi \neq \chi_0 \bmod q} \left| \sum_{n \leqslant N} \chi(n) \right|.$$

Non-trivial upper bounds for $\Delta(N, q)$ are of great interest in number theory due to the wealth of applications, but such bounds have proven formidable: essentially the only results in this direction are the classical Pólya–Vinogradov inequality (non-trivial for $N > \sqrt{q} \log q$), Burgess's work [1] ($N > q^{1/4+\epsilon}$), and the work of Graham and Ringrose [3], which is effective in a wider range, but only when q has many small factors.

In this paper we consider instead what lower bounds may be given for large character sums. An easy bound comes from the L^2-norm

$$\sqrt{N} \ll_\delta \|S(N, \chi)\|_2 \leqslant \Delta(N, q), \quad N < (1 - \delta)q,$$

while in [4], Granville and Soundararajan have improved on this for all $N < \sqrt{q}$ by estimating with the L^{2k}-norm, $k = \lfloor \log q / \log N \rfloor$. When $\log N$ is very small this yields impressive results, but for N a power of q, $N = q^\theta$ with $0 < \theta < 1$, the estimate obtained is of shape $\Delta(N, q) \geqslant \sqrt{N} (\log q)^{A(\theta)}$, improving on the trivial one by only a power of $\log q$. In our primary result we repair this defect, proving a bound of

$$\Delta(N, q) \geqslant \sqrt{N} \exp\left((1 + o(1)) \sqrt{(1 - \theta) \frac{\log q}{\log \log q}} \right), \quad N = q^\theta, 0 < \theta < 1. \tag{1}$$

This result is in keeping with the available omega result for the corresponding Dirichlet L-functions at the central point.

Our proof of (1) adapts the resonance method, which was originally introduced by Soundararajan [12] to produce the best known large values

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of L-functions on the half line, and has subsequently been applied in several other contexts (see [9, 11]). Actually, as we will explain in the next section, the resonance method strictly contains the moment estimate of [4], and so we update the previously known bounds on $\Delta(N, q)$ in almost all ranges of N.

When studying large character sums, the range in which $\log N = (\log q)^\theta$, $\theta = 1/2 + o(1)$ appears to mark a transition in the behavior. For instance, a conjecture of Farmer *et al* [2] on the extreme values of L-functions on the half line would imply the upper bound

$$\Delta(N, q) \leqslant \sqrt{N} \exp((1/\sqrt{2} + o(1))\sqrt{\log q \log\log q}),$$

which gives the square-root cancelation estimate $\Delta(N, q) \ll_{\theta,\epsilon} N^{1/2+\epsilon}$ for $\log N = (\log q)^\theta$ for any $\theta > 1/2$, but gives nothing for $\theta \leqslant 1/2$. By contrast, in [4] it is demonstrated that $\Delta(N, q) \geqslant N^{1/2+\delta(\theta)}$, $\delta > 0$ when $\log N = (\log q)^\theta$ with $\theta < 1/2$. Our results also follow this dichotomy. Thus, although above we stated (1) only for N a fixed power of q, in Theorem 3.1 we show that the result actually holds when $\log N > (\log q)^{1/2}(\log\log q)^{1/2+o(1)}$. Our Theorem 3.2 treats the transition region and Theorem 3.4 considers small N, with $\log N = (\log q)^\theta$ for $\theta < 1/2$.

In order to highlight our results in the small range, we first pause to briefly describe the somewhat different situation of large real character sums. An analogue of $\Delta(N, q)$ for real characters is

$$\Delta_{\mathbb{R}}(N, q) = \sup_{D\in\mathcal{D}, q<|D|\leqslant 2q} \left|\sum_{n\leqslant N} \left(\frac{D}{n}\right)\right|,$$

where $\mathcal{D}$ denotes the set of fundamental discriminants. An idea going back to Littlewood [8] is that one can generate large sums involving real characters by considering those D in prescribed residue classes modulo $Q = \prod_{p<P} p$, with the residues chosen to ensure that $(D/p) = 1$ for all $p < P$. Since, by this arrangement, $(D/n) = 1$ whenever n is composed of primes smaller than P, the character sum has a large positive bias on these P-smooth numbers, and one hopes to prove that the remaining part of the sum is of smaller magnitude (actually the positive bias is on numbers that are P-smooth after division by a square, an important distinction when the sum is long). This idea was carried out in [4] in parallel to the estimate for $\Delta(N, q)$, with comparable and generally stronger results.

An obvious barrier for the size of the parameter P in Littlewood's method is $P = \log q$, since taking $P = c \log q$ requires working with D in the range $q < |D| \leqslant 2q$ modulo $Q = q^{c+o(1)}$. Although there are many square residues for D modulo Q, a c larger than 1 requires a subtle understanding of their distribution. In fact, in [4], $c = 1/3$ is chosen for technical reasons. With $P = c \log q$, $c < 1$, and $\log N = (\log q)^\theta$, $\theta < 1/2$, the number of numbers less than N that are P-smooth after division by a square is, up to constants, the same as the number that are actually P-smooth†. Thus, in the range

† When $\theta > 1/2$ the additional numbers make up a dominant proportion.

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$\log N = (\log q)^\theta$, $0 < \theta < 1/2$, [4] obtains

$$\Delta_{\mathbb{R}}(N, q) \geqslant \Psi\left(N, \frac{1}{3}\log q\right), \qquad \Psi(x, y) = \sum_{n \leqslant x,\, p \mid n \Rightarrow p < y} 1,$$

and it would seemingly require a great effort to make this method yield, for any fixed $\theta < 1/2$, the same result with $1/3$ replaced by a constant a little greater than 1. Meanwhile, we prove in Theorem 3.4 that for all sufficiently large q,

$$\Delta(N, q) \geqslant \Psi(N, (c(\theta) - \epsilon)\log q), \qquad \log N = (\log q)^\theta$$

with c increasing, $c(\theta) \to \infty$ as $\theta \to 1/2$ and $c(0) > 0.398\,29 > 1/3$. Our bounds for $\Delta(N, q)$ are also generally stronger than those presently known for $\Delta_{\mathbb{R}}(N, q)$ in the other regimes. Corresponding improvements for $\Delta_{\mathbb{R}}(N, q)$ are, however, available.

§2. *Discussion of method.* In the next section we will give precise statements of our results; here we outline the resonance method for character sums, and its advantage over the previous methods in the general and real character setting.

Observe that by pairing L^∞ with L^1, we may write

$$\Delta(N, q)^2 = \sup_{w(\chi) \neq 0} \frac{\langle |S(N, \chi)|^2, w(\chi)\rangle}{\|w(\chi)\|_1}, \tag{2}$$

and therefore a lower bound is obtained by restricting the class of functions w. In the resonance method, the "resonating" weight w is required to be the square of a Dirichlet polynomial with multiplicative coefficients,

$$w(\chi) = \left|\sum_{n \leqslant q/N} r(n)\chi(n)\right|^2, \quad r \text{ multiplicative.}$$

Expanding the sums and using the orthogonality of characters, one is led to the maximization of the ratio of two forms in variables indexed by the primes. When N is sufficiently large compared to q, in the regime $\log N > (\log q)^{1/2+\epsilon}$, this optimization problem is closely related to the one solved by Soundararajan in [12] and so our proof of (1) is by a reduction to the earlier result. When $\log N < (\log q)^{1/2-\epsilon}$ the optimization problem changes, becoming more closely related to the distribution of smooth numbers, and so in this regime we use ideas from that theory, e.g. the saddle point analysis developed in [6].

As regards the relation to previous work, note that with the choice of weights

$$w(\chi) = \left|\left(\sum_{n \leqslant N} \chi(n)\right)^{k-1}\right|^2$$

we recover the previous L^{2k} estimate for $\Delta(N, q)$, while, in the context of real characters, a choice of

$$w(\chi) = \left|\prod_{p < P}(1 + \chi(p))\right|^2$$

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yields the result for $\Delta_{\mathbb{R}}(N, q)$. Thus the resonance method can be understood as simultaneously optimizing the two seemingly unrelated methods used to treat large real and general character sums in [4].

§3. *Statement of results.* We streamline our presentation by assuming that the modulus q is prime. The restriction is needed only when the character sum is quite long: when $N > q^{2/3}$ it becomes advantageous to use Pólya's "Fourier expansion" for primitive character sums to transform sums of length N into dual sums of length q/N. Here the assumption that q is prime ensures that all non-trivial $\chi \bmod q$ are primitive. However, using the same techniques without passing to the dual sum, it is possible to obtain new results on $\Delta(N, q)$ for composite q whenever $N < q^{1-\epsilon}$. The interested reader may find the details in [7].

As discussed in the introduction, our results split into a long, transition and short range. Our theorem in the long regime is as follows.

THEOREM 3.1. *Let q be a large prime and let $4\sqrt{\log q \log_2 q}\log_3 q < \log N$ and $N = q^\theta$ with $\theta \leqslant 2/3$. Then*

$$\Delta(N, q) \geqslant \sqrt{N}\exp\left((1 + o(1))\sqrt{\frac{(1-\theta)\log q}{\log_2 q}}\right).$$

Also,

$$\Delta\left(\frac{q}{N}, q\right) \geqslant \sqrt{\frac{q}{N}}\exp\left((1 + o(1))\sqrt{\frac{(1-\theta)\log q}{2\log_2 q}}\right).$$

These bounds markedly improve those of [4, Theorems 6–8] for $\Delta(N, q)$ in the corresponding range. Our bound is also larger than the one for real characters in [4, Theorem 10]:

$$\Delta_{\mathbb{R}}(N, q) \geqslant \sqrt{N}\exp\left((1 + o(1))\frac{\sqrt{\log q}}{\log_2 q}\right).$$

Next we consider the transition range $\log\log N = (\frac{1}{2} + o(1))\log\log q$.

THEOREM 3.2. *Let q be a large prime and let $\log N = \tau\sqrt{\log q \log_2 q}$ with $\tau = (\log_2 q)^{O(1)}$. Define A and τ' by solving*

$$\tau = \int_A^\infty \mathrm{e}^{-x}\frac{dx}{x}, \qquad \tau' = \int_A^\infty \mathrm{e}^{-x}\frac{dx}{x^2}.$$

Then

$$\Delta(N, q) \geqslant \sqrt{N}\exp\left((1 + o(1))A(\tau + \tau')\sqrt{\frac{\log q}{\log_2 q}}\right).$$

For dual sums of length q/N instead define A and τ' by

$$\sqrt{2}\tau = \int_A^\infty \mathrm{e}^{-x}\frac{dx}{x}, \qquad \tau' = \int_A^\infty \mathrm{e}^{-x}\frac{dx}{x^2}.$$

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With this new definition we have

$$\Delta\left(\frac{q}{N}, q\right) \geqslant \sqrt{\frac{q}{N}} \exp\left((1+o(1))A\left(\tau + \frac{\tau'}{\sqrt{2}}\right)\sqrt{\frac{\log q}{\log_2 q}}\right).$$

Remark. With respect to $\Delta(N, q)$, as $\tau \to \infty$, $A\tau \to 0$ and $A\tau' \to 1$, so that for growing τ we obtain

$$\Delta(N, q) \geqslant \sqrt{N} \exp\left((1-o(1))\sqrt{\frac{\log q}{\log_2 q}}\right), \quad \tau \to \infty.$$

Meanwhile, as $\tau \to 0$, $A \to \infty$ and $\tau \sim \mathrm{e}^{-A}/A$, $\tau' \sim \mathrm{e}^{-A}/A^2$ so that for small τ we obtain

$$\Delta(N, q) \geqslant \sqrt{N} \exp\left((1-o(1))\tau(-\log \tau)\sqrt{\frac{\log q}{\log_2 q}}\right), \quad \tau \to 0.$$

These results should be compared to [**4**, Theorem 5], where, in the same range, it is shown that there exists a constant $c > 0$ such that

$$\Delta(N, q) \geqslant \sqrt{N} \exp\left(c\frac{\tau + \tau\log(1+\tau)}{1+\tau^2}\sqrt{\frac{\log q}{\log_2 q}}\right).$$

Although the constant c is not explicitly given, our result is asymptotically stronger for $\tau \to 0$ and $\tau \to \infty$. Our result for the dual sum $\Delta(q/N, q)$ gives a comparable improvement to [**4**, Theorem 8].

Finally, we consider the range of small N, $(\log q)^{O(1)} \leqslant N < \mathrm{e}^{\sqrt{\log q}}$, where our result is closely related to the distribution of smooth numbers. For reference, recall that for N in this range, the counting function $\Psi(N, \theta \log q)$ for $\theta \log q$-smooth numbers satisfies

$$\Psi(N, \theta \log q) = N \exp(-u \log u - u \log_2(u+2) + u + o(u)),$$
$$u = \frac{\log N}{\log(\theta \log q)},$$

uniformly in $1 \leqslant \theta \ll N/\log q$. The following corollary summarizes our main qualitative results on $\Delta(N, q)$ in this range.

COROLLARY 3.3. *Let q be a large prime and suppose that $\log N = (\log q)^\theta$ for a fixed $0 < \theta < 1/2$. Then*

$$\Delta(N, q) \geqslant \Psi(N, (\kappa(1-\theta) + o(1)) \log q),$$

where $\kappa(t)$ is defined by

$$\kappa(t) = 2^{-1/(1-t)} G(t)^{t/(1-t)}, \qquad G(t) = \frac{\Gamma(1-1/2t)\Gamma(2/t-1)}{\Gamma(3/2t)}. \quad (3)$$

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The function $\kappa(1-\theta)$ *is increasing for* $\theta \in (0, 1/2)$ *with* $\lim_{\theta\to 0}\kappa(1-\theta) = 8/e^3 = 0.398\,296\,5\ldots$ *and* $\kappa(1-\theta) \sim 1/(4-8\theta)$ *as* $\theta \uparrow 1/2$.

When θ *is growing*† *with* q, *for* $\log N = \sqrt{\log q}\exp(-(\log_2 q)^{o(1)})$,

$$\Delta(N, q) \geqslant \Psi(N, \log q \log_2^{1-o(1)} q).$$

Remark. Granville and Soundararajan have conjectured [4, Conjecture 1] that character sums are well approximated by their restricted sum over smooth numbers, in the sense that

$$\sum_{n\leqslant N}\chi(n) = \sum_{\substack{n\leqslant N\\ p|n\Rightarrow p<y}}\chi(n) + o(\Psi(N, y))$$

when $y = (\log q + (\log N)^2)(\log\log q)^A$ for some constant A. Their work shows that if the conjecture is to be true, one must take $A \geqslant 0$, while the above corollary improves this to $A \geqslant 1$.

The results of this corollary may be compared to [4, Theorem 4], its corollaries, and the result [4, Theorem 9] for real characters.

Corollary 3.3 is a direct consequence of the following more technical theorem, which also contains results on the dual problem of long character sums $\Delta(q/N, q)$.

THEOREM 3.4. *Let* q *be a large prime and assume that* $0 < \log N < \sqrt{\log q}$. *Keep the definitions of* $\kappa(t)$ *and* $G(t)$ *in (3). Let* $\sigma > 1/2$ *solve*

$$\log N = \frac{(\log q)^{1-\sigma}}{2(1-\sigma)}G(\sigma)^{\sigma}.$$

Then

$$\Delta(N, q) \geqslant \Psi(N, (1+o(1))\kappa(\sigma)\log q).$$

Let $\sigma' > \frac{1}{2}$ *solve*

$$\log N = \frac{(\log q)^{1-\sigma'}}{2^{2-\sigma'}(1-\sigma')}G(\sigma')^{\sigma'}.$$

Then

$$\Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N}\Psi(N, (1/2+o(1))\kappa(\sigma')\log q).$$

Furthermore, if $\log N < \log_2^2 q \log_3^{-10} q$ *then*

$$\Delta\left(\frac{q}{N}, q\right) \gg \sqrt{q}\frac{\log\log q}{\log(\log N/(\log\log q))}\frac{\Psi(N, \log q)}{N}.$$

† The $o(\cdots)$ notation in this statement may be read as follows: for each $\delta(q)$ tending to 0 with growing q, there exists an $\epsilon(q)$, also tending to 0, such that for $\log N = \sqrt{\log q}\exp(-(\log_2 q)^{\delta(q)})$, we have

$$\Delta(N, q) \geqslant \Psi(N, \log q \log_2^{1-\epsilon(q)} q).$$

A careful quantification of our proof shows that $\epsilon(q) \geqslant \log_4 q/\log_3 q$, and, to within lower-order terms, this is attained.

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By taking $N = O(1)$ in the last bound we recover Granville and Soundararajan's result in [5] that for every large prime q there exists a non-trivial character $\chi \bmod q$ having character sum of size much greater than $\sqrt{q}\log\log q$. Assuming the generalized Riemann hypothesis, Montgomery and Vaughan [10] have given a matching upper bound so that this result is best possible, up to the constant.

The dual bound may be compared to [4, Theorems 8 and 11] for real characters.

§4. *The large N regime.* We first consider Theorem 3.1. Following (2), we prove the basic proposition.

PROPOSITION 4.1. *Let q be a large prime and $1 < N < q$. Set $x = q/N$. Let $r(n)$ be a non-negative multiplicative function supported on squarefree n. Then*

$$\Delta(N,q)^2 + O(N) \gtrsim N \sum_{\substack{n_1,n_2\leqslant N\\(n_1,n_2)=1}} \frac{r(n_1)r(n_2)}{\max(n_1,n_2)}\Bigg[\sum_{\substack{g\leqslant x/\max(n_1,n_2)\\(g,n_1n_2)=1}} r(g)^2 \Big/ \prod_p (1+r(p)^2)\Bigg].$$

Proof. Define a "resonator" $R(\chi) = (1/\sqrt{q-1})\sum_{n\leqslant x} r(n)\chi(n)$. Plainly

$$\Delta(N,q)^2 \geqslant \sum_{\chi\neq\chi_0} |R(\chi)|^2 \Big|\sum_{n\leqslant N}\chi(n)\Big|^2 \Big/ \sum_{\chi}|R(\chi)|^2. \tag{4}$$

By orthogonality of characters, the denominator is $\sum_{n\leqslant x} r(n)^2 \leqslant \prod_p(1+r(p)^2)$. Meanwhile, the numerator is

$$\sum_{m_1,m_2\leqslant x} r(m_1)r(m_2) \sum_{\substack{n_1,n_2\leqslant N\\(n_1n_2,q)=1\\m_1n_1=m_2n_2}} 1 - |R(\chi_0)|^2\Big|\sum_{n\leqslant N}\chi_0(n)\Big|^2.$$

By Cauchy–Schwarz, $|R(\chi_0)|^2 \leqslant (x/(q-1))\sum_{m\leqslant x} r(m)^2$ and therefore the negative term contributes $O(N)$ to the ratio (4). In the main term, let $g = (m_1,m_2)$, $h = (n_1,n_2)$ and replace $m_i := m_i/g$ to obtain

$$\sum_{\substack{m_1,m_2\leqslant x\\(m_1,m_2)=1}} r(m_1)r(m_2) \sum_{\substack{g\leqslant x/\max(m_1,m_2)\\(g,m_1m_2)=1}} r(g)^2 \sum_{h<N/\max(m_1,m_2)} 1$$
$$\gtrsim N \sum_{\substack{m_1,m_2\leqslant N\\(m_1,m_2)=1}} \frac{r(m_1)r(m_2)}{\max(m_1,m_2)} \sum_{\substack{g\leqslant x/\max(m_1,m_2)\\(g,m_1m_2)=1}} r(g)^2,$$

which proves the proposition. □

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When the character sum is very long it is convenient to use Pólya's Fourier expansion (see [10, Lemma 1]) to work with a dual sum.

Recall that a character χ is said to be even (respectively, odd) if $\chi(-1)=1$ (respectively, $\chi(-1)=-1$). For sums of primitive characters Pólya proved that

$$\sum_{n\leqslant q/N}\chi(n)=\frac{\tau(\chi)}{2\pi\mathrm{i}}\sum_{\substack{|h|\leqslant H\\ h\neq 0}}\frac{\overline{\chi}(h)}{h}\left(1-e\left(-\frac{h}{N}\right)\right)+O(1+qH^{-1}\log q), \quad (5)$$

where $e(x)=\mathrm{e}^{2\pi\mathrm{i}x}$ and $\tau(\chi)$ is the usual Gauss sum, of size $|\tau(\chi)|=\sqrt{q}$. Choose $H=\sqrt{Nq}\log q$ and define

$$S(\chi)=\sum_{\substack{|h|\leqslant H\\ h\neq 0}}\frac{\overline{\chi}(h)}{h}\left(1-c\left(\frac{h}{N}\right)\right), \qquad c(x)=\cos(2\pi x).$$

If χ is even $S(\chi)=0$, but for odd primitive characters χ we have

$$\left|\sum_{n\leqslant q/N}\chi(n)\right|=\frac{\sqrt{q}}{2\pi}|S(\chi)|+O\left(\sqrt{\frac{q}{N}}\right). \quad (6)$$

Thus

$$\Delta\left(\frac{q}{N},q\right)\geqslant\frac{\sqrt{q}}{2\pi}\sup_{\chi}|S(\chi)|+O\left(\sqrt{\frac{q}{N}}\right). \quad (7)$$

Using this relation we now prove a dual version of the main proposition.

PROPOSITION 4.2. *Let q be prime, $N\leqslant q/\log^2 q$ and set*

$$x=\frac{1}{2\log q}\sqrt{\frac{q}{N}}.$$

Let $r(n)$ be a multiplicative function supported on squarefree numbers. Then

$$\left(\Delta\left(\frac{q}{N},q\right)+O\left(\sqrt{\frac{q}{N}}\right)\right)^2\gg\frac{q}{N}\sum_{\substack{m_1,m_2\leqslant\min(x,N/2)\\ (m_1,m_2)=1}}\frac{r(m_1)r(m_2)m_1m_2}{\max(m_1,m_2)^3}\times\left[\sum_{\substack{g\leqslant x/\max(m_1,m_2)\\ (g,m_1m_2)=1}}r(g)^2\Big/\prod_p(1+r(p)^2)\right].$$

Proof. We use (7). Set $R(\chi)=(1/\sqrt{q-1})\sum_{m<x}r(m)\chi(m)$. Then

$$\sup_{\chi}|S(\chi)|^2\geqslant\sum_{\chi}|R(\chi)S(\chi)|^2\Big/\sum_{\chi}|R(\chi)|^2.$$

The denominator is bounded by $\prod_p(1+r(p)^2)$ while the numerator is equal to

$$\sum_{m_1,m_2\leqslant x}r(m_1)r(m_2)\sum_{\substack{0\neq|n_1|,|n_2|\leqslant H\\ m_1n_2\equiv m_2n_1 \bmod q}}\frac{(1-c(n_1/N))(1-c(n_2/N))}{n_1n_2}.$$

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Because $xH < q/2$, the congruence modulo q is possible only if n_1 and n_2 have the same sign. Discarding those (positive) terms with $\max(n_1, n_2) > N/2$, the numerator is

$$\begin{aligned}
&\gg N^{-4} \sum_{m_1,m_2\leqslant x} r(m_1)r(m_2) \sum_{\substack{0<n_1,n_2\leqslant N/2\\ n_1m_2=n_2m_1}} n_1n_2\\
&= N^{-4} \sum_{\substack{m_1,m_2\leqslant x\\ (m_1,m_2)=1}} r(m_1)r(m_2)m_1m_2\\
&\quad\times \sum_{\substack{g\leqslant x/\max(m_1,m_2)\\ (g,m_1m_2)=1}} r(g)^2 \sum_{h\leqslant N/(2\max(m_1,m_2))} h^2\\
&\gg \frac{1}{N} \sum_{\substack{m_1,m_2\leqslant \min(x,N/2)\\ (m_1,m_2)=1}} \frac{r(m_1)r(m_2)m_1m_2}{\max(m_1,m_2)^3} \sum_{\substack{g\leqslant x/\max(m_1,m_2)\\ (g,m_1m_2)=1}} r(g)^2. \qquad \square
\end{aligned}$$

4.1. *Choice of resonator, and some lemmas.* Let either $x = q/N$ for $\Delta(N, q)$ or $x = (1/2\log q)\sqrt{q/N}$ for $\Delta(q/N, q)$. In either case, set $\lambda = \sqrt{\log x \log\log x}$ and, as in [12], define a multiplicative function $r(n)$ at prime powers by

$$r(p) = \begin{cases} \dfrac{\lambda}{\sqrt{p}\log p} & \lambda^2 \leqslant p \leqslant \exp((\log\lambda)^2)\\ 0 & \text{otherwise,}\end{cases}$$
$$r(p^n) = 0, \quad n \geqslant 2.$$

We also define a multiplicative function t by $t(p^n) = r(p^n)/(1 + r(p^n)^2)$.

The following two estimates are extrapolated from those used in [12, proof of Theorem 2.1].

LEMMA 4.3. *Assume that $z > \exp(3\lambda\log\lambda)$. As $x \to \infty$, we have*

$$\sum_{m\leqslant z} \frac{t(m)}{\sqrt{m}} \sim \prod_p \left(1 + \frac{t(p)}{\sqrt{p}}\right) = \exp\left((1 + o(1))\frac{\lambda}{2\log\lambda}\right). \tag{8}$$

Also, for $\alpha = 1/(\log\lambda)^3$,

$$\begin{aligned}
&x^{-\alpha} \sum_{\substack{m_1,m_2\leqslant z\\ (m_1,m_2)=1}} \frac{r(m_1)r(m_2)}{(m_1m_2)^{\frac12-\alpha}} \sum_{(d,m_1m_2)=1} r(d)^2d^\alpha \Big/ \left(\sum_{m\leqslant x} \frac{t(m)}{\sqrt{m}}\right)^2 \sum_d r(d)^2\\
&\leqslant \exp\left(-(1 + o(1))\frac{32\log x}{(\log\log x)^4}\right).
\end{aligned} \tag{9}$$

Proof. First (8): by "Rankin's trick", for any $\alpha > 0$ the sum is

$$\prod_p \left(1 + \frac{t(p)}{\sqrt{p}}\right) + O\left(z^{-\alpha} \prod_p \left(1 + \frac{t(p)}{p^{\frac12-\alpha}}\right)\right).$$

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The main term is of the desired size [12, top of p. 6] so it suffices to bound the error. Choose $\alpha = 1/(\log\lambda)^3$. Then the ratio of the error term to the main term is

$$\ll z^{-\alpha}\prod_p\Big(1+\frac{t(p)}{p^{\frac12-\alpha}}\Big)\Big/\Big(1+\frac{t(p)}{p^{\frac12}}\Big)\leqslant z^{-\alpha}\exp\Big(\sum_p\frac{t(p)}{\sqrt{p}}\Big(p^\alpha-1\Big)\Big),$$

and since $\alpha\log p\leqslant 1/\log\lambda$ for all p with $t(p)\neq 0$, the logarithm of the right-hand side is bounded by

$$\leqslant\alpha\Big(-\log z+2\sum_p\frac{t(p)\log p}{\sqrt{p}}\Big)\leqslant\alpha\Big(-\log z+2\lambda\sum_{p<e^{(\log\lambda)^2}}\frac{1}{p}\Big),$$

which tends to $-\infty$ as $x\to\infty$.

Now (9): by our calculation for (8), the denominator of (9) is

$$(1+o(1))\prod_p\Big(1+\frac{t(p)}{\sqrt{p}}\Big)\prod_p(1+r(p)^2).$$

Meanwhile, the double sum in the numerator is bounded by

$$\begin{aligned}&\leqslant\sum_{\substack{m_1,m_2\leqslant z\\(m_1,m_2)=1}}\frac{t(m_1)t(m_2)}{(m_1m_2)^{\frac12-\alpha}}\sum_d r(d)^2d^\alpha\\&\leqslant\Big(\prod_p\Big(1+\frac{t(p)}{p^{\frac12-\alpha}}\Big)\Big)^2\prod_p(1+r(p)^2p^\alpha).\end{aligned}$$

Therefore the ratio in (9) is

$$\ll x^{-\alpha}\prod_p\Big(\frac{1+t(p)p^{-\frac12+\alpha}}{1+t(p)p^{-\frac12}}\Big)^2\prod_p\Big(\frac{1+r(p)^2p^\alpha}{1+r(p)^2}\Big),$$

and this has logarithm

$$\leqslant-\alpha\log x+2\alpha\sum_{\lambda^2\leqslant p\leqslant\exp((\log\lambda)^2)}\frac{r(p)\log p}{\sqrt{p}}+\alpha\sum_{\lambda^2\leqslant p\leqslant\exp((\log\lambda)^2)}r(p)^2\log p.$$

Substituting the definition of $r(p)$ and α, and using the prime number theorem, the last expression is bounded by $-\lambda^2(1+o(1))/\log^5\lambda$. □

We record one more elementary estimate.

LEMMA 4.4. *Uniformly in $y\geqslant 1$ and for any $k>0$,*

$$\sum_{\substack{d\leqslant y\\(d,k)=1}}\frac{\mu(d)t(d)^2}{d}=1+o(1)\quad(x\to\infty).\tag{10}$$

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Proof. The sum is†

$$1+\sum_{\substack{\lambda^2<d<y\\(d,k)=1}}\frac{\mu(d)t(d)^2}{d}=1+O\Big(\lambda^2\sum_{d>\lambda^2}^{\flat}\frac{1}{d^2(\log d)^2}\Big)=1+o(1). \qquad \square$$

Combining the above two lemmas we obtain our basic estimate.

LEMMA 4.5. *Uniformly in* $z\geqslant 1$,

$$\sum_{\substack{m_1,m_2\leqslant z\\(m_1,m_2)=1}}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,m_2)^3}\gg\frac{1}{\log z}\Big(\sum_{m\leqslant z}\frac{t(m)}{\sqrt{m}}\Big)^2.$$

In particular, for $\log z>3\lambda\log\log\lambda$, *we have*

$$\sum_{\substack{m_1,m_2\leqslant z\\(m_1,m_2)=1}}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,m_2)^3}\geqslant\exp\Big((1+o(1))\frac{\lambda}{\log\lambda}\Big).$$

Proof. The left-hand side is equal to

$$\begin{aligned}&\sum_{m_1,m_2\leqslant z}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,m_2)^3}\sum_{d\mid(m_1,m_2)}\mu(d)\\&=\sum_{m_1,m_2\leqslant z}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,m_2)^3}\sum_{\substack{d\leqslant z/\max(m_1,m_2)\\(d,m_1m_2)=1}}\frac{\mu(d)t(d)^2}{d}\\&=(1+o(1))\sum_{m_1,m_2\leqslant z}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,m_2)^3}\end{aligned}\tag{11}$$

by (10) of Lemma 4.4. This last sum is

$$\gg\sum_{0\leqslant k<\log z}\Big(\sum_{z/e^{k+1}<m\leqslant z/e^k}\frac{t(m)}{\sqrt{m}}\Big)^2\geqslant\frac{1}{\log z}\Big(\sum_{m\leqslant z}\frac{t(m)}{\sqrt{m}}\Big)^2$$

by Cauchy–Schwarz. $\square$

4.2. *Proof of Theorem 3.1.* Recall that we set either $x=q/N$ for $\Delta(N,q)$, or $x=(1/2\log q)\sqrt{q/N}$ for $\Delta(q/N,q)$ and in either case $\lambda=\sqrt{\log x\log_2 x}$. Let $z=\min(N,x)^{\frac{4}{5}}$. Since we assume that $\log N\geqslant 4\sqrt{\log q\log_2 q}\log_3 q$ this guarantees that $z\geqslant\exp(3\lambda\log_2\lambda)$.

In view of Propositions 4.1 and 4.2, both the bound for $\Delta(N,q)$ and $\Delta(q/N,q)$ will follow from the estimate

$$\begin{aligned}&\sum_{\substack{n_1,n_2<z\\(n_1,n_2)=1}}\frac{r(n_1)r(n_2)n_1n_2}{\max(n_1,n_2)^3}\Big[\sum_{\substack{g\leqslant x/\max(n_1,n_2)\\(g,n_1n_2)=1}}r(g)^2\Big/\prod_p(1+r(p)^2)\Big]\\&\geqslant\exp\Big((2+o(1))\sqrt{\frac{\log x}{\log_2 x}}\Big).\end{aligned}$$

† Here and throughout a flat indicates that summation is restricted to squarefree numbers.

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Applying Rankin's trick, the sum over g on the left is

$$\prod_{p\nmid m_1m_2}(1+r(p)^2)+O\bigg(\bigg(\frac{x}{\max(m_1,\,m_2)}\bigg)^{-\alpha}\prod_{p\nmid m_1m_2}(1+r(p)^2p^{\alpha})\bigg).$$

Thus we obtain the desired main term of

$$\sum_{\substack{m_1,m_2\leqslant z\\(m_1,m_2)=1}}\frac{t(m_1)t(m_2)m_1m_2}{\max(m_1,\,m_2)^3}\gg\frac{1}{\log z}\bigg(\sum_{m<z}\frac{t(m)}{\sqrt{m}}\bigg)^2\geqslant\exp\bigg((1+o(1))\frac{\lambda}{\log\lambda}\bigg)$$

with an error term of

$$\bigg(\prod_p(1+r(p)^2)\bigg)^{-1}x^{-\alpha}\sum_{\substack{m_1,m_2\leqslant z\\(m_1,m_2)=1}}\frac{r(m_1)r(m_2)(m_1m_2)^{1+\alpha}}{\max(m_1,\,m_2)^3}\sum_{(g,m_1m_2)=1}r(g)^2g^{\alpha}.$$

By Lemmas 4.3 and 4.5, the ratio of this error term to the main term is bounded by $\log z$ times the expression in (9), and thus this ratio is $o(1)$.

§5. *Structure of argument for shorter sums.* For transition and small N we use the following variant of Proposition 4.1.

PROPOSITION 5.1. *Let q be a large prime, $N<\sqrt{q}$ and $x=(q-1)/N$. Let $r(n)$ be a non-negative completely multiplicative function. Write*

$$B=\frac{\sum_{n=1}^{\infty}r(n)^2}{\sum_{n\leqslant x/N}r(n)^2}.\tag{12}$$

Then

$$\Delta(N,q)\geqslant O(1)+\frac{1}{B}\sum_{n\leqslant N}r(n).$$

Proof. Choose the "resonator" $R(\chi)=(1/\sqrt{q-1})\sum_{n\leqslant x}r(n)\chi(n)$. Plainly

$$\Delta(N,q)\geqslant\sum_{\chi\neq\chi_0}|R(\chi)|^2\sum_{n\leqslant N}\chi(n)\Big/\sum_{\chi}|R(\chi)|^2.$$

By orthogonality of characters, the denominator is $\sum_{n\leqslant x}r(n)^2$. Meanwhile the numerator is

$$\begin{aligned}&\sum_{\chi}|R(\chi)|^2\sum_{n\leqslant N}\chi(n)-|R(\chi_0)|^2\sum_{n\leqslant N}\chi_0(n)\\&\quad=\sum_{n\leqslant N}\sum_{m\leqslant x/n}r(m)r(mn)-O\bigg(\sum_{n\leqslant x}r(n)^2\bigg)\end{aligned}$$

after applying Cauchy–Schwarz to the error term. Since r is non-negative and completely multiplicative, this proves the proposition. □

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As before, we also use a dual version of the above proposition to treat long sums, of length q/N (N small).

PROPOSITION 5.2. *Let q be a large prime, $N < \sqrt{q}$, $H = \sqrt{qN}\log q$, $x = q/2H$ and $N' < H$ a parameter. Let r be a non-negative completely multiplicative function and put*

$$B_{N'} = \frac{\sum_n r(n)^2}{\sum_{n\leqslant x/N'} r(n)^2}.$$

We have

$$\Delta\left(\frac{q}{N}, q\right) \geqslant O\left(\sqrt{\frac{q}{N}}\right) + \frac{\sqrt{q}}{\pi}\frac{1}{B_{N'}}\sum_{0<n\leqslant N'}\frac{r(n)}{n}\left(1 - c\left(\frac{n}{N}\right)\right).$$

Specializing to $N' = N/2$ we obtain

$$\Delta\left(\frac{q}{N}, q\right) \gg O\left(\sqrt{\frac{q}{N}}\right) + \frac{\sqrt{q}}{N^2}\frac{1}{B_{N/2}}\sum_{n\leqslant N/2} nr(n).$$

Proof. Define the "resonator" $R(\chi) = (1/\sqrt{q-1})\sum_{n\leqslant x} r(n)\chi(n)$. Then (recall (6))

$$\begin{aligned}\Delta(N, q) + O\left(\sqrt{\frac{q}{N}}\right) &\geqslant \frac{\sqrt{q}}{\pi}\sup_{\chi}|S(\chi)| \\ &\geqslant \frac{\sqrt{q}}{\pi}\sum_{\chi}|R(\chi)|^2 S(\chi)\Big/\sum_{\chi}|R(\chi)|^2.\end{aligned}$$

The denominator is $\sum_{n\leqslant x} r(n)^2$. Since $x = q/2H$, the sum in the numerator is

$$\begin{aligned}&\sum_{0\neq|h|\leqslant H}\frac{(1 - c(h/N))}{h}\sum_{\substack{n_1,n_2\leqslant x\\ n_1h\equiv n_2 \bmod q}} r(n_1)r(n_2) \\ &\quad= \sum_{0<h\leqslant H}\frac{r(h)(1 - c(h/N))}{h}\sum_{n\leqslant x/h} r(n)^2.\end{aligned}$$

Since all terms in the numerator are positive, we can discard those $h > N'$ to obtain the result. □

5.1. *Outline of proofs, and lemmas.* We now provide an outline of the argument used in proving Theorems 3.2 and 3.4, and review the facts that we will need from the theory of smooth numbers.

After introduction of the multiplicative function $r(n)$, the proofs proceed in the same three steps.

(A) Let x be the length of the resonator and N the length of the sum. We check that

$$\sum_{n\leqslant x/N} r(n)^2 = (1 + o(1))\prod_p (1 - r(p)^2)^{-1}$$

so that B in Proposition 5.1 may be taken as $1 + o(1)$.

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(B) From Proposition 5.1 and part (A) it follows that $\Delta(N, q) \gtrsim \sum_{n \leqslant N} r(n)$. We determine the asymptotic shape of this sum via the Perron integral

$$\sum_{n \leqslant N} r(n) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} R(s) N^s \frac{ds}{s}, \qquad R(s) = \prod_p (1 - r(p) p^{-s})^{-1}$$

and a saddle point calculation.

(C) We analyze the various implicitly defined parameters that arise in the saddle point calculation of part (B) in order to obtain explicit lower bounds for $\Delta(N, q)$.

Remark. Via the saddle point method, the condition

$$\sum_{n \leqslant x/N} r(n)^2 \geqslant (1 - o(1)) \sum_n r(n)^2$$

in step (A) is closely related to the condition at primes

$$\sum_p \log p \frac{r(p)^2}{1 - r(p)^2} < \log x.$$

Typically part (B) leads to an asymptotic $\sum_{n \leqslant N} r(n) = c \cdot N^\sigma$, and from there to $\sum_{n \leqslant N/p} r(n) \doteq c \cdot (N/p)^\sigma$. Thus applying Lagrange multipliers to the optimization problem

$$\begin{aligned} &\text{maximize:} \quad \sum_{n \leqslant N} r(n) \\ &\text{subject to:} \quad \sum_{n \leqslant x/N} r(n)^2 = (1 + o(1)) \sum_n r(n)^2 \end{aligned}$$

suggests the heuristic solution

$$\frac{r(p)}{(1 - r(p)^2)^2} \doteq \frac{\lambda}{p^\theta \log p}, \tag{13}$$

which guides our choice of resonator functions.

The bound in the first step (A) is accomplished by an appeal to the following simple lemma.

LEMMA 5.3. *Let $f_i(n)$ be a sequence of non-negative, completely multiplicative functions satisfying $f_i(n) < 1$, and let $y_i \to \infty$ be a growing sequence of parameters. Define*

$$\alpha_i = \frac{(\log \log y_i)^2}{\log y_i}.$$

Suppose that

$$\sum_p \log p \frac{f_i(p)}{1 - f_i(p)} < \log y_i - \frac{\log y_i}{\log \log y_i}$$

and

$$\sum_{k \log p > \log y_i / (\log \log y_i)^4} f(p_i)^k p_i^{k\alpha} = o(1), \quad i \to \infty.$$

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Then

$$\sum_{n\leqslant y_i} f_i(n) = (1+o(1))\sum_{n=1}^{\infty} f_i(n), \quad i\to\infty.$$

Proof. By Rankin's trick,

$$\sum_{n\leqslant y_i} f_i(n) = \sum_{n=1}^{\infty} f_i(n) + O\Big(y_i^{-\alpha_i}\sum_{n=1}^{\infty} f_i(n)n^{\alpha_i}\Big).$$

The logarithm of the ratio between the error term and the main term is

$$-\alpha_i \log y_i + \log\prod_p\Big(\frac{1-f_i(p)}{1-f_i(p)p^{\alpha_i}}\Big). \tag{14}$$

The logarithm of the infinite product is

$$\begin{aligned}\sum_{p,k}\frac{f_i(p)^k}{k}[p^{k\alpha_i}-1] &\leqslant (1+O(\log\log^{-2} y_i))\alpha_i\sum_p \log p\frac{f_i(p)}{1-f_i(p)}\\ &\quad+\sum_{k\log p>\log y_i/(\log\log y_i)^4}\frac{f_i(p)p^{k\alpha_i}}{k}\\ &\leqslant \alpha_i\Big(\log y_i - \frac{\log y_i}{\log\log y_i}+O\Big(\frac{\log y_i}{(\log\log y_i)^2}\Big)\Big)+o(1),\end{aligned}$$

which proves that the quantity in (14) tends to $-\infty$ as $i\to\infty$. □

The analysis in steps (B) and (C) is connected to the corresponding analysis of smooth numbers contained in [**6**]. Recall that we set $\Psi(x,y)=\#\{n\leqslant x : p|n\Rightarrow p\leqslant y\}$ for the number of y-smooth numbers less than x. The generating function is

$$\zeta(s,y)=\prod_{p<y}(1-p^{-s})^{-1},$$

with logarithm and logarithmic derivatives given by

$$\psi(s,y)=\log\zeta(s,y), \qquad \psi_j(s,y)=(-1)^j\frac{d^j}{ds^j}\psi(s,y), \quad j=0,1,2,\ldots.$$

Let $\alpha=\alpha(x,y)>0$ solving $\psi_1(\alpha,y)=\log x$ be the saddle point. The basic result is as follows.

THEOREM 5.4. *Uniformly in the range $x\geqslant y\geqslant 2$, we have*

$$\Psi(x,y)=\frac{x^\alpha\zeta(\alpha,y)}{\alpha\sqrt{2\pi\psi_2(\alpha,y)}}\Big\{1+O\Big(\frac{\log y}{\log x}+\frac{\log y}{y}\Big)\Big\}.$$

Proof. This is [**6**, Theorem 1.6]. □

The first stage in the proof of Theorem 5.4 is to write

$$\Psi(x,y)=\frac{1}{2\pi\mathrm{i}}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\zeta(s,y)x^s\frac{ds}{s}$$

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and to truncate the integral at some height T. The following lemma, which we use, is essentially the one given in [6] to bound the error from truncation.

LEMMA 5.5. *Let $f(n)$ be a bounded, non-negative arithmetic function with Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)/n^s$, which converges absolutely at $s = \sigma_0 > 0$. Uniformly in $x \geqslant 2$, $T \geqslant 1$, $\sigma \geqslant \sigma_0$ and $0 < \tau < T$ we have*

$$\sum_{n \leqslant x} f(n) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\tau}^{\sigma + \mathrm{i}\tau} F(s) \frac{x^s}{s} ds + R$$

where R is bounded by

$$R \ll 1 + x^{\sigma} F(\sigma) \left(T^{-\frac{1}{2}} + \sup_{\tau \leqslant t \leqslant T} \frac{|F(\sigma + \mathrm{i}t)|}{F(\sigma)} \log T \right).$$

Proof. See [6, Lemma 3.3]. □

For the purpose of making comparisons in Theorem 3.4 the asymptotic behavior of $\log \Psi(x, y)$ suffices. The following is a simple consequence of the analysis in [6].

COROLLARY 5.6. *Let $u = \log x / \log y$. When $u < \sqrt{y}$ and for $|\kappa| < 1$ we have*

$$\log \frac{\Psi(x, \mathrm{e}^{\kappa} y)}{\Psi(x, y)} = \left(\frac{\kappa + O(\log^{-1} u)}{\log y} \right) u (\log u + \log_2(u + 2)). \tag{15}$$

Proof. Theorem 1.2 of [6] gives, for any fixed $\epsilon > 0$, and uniformly in $y \geqslant 2$, $1 \leqslant u \leqslant y^{1-\epsilon}$,

$$\log(\Psi(x, y)/x) = \{1 + O(\exp\{-(\log u)^{3/5-\epsilon}\})\} \log \rho(u)$$

with $\rho(u)$ the Dickman–de Bruijn function. Corollary 2.3 of [6] gives, for $u \geqslant 1$,

$$\rho(u) = \exp\left\{ -u \left(\log u + \log_2(u + 2) - 1 + O\left(\frac{\log_2(u + 2)}{\log u} \right) \right) \right\}.$$

Combining these evaluations proves the corollary. □

§6. *The transition range, proof of Theorem 3.2.* In this section we set $\log N = \tau \log^{\frac{1}{2}} q \log_2^{\frac{1}{2}} q$. We assume that $\tau \geqslant \log_2^{-A} q$ for some large constant A, and we may assume that $\tau \ll \log_3 q$ since the case of larger τ is contained in Theorem 3.1.

We treat the main case $\Delta(N, q)$ and the dual case $\Delta(q/N, q)$ simultaneously. In the first place put $x = q/N$ and in the dual case choose $x = \sqrt{q/N} \log^{-1} q$. In either case, set $\lambda = \log^{\frac{1}{2}} x \log_2^{\frac{1}{2}} x$ and define the completely multiplicative function $r(n)$ by

$$r(p) = \begin{cases} \dfrac{\lambda}{p^{\frac{1}{2}} \log p} & \lambda^2 < p < \exp(\log^2 \lambda) \\ 0 & \text{otherwise.} \end{cases}$$

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The generating function is defined for $\Re(s) > 0$ by $R(s) = \prod_p (1 - r(p)/p^s)^{-1}$; it has logarithm and derivatives

$$\phi_0(s) = \log R(s), \qquad \phi_j(s) = (-1)^j \frac{d^j}{ds^j}\phi_0(s).$$

Let $\sigma > 0$ solving $\phi_1(\sigma) = \log N$ be the saddle point. We localize σ as follows.

LEMMA 6.1. *Let*

$$\log N = \tau \log^{1/2} q \log_2^{1/2} q \quad \textit{with } (\log_2 q)^{-A} < \tau \ll \log_3 q.$$

Then σ satisfies

$$\frac{1}{2} - O\left(\frac{\log_3 q}{\log_2 q}\right) \leqslant \sigma \leqslant \frac{1}{2} + \frac{\log_3^2 q}{\log_2 q}.$$

Moreover, in the range

$$(\log_2 q \log_3^2 q)^{-1} < \sigma - \frac{1}{2} < \frac{\log_3^2 q}{\log_2 q},$$

we have $\mathrm{e}^{-O(\log_3^2 q)} \leqslant \tau \ll \log_4 q$.

Proof. When $\sigma < \frac{1}{2}$, since $\log N = \phi_1(\sigma) \ll \sqrt{\log q \log_2 q}\, \log_3 q$ and $\phi_2(\alpha)$ is decreasing in $\alpha > 0$,

$$\frac{1}{2} - \sigma \ll \frac{\sqrt{\log q \log_2 q}\, \log_3 q}{\phi_2(\frac{1}{2})};$$

$$\phi_2\left(\frac{1}{2}\right) \geqslant \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{\lambda \log p}{p} \gg \lambda \log^2 \lambda$$

so that $1/2 - \sigma \ll \log_3 q / \log_2 q$.

Meanwhile, in the range $(\log_2 q \log_3^2 q)^{-1} < \sigma - 1/2 < \log_3^2 q / \log_2 q$

$$\log N = \phi_1(\sigma) \sim \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p^{\frac{1}{2}+\sigma}} \sim \int_{(2\sigma-1)\log\lambda}^{\infty} \mathrm{e}^{-x} \frac{dx}{x} \tag{16}$$

by partial summation against the prime number theorem. Since $\tau \sim \log N/\lambda$ in the case of $\Delta(N, q)$ and $\tau \sim \log N/\sqrt{2}\lambda$ in the dual case of $\Delta(N/q, q)$, both the upper bound for σ and the second statement of the lemma are deduced by estimating

$$\int_{(2\sigma-1)\log\lambda}^{\infty} \mathrm{e}^{-x} \frac{dx}{x} \quad \text{at } \sigma = \frac{1}{2} + \frac{\log_3^2 q}{\log_2 q} \text{ and } \sigma = \frac{1}{2} + (\log_2 q \log_3^2 q)^{-1}. \quad \square$$

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The standard facts that we require regarding $R(s)$ near its saddle point are contained in the following lemma.

LEMMA 6.2. *Continue to let σ solve $\phi_1(\sigma) = \log N$. Uniformly in*

$$O\left(\frac{\log_3 q}{\log_2 q}\right) < \sigma - 1/2 < \frac{\log_3^2 q}{\log_2 q},$$

we have the following estimates regarding the functions ϕ_j.

(1) $|\phi_3(\sigma + it)| \leqslant \phi_3(\sigma)$.
(2) $\phi_2(\sigma) \ll \log N \log^2 \lambda$.
(3) *For $|t| < \log^{-2} \lambda$,*

$$\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg t^2 \phi_2(\sigma).$$

(4) *For all t satisfying $\log^{-2} \lambda < |t| < \exp((\log \lambda)^{21/20})$ and $\delta > 0$, we have*

$$\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg_\delta \min(t^2, 1)\lambda^{1-\delta}.$$

Proof. The first three items are straightforward, so we show the proof of (4). For $|t| < 1$ write

$$\begin{aligned}\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] &\geqslant \sum_{\lambda^2 < p < \lambda^3} \frac{\lambda}{p^{\frac{1}{2}+\sigma} \log p}[1 - \cos(t \log p)] \\ &\gg \frac{\lambda}{\log^2 \lambda} \Re\left[\sum_{\lambda^2 < p < \lambda^3} \frac{(1 - p^{it}) \log p}{p^{1+\delta/2}}\right] \gg_\delta t^2 \lambda^{1-\delta}\end{aligned}$$

by partial summation against the prime number theorem.

When $|t| > 1$ choose $Y = \exp((\log \lambda)^{11/6})$ and write

$$\begin{aligned}\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] &\geqslant \sum_{Y < p < Y^2} \frac{\lambda}{p^{\frac{1}{2}+\sigma} \log p}[1 - \cos(t \log p)] \\ &\gg_\delta \lambda^{1-\delta} \Re\left\{\sum_{Y < p < Y^2} \frac{\log p}{p}[1 - p^{-it}]\right\} \\ &\gg_\delta \lambda^{1-\delta}(\log Y + O(1) + O_\epsilon(t \exp(-(\log Y)^{3/5-\epsilon}))).\end{aligned}$$

This suffices in view of the restriction on t. □

We now proceed to the proof of Theorem 3.2.

Step A. We apply Lemma 5.3 with $y = x/N$. Recall that $\alpha = \log_2^2 y_i / \log y_i$.

$$\begin{aligned}\sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \log p \frac{r(p)^2}{1 - r(p)^2} &\leqslant (1 + O(\log_2^2 q))\lambda^2 \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p \log p} \\ &\leqslant (1 + O(\log_2^2 q))\lambda^2[(2 \log \lambda)^{-1} - \log^{-2} \lambda] \\ &\leqslant \log x - (1 - o(1)) \frac{\log x \log_3 x}{\log_2 x}.\end{aligned}$$

Since $\log N = O(\log^{\frac{1}{2}+\epsilon} x)$, the first condition in Lemma 5.3 is satisfied.

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Meanwhile, for some $c > 0$,

$$\sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \sum_{k > \log y/\log p \log_2^4 y} r(p)^{2k} p^{\alpha k} \ll \exp(\log^2 \lambda)(\log \lambda)^{-c(\log y/\log_2^6 y)} = o(1)$$

so that the second condition is also satisfied. Thus by Lemma 5.3,

$$\sum_{n \leqslant x/N} r(n)^2 = (1 + o(1)) \sum_n r(n)^2$$

and therefore

$$\Delta(N, q) \geqslant (1 + o(1)) \sum_{n \leqslant N} r(n), \qquad \Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N^2} \sum_{n \leqslant N/2} nr(n).$$

Step B. We have the following asymptotic expansions of $\sum_{n \leqslant N} r(n)$, $\sum_{n \leqslant N} nr(n)$.

PROPOSITION 6.3. *Let $\sigma > 0$ be the unique solution to $\phi_1(\sigma) = \log N$. We have*

$$\sum_{n \leqslant N} r(n) \sim \frac{N^\sigma e^{\phi_0(\sigma)}}{\sigma \sqrt{2\pi \phi_2(\sigma)}}, \qquad \sum_{n \leqslant N} nr(n) \sim \frac{N^{1+\sigma} e^{\phi_0(\sigma)}}{(1+\sigma)\sqrt{2\pi \phi_2(\sigma)}}.$$

Given Lemma 5.5 and the estimates in Lemma 6.2, this is a straightforward application of the saddle point method. One consequence is that $\sum_{n \leqslant N/2} nr(n) \asymp \sum_{n \leqslant N} nr(n)$.

Step C. By (2) of Lemma 6.2, $\phi_2(\sigma) = \log^{O(1)} q$, and therefore Proposition 6.3 implies that for $\sigma > 0$ solving $\phi_1(\sigma) = \log N$ we have

$$\Delta(N, q) \geqslant N^\sigma e^{\phi_0(\sigma) + O(\log_2 q)}, \qquad \Delta\left(\frac{q}{N}, q\right) \geqslant \frac{\sqrt{q}}{N} N^\sigma e^{\phi_0(\sigma) + O(\log_2 q)}.$$

The error terms are negligible, so it suffices to determine N^σ and $\phi_0(\sigma)$.

By Lemma 6.1, we have

$$\frac{1}{2} - O\left(\frac{\log_3 q}{\log_2 q}\right) \leqslant \sigma \leqslant \frac{1}{2} + \frac{\log_3^2 q}{\log_2 q}.$$

We first show that for N such that $\sigma < 1/2 + 1/\log_2 q \log_3^2 q$,

$$\Delta(N, q) \geqslant \sqrt{N} \exp\left((1 + o(1))\sqrt{\frac{\log q}{\log_2 q}}\right), \tag{17}$$

$$\Delta\left(\frac{q}{N}, q\right) \geqslant \sqrt{\frac{q}{N}} \exp\left((1 + o(1))\sqrt{\frac{\log q}{2 \log_2 q}}\right).$$

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Note that for $\sigma < 1/2 + 1/\log_2 q \log_3^2 q$, (16) in the proof of Lemma 6.1 shows that

$$\log N \gg \sqrt{\log q \log_2 q} \log_4 q$$

so that $\tau \gg \log_4 q$ for all such N. Since $A\tau \to 0$ and $A\tau' \to 1$ as $\tau \to \infty$, the bounds (17) verify the theorem in this range.

For $\sigma < 1/2$, $(1/2 - \sigma) \log N = o(\sqrt{\log x / \log_2 x})$, while

$$\phi_0(\sigma) \geqslant \phi_0\left(\frac{1}{2}\right) \geqslant \lambda \sum_{\lambda^2 \leqslant p < \exp(\log^2 \lambda)} \frac{1}{p \log p} \gtrsim \frac{\lambda}{2 \log \lambda} \sim \sqrt{\frac{\log x}{\log_2 x}}.$$

Combining these estimates, we obtain (17) for the case $\sigma < \frac{1}{2}$. Meanwhile, in the range $\frac{1}{2} \leqslant \sigma < \frac{1}{2} + \log_2^{-1} q \log_3^{-2} q$, we have

$$\phi_0(\tfrac{1}{2}) - \phi_0(\sigma) \leqslant (\sigma - \tfrac{1}{2}) \phi_1(\tfrac{1}{2}).$$

Since

$$\phi_1\left(\frac{1}{2}\right) \sim \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p} \asymp \lambda \log_2 \lambda,$$

this shows that

$$\phi_0\left(\frac{1}{2}\right) - \phi_0(\sigma) = o\left(\sqrt{\frac{\log x}{\log_2 x}}\right).$$

Thus we also have (17) for $\sigma < 1/2 + 1/\log_2 q \log_3^2 q$.

We now consider the case $\log_2^{-1} q \log_3^{-2} q < \sigma - \frac{1}{2} < \log_2^{-1} q \log_3^2 q$. Let A solve

$$\frac{\log N}{\lambda} = \int_A^\infty \mathrm{e}^{-x} \frac{dx}{x}$$

and let $A' = (2\sigma - 1) \log \lambda$. By putting in standard explicit estimates in (16) of Lemma 6.1, it is readily seen that $A \sim A'$ even in the case $A \to 0$ with growing q (use $\tau \ll \log_4 q \Rightarrow A \geqslant \log_3^{-O(1)} q$). Thus

$$\sigma \log N = \frac{1}{2} \log N + (1 + o(1)) \frac{A \log N}{2 \log \lambda}.$$

Meanwhile, the prime number theorem gives

$$\phi_0(\sigma) \sim \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p^{\frac{1}{2}+\sigma} \log p} \sim \lambda \int_{2 \log \lambda}^{\log^2 \lambda} \mathrm{e}^{(\frac{1}{2}-\sigma)y} \frac{dy}{y^2}$$

and after a change of variables, this says that

$$\left(\sigma - \frac{1}{2}\right)\lambda \int_{2(\sigma - \frac{1}{2}) \log \lambda}^\infty \mathrm{e}^{-x} \frac{dx}{x^2} \sim \left(\sigma - \frac{1}{2}\right)\lambda \int_A^\infty \mathrm{e}^{-x} \frac{dx}{x^2}.$$

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Since $A \sim (\sigma - \frac{1}{2})2\log\lambda$ and recalling that we set $\tau' = \int_A^\infty \mathrm{e}^{-x}\,(dx/x^2)$, we obtain $\phi_0(\sigma) \sim A\tau'\lambda/2\log\lambda$. Thus

$$N^\sigma \mathrm{e}^{\phi_0(\sigma)} = N^{\frac{1}{2}} \exp\left((1+o(1))\left(\frac{A\log N}{2\log\lambda} + \frac{A\tau'\lambda}{2\log\lambda}\right)\right),$$

which proves the theorem.

§7. *Short character sums, proof of Theorem 3.4.* We quickly dispose of the case of small N, $\log N < \log_2^2 q\,\log_3^{-10} q$ for both $\Delta(N, q)$ and $\Delta(q/N, q)$. The main work of Theorem 3.4 will then be to consider the range $\log_2^2 q\,\log_3^{-10} q < \log N = o(\sqrt{\log q\,\log_2 q})$.

7.1. *Case of small N.* When $\log N < \log_2^2 q\,\log_3^{-10} q$, notice that for all $n \leqslant N\log q$, the number of prime divisors of n counted with multiplicity satisfies $\Omega(n) \ll \log_2^2 q\,\log_3^{-10} q$. Thus for completely multiplicative function r defined at primes by

$$r(p) = \begin{cases} 1 - (\log_2 q)^{-2} & 1 < p \leqslant \dfrac{\log q}{\log_2^5 q} \\ 0 & \text{otherwise,} \end{cases}$$

we have $r(n) = 1 - o(1)$ for all smooth $n < N\log q$ that satisfy $p|n \Rightarrow p < \log q/\log_2^5 q$.

For this choice of r we check that

$$\sum_p \log p \frac{r(p)^2}{1-r(p)^2} \leqslant \log_2^4 q \sum_{p < \log q/\log_2^5 q} \log p \ll \frac{\log q}{\log_2 q}$$

and

$$\sum_{k\log p > (1/4)\log q/\log_2^4 q} r(p)^{2k} p^{k\log_2^2 q/(1/4)\log q} \ll \log q\,\log_2 q\,(1-\log_2^{-2} q)^{2\log q/\log_2^6 q} = o(1),$$

so that by Lemma 5.3,

$$\sum_{n \leqslant q^{\frac{1}{4}}} r(n)^2 = (1+o(1))\prod_p (1-r(p)^2)^{-1}.$$

In particular, choosing $x = q/N$, we have

$$\sum_{n \leqslant x/N} r(n)^2 = (1+o(1))\sum_n r(n)^2,$$

and so by Proposition 5.1,

$$\Delta(N,q) \geqslant (1+o(1))\sum_{n<N} r(n) \geqslant (1+o(1))\Psi\left(N, \frac{\log q}{\log_2^5 q}\right) \geqslant (1+o(1))\Psi(N, \log q).$$

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Choosing $x = (1/\log q)\sqrt{q/N}$ and setting $N' = N \log q$, we have

$$\sum_{n \leqslant x/N'} r(n)^2 = (1 + o(1)) \sum_n r(n)^2,$$

and so by Proposition 5.2 we obtain

$$\begin{aligned}\Delta\left(\frac{q}{N}, q\right) &\gg \sqrt{q} \sum_{n \leqslant N \log q} \frac{r(n)}{n}\left(1 - c\left(\frac{n}{N}\right)\right) \\ &\gg \sqrt{q} \sum_{\substack{n \leqslant N \log q \\ p|n \Rightarrow p < \log q/\log_2^5 q}} \frac{1 - c(n/N)}{n} \\ &\gg \frac{\sqrt{q}}{N} \Psi(N, \log q) \frac{\log\log q}{\log(\log N/\log\ \log q)}\end{aligned}$$

by following [**4**, proof of Theorem 11, p. 394] in the last step.

7.2. *Main case.* Henceforth we assume that

$$\log_2^2 q\ \log_3^{-10} q < \log N = o(\sqrt{\log q\ \log_2 q}).$$

We will describe the analysis of $\Delta(N, q)$ in detail. Afterwards we will sketch the necessary modifications in order to handle the dual case of $\Delta(q/N, q)$.

Throughout this subsection $x = (q-1)/N$, $\epsilon = \epsilon(q) > 0$ is a parameter tending to 0 as $q \to \infty$ and $M = (1-\epsilon)\log q$. We choose the resonator coefficients $r(n) = r_\sigma(n)$ in accordance with the heuristic solution (13), where here $\sigma = \sigma(N, q) > 1/2 + 1/\log q$ is the location of the relevant saddle point†. Thus we define $r_\sigma(n)$ to be completely multiplicative, with values given at primes by

$$r_\sigma(p) = f_\sigma\left(\frac{p}{M}\right),$$

with $0 < f_\sigma(x) < 1$ the unique continuous solution to the equations

$$\frac{f_\sigma(x)}{(1 - f_\sigma(x)^2)^2} = \left(\frac{c_\sigma}{x}\right)^\sigma, \qquad \int_0^\infty \frac{f_\sigma(x)^2}{1 - f_\sigma(x)^2}\, dx = 1.$$

Note that the second equation implicitly defines the constant c_σ.

The following basic properties of the function f_σ are straightforward to establish.

LEMMA 7.1. *For each $\sigma \in (\frac{1}{2}, 1)$, the function f_σ satisfies the following properties.*

(1) *f_σ is smooth, decreasing, and a bijection $(0, \infty) \to (0, 1)$.*
(2) *$f_\sigma(x) \leqslant (c_\sigma/x)^\sigma$.*

† The analysis is somewhat complicated by the fact that the resonator function and the saddle point are mutually dependent.

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(3) $\min(1/2,\ 1/4(x/c_\sigma)^{\sigma/2}) \leqslant 1 - f_\sigma(x) \leqslant (x/c_\sigma)^{\sigma/2}$.

(4) *The value of* c_σ *is*

$$c_\sigma = \frac{\Gamma(3/2\sigma)}{\Gamma(1-1/2\sigma)\Gamma(2/\sigma-1)}.$$

In particular, $c_\sigma < 1$, $c_\sigma \to \frac{1}{2}$ *as* $\sigma \uparrow 1$ *and* $c_\sigma \sim (2\sigma-1)$ *as* $\sigma \downarrow \frac{1}{2}$.

(5) *The Mellin transform of* f_σ *is*

$$\hat{f}_\sigma(s) = \int_0^\infty f_\sigma(x)x^{s-1}\,dx = \frac{c_\sigma^s}{2\sigma}\frac{\Gamma(1/2-s/2\sigma)\Gamma(2s/\sigma+1)}{\Gamma(3/2+3s/2\sigma)},$$

which converges absolutely in $0 < \Re(s) < \sigma$. *In particular,*

$$\hat{f}_\sigma(1-\sigma) = \frac{1}{2(1-\sigma)}c_\sigma^{-\sigma}.$$

(6) *The Mellin transform of* $g_\sigma(x) = f_\sigma(x)^2/(1-f_\sigma(x)^2)$ *is*

$$\hat{g}(s) = \int_0^\infty g(x)x^{s-1}\,dx = \frac{c_\sigma^s}{s}\frac{\Gamma(1-s/2\sigma)\Gamma(2s/\sigma-1)}{\Gamma(3s/2\sigma)}.$$

This integral converges absolutely in $\sigma/2 < \Re(s) < 2\sigma$.

Proof. To compute the integrals, substitute

$$\frac{dx}{x} = -\left[\frac{4f}{1-f^2} + \frac{1}{f}\right]df. \qquad \square$$

LEMMA 7.2. *Uniformly in* $\frac{1}{2} < \sigma < 1$ *there is a constant* $c > 0$ *such that*

$$\left|\frac{\hat{f}_\sigma(1-\sigma+\mathrm{i}t)}{\hat{f}_\sigma(1-\sigma)}\right| \leqslant \begin{cases} \left(1+\dfrac{ct^2}{(1-\sigma)^2}\right)^{-\frac{1}{2}} & |t| < \dfrac{1}{4} \\ 1-c & |t| \geqslant \dfrac{1}{4}. \end{cases}$$

Proof. This follows on Taylor expanding $\hat{f}(s)$ about $s = 1-\sigma$. $\square$

We now follow the outline at the beginning of §5.1 to complete the proof.

Step A. We apply Lemma 5.3 with $y_q = x/N$, $\alpha = (\log\log y)^2/\log y$.

Recall that we assume $\sigma > 1/2 + 1/\log_2 q$ and have set $M = (1-\epsilon)\log q$. For $\epsilon > C/\log\log q$, C fixed but sufficiently large, we have, uniformly in $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned}\sum_p \log p \frac{r(p)^2}{1-r(p)^2} &\leqslant \sum_n \Lambda(n)\frac{f_\sigma(n/M)^2}{1-f_\sigma(n/M)^2} \\ &= \frac{1}{2\pi\mathrm{i}}\int_{(\frac{1}{2}+\sigma)} \frac{-\zeta'}{\zeta}(s)M^s\hat{g}(s)\,ds \\ &\leqslant M(1+O(\exp(-\sqrt{\log M}))) \leqslant \log\frac{x}{N} - \frac{\log(x/N)}{\log\log q}\end{aligned}$$

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by shifting the contour into the standard zero-free region for zeta and passing the pole at 1 of $-\zeta'/\zeta$. Thus the first condition of the lemma is satisfied. Meanwhile,

$$\sum_{k\log p>\log(x/N)/(\log\log q)^4} r(p)^{2k}p^{2k\alpha} \ll \sum_{p<\exp(\log^3 M)} \frac{(1-(1-f_\sigma(p/M)^2))^{\log q/\log_2^8 q}}{1-f_\sigma(p/M)^2} + \sum_{p>\exp(\log^3 M)} \frac{f_\sigma(p/M)^2p^\alpha}{1-f_\sigma(p/M)^2p^\alpha}.$$

Substituting the lower bound (Lemma 7.1(3)) for $1-f_\sigma$ in the first sum and the upper bound (Lemma 7.1(2)) for f_σ in the second sum proves that each sum is $o(1)$, uniformly in $\sigma > 1/2 + 1/\log_2 q$. This verifies the second condition of Lemma 5.3. It follows that

$$\sum_{n\leqslant x/N} r_\sigma(n)^2 = (1+o(1))\prod_p (1-r_\sigma(p)^2)^{-1}, \quad q\to\infty,$$

uniformly for $\sigma\in[1/2+1/\log_2 q, 1]$.

Before proceeding to steps B and C we prove a sequence of lemmas regarding the multiplicative function r_σ.

Parallel to the study of smooth numbers in [6], our analysis of sums involving $r_\sigma(n)$ relies on properties of the generating Dirichlet series

$$R_\sigma(s) = \prod_p \left(1-\frac{r_\sigma(p)}{p^s}\right)^{-1}, \qquad \Re(s)\geqslant 1-\sigma$$

and its logarithm and logarithmic derivatives

$$\phi_{0,\sigma}(s) = \log R_\sigma(s), \qquad \phi_{j,\sigma}(s) = (-1)^j\frac{d^j}{ds^j}\phi_0(s), \quad j>0,$$

on the line $\Re(s)=\sigma$. Note that explicitly

$$\phi_{j,\sigma}(s) = \sum_p \log^j p \sum_{n=1}^\infty n^{j-1}\left(\frac{r_\sigma(p)}{p^s}\right)^n \quad (j\geqslant 0).$$

It will be convenient to work with the "non-multiplicative" approximations

$$\tilde\phi_{j,\sigma}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)f_\sigma(n/M)\log^{j-1}n}{n^s}. \tag{18}$$

Our first lemma demonstrates that $\tilde\phi_j$ is in fact a strong approximation to ϕ_j.

LEMMA 7.3. *Let $s=\sigma+it$. Uniformly in*

$$\frac12+\frac{1}{\log_2 q}\leqslant\sigma\leqslant 1-\frac{1}{\log_2 q},$$

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we have

$$|\phi_{j,\sigma}(s) - \tilde{\phi}_{j,\sigma}(s)| \ll \frac{\log^j M}{M^{\min(\sigma/2,\sigma-1/2)}}.$$

Proof. We have

$$\begin{aligned}|\phi_{j,\sigma}(s) - \tilde{\phi}_{j,\sigma}(s)| &= \left|\sum_p \log^j p \sum_{n=2}^{\infty} n^{j-1} \frac{f_\sigma(p/M)^n - f_\sigma(p^n/M)}{p^{ns}}\right| \\ &\leqslant \sum_p \log^j p \sum_{n=2}^{\infty} n^{j-1} \frac{|1 - f_\sigma(p/M)^n| + |1 - f_\sigma(p^n/M)|}{p^{n\sigma}} \\ &\leqslant \sum_p \log^j p \left|1 - f_\sigma\left(\frac{p}{M}\right)\right| \left(\sum_{n=2}^{\infty} \frac{n^j}{p^{n\sigma}}\right) \\ &\quad + \sum_p \log^j p \sum_{n=2}^{\infty} \frac{n^{j-1}}{p^{n\sigma}} \left|1 - f_\sigma\left(\frac{p^n}{M}\right)\right|,\end{aligned}$$

and the claimed bound follows on bounding the absolute value terms by the minimum of 1 and the upper bound given in (3) of Lemma 7.1. □

We will impose $\phi_{1,\sigma}(\sigma) = \log N$, with the effect that $s = \sigma$ is the saddle point in the Perron contour for $\sum_{n \leqslant N} r_\sigma(n)$. Our next lemma gives the relationship between N and σ.

LEMMA 7.4. *We have, uniformly in* $1/2 + 2/\log_2 q < \sigma < 1$ *and* $|t| < \log_2 q$,

$$\begin{aligned}\phi_{1,\sigma}(\sigma + \mathrm{i}t) = \hat{f}_\sigma(1 - \sigma - \mathrm{i}t) M^{1-\sigma-\mathrm{i}t} - \frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) \\ + O(M^{1-\sigma} \exp(-\sqrt{\log M})).\end{aligned}$$

In particular, for σ *solving* $\phi_{1,\sigma}(\sigma) = \log N$, *the bounds* $\log_2^q \log_3^{-10} q < \log N < \sqrt{\log q}$ *imply that*

$$(1 - \sigma) \gg \frac{\log_3 q}{\log_2 q} \quad \text{and} \quad \left(\sigma - \frac{1}{2}\right) \gg \frac{\log_3 q}{\log_2 q} \quad \text{as } q \to \infty.$$

Proof. It suffices to prove the corresponding result for $\tilde{\phi}_1$ since the previous lemma implies that the resulting error is contained in the error term. To prove this lemma, write

$$\tilde{\phi}_{1,\sigma}(\sigma + \mathrm{i}t) = \frac{1}{2\pi \mathrm{i}} \int_{(\frac{1}{2})} \left(-\frac{\zeta'}{\zeta}(s + \sigma + \mathrm{i}t)\right) M^s \hat{f}_\sigma(s)\, ds,$$

shift contours, and use the standard zero-free region for ζ. □

We also require control of the other small derivatives $\phi_{j,\sigma}(\sigma)$, $j = 0, 1, 2, \ldots$.

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LEMMA 7.5. *Suppose that σ varies with q in such a way that*

$$(1-\sigma)\log_2 q \to \infty, \qquad (2\sigma-1)\log_2 q \to \infty \quad \text{as } q \to \infty.$$

Then $\phi_{j,\sigma}(\sigma) \sim \phi_{0,\sigma}(\sigma)\log_2^j q$ as $q \to \infty$.

Proof. Fix Y such that $\log Y = \log M + \sqrt{\log M/(2\sigma-1)}$. Then

$$\begin{aligned}\tilde{\phi}_{j,\sigma}(\sigma) &= \sum_n \frac{\Lambda(n) f_\sigma(n/M)\log^{j-1}(n)}{n^\sigma} \\ &\leqslant \tilde{\phi}_{0,\sigma}(\sigma)\log^j Y + \sum_{n>Y} \frac{\Lambda(n) f_\sigma(n/M)\log^{j-1}(n)}{n^\sigma}.\end{aligned}$$

Since

$$f_\sigma\left(\frac{n}{M}\right) \leqslant \left(\frac{Mc_\sigma}{n}\right)^\sigma \quad \text{and} \quad \log Y \geqslant \frac{1}{2\sigma-1},$$

the sum on the right is bounded by

$$\begin{aligned}(Mc_\sigma)^\sigma \frac{\log^{j-1} Y}{(2\sigma-1)Y^{2\sigma-1}} &\ll \log^{j-1} Y\, \hat{f}_\sigma(1-\sigma) M^{1-\sigma} \cdot \left(\frac{M}{Y}\right)^{2\sigma-1} \\ &\ll \log^{j-1} Y \tilde{\phi}_1(\sigma)\left(\frac{M}{Y}\right)^{2\sigma-1} \qquad (19)\end{aligned}$$

(note that $c_\sigma < 1$), and this is $o(\tilde{\phi}_{1,\sigma}(\sigma)\log^{j-1} Y)$. In particular, applying this with $j=1$ gives

$$\tilde{\phi}_{1,\sigma}(\sigma) \leqslant \tilde{\phi}_{0,\sigma}(\sigma)\log Y + o(\tilde{\phi}_{1,\sigma}(\sigma))$$

and so $\phi_{1,\sigma}(\sigma) \leqslant (1+o(1))\tilde{\phi}_{0,\sigma}(\sigma)\log_2 q$. Reinserting this above yields the general upper bound $\tilde{\phi}_{j,\sigma}(\sigma) \leqslant (1+o_j(1))\tilde{\phi}_{0,\sigma}(\sigma)(\log_2 q)^j$.

Now choose Z so that $\log Z = \log M - \sqrt{\log M/(1-\sigma)}$. Then

$$\tilde{\phi}_{j,\sigma}(\sigma) \geqslant \log^j Z\left[\tilde{\phi}_{0,\sigma}(\sigma) - \sum_{n<Z} \frac{\Lambda(n) f_\sigma(n/M)}{n^\sigma \log n}\right].$$

Bounding $f_\sigma < 1$ and using $\log Z \geqslant 1/(1-\sigma)$, we have

$$\begin{aligned}\sum_{n<Z} \frac{\Lambda(n) f_\sigma(n/M)}{n^\sigma \log n} &\ll \frac{Z^{1-\sigma}}{(1-\sigma)\log Z} \\ &\ll \frac{M^{1-\sigma}}{(1-\sigma)\log M}\left(\frac{Z}{M}\right)^{1-\sigma} \\ &\ll \frac{\tilde{\phi}_{1,\sigma}(\sigma)}{\log M}\left(\frac{Z}{M}\right)^{1-\sigma} \ll \tilde{\phi}_{0,\sigma}(\sigma)\left(\frac{Z}{M}\right)^{1-\sigma} = o(\tilde{\phi}_{0,\sigma}(\sigma)).\end{aligned}$$

Thus $\tilde{\phi}_{j,\sigma}(\sigma) \sim \tilde{\phi}_{0,\sigma}(\sigma)\log^j M$. This completes the proof, since Lemmas 7.3 and 7.4 imply that, for each j, $\phi_{j,\sigma}(\sigma) \sim \tilde{\phi}_{j,\sigma}(\sigma)$. □

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The final lemma needed for step B guarantees decay of the generating function $R_\sigma(\sigma+\mathrm{i}t)$ away from the real axis. This is a standard ingredient of the saddle point method.

LEMMA 7.6. *Let $\phi_{1,\sigma}(\sigma)=\log N$. Then uniformly for N varying in the range $\log_2^2 q\,\log_3^{-10} q<\log N<\sqrt{\log q}$, we have*

$$\Re[\tilde\phi_{0,\sigma}(\sigma)-\tilde\phi_{0,\sigma}(\sigma+\mathrm{i}t)] \gg \begin{cases} t^2\tilde\phi_{2,\sigma}(\sigma), & |t|<(\log Y)^{-1} \\ \dfrac{\tilde\phi_{1,\sigma}(\sigma)\min(t^2/(1-\sigma)^2,1)}{\log M+3\log_2 M/(2\sigma-1)}, & (\log Y)^{-1}<|t|<M \end{cases}$$

where $\log Y=\log M+\sqrt{\log M/(2\sigma-1)}$.

Proof. For all t, we have

$$\Re[\tilde\phi_{0,\sigma}(\sigma)-\tilde\phi_{0,\sigma}(\sigma+\mathrm{i}t)]=\sum_n \frac{\Lambda(n)f_\sigma(n/M)}{n^\sigma\log n}(1-\cos(t\log n)). \tag{20}$$

For $|t|<(\log Y)^{-1}$ this is

$$\gg t^2\tilde\phi_{2,\sigma}(\sigma)-O\left(t^2\sum_{n>Y}\frac{\Lambda(n)f_\sigma(n/M)\log n}{n^\sigma}\right)$$
$$\gg t^2\tilde\phi_{2,\sigma}(\sigma)\left(1-O\left(\left(\frac{M}{Y}\right)^{2\sigma-1}\right)\right)\gg t^2\phi_{2,\sigma}(\sigma),$$

by bounding the tail as in the previous lemma.

For $(\log Y)^{-1}<|t|<M$, the expression (20) is at least as large as

$$\frac{1}{\log Y}\Re\left\{\sum_n\frac{\Lambda(n)f_\sigma(n/M)}{n^\sigma}(1-n^{-\mathrm{i}t})-\sum_{n>Y}\frac{\Lambda(n)f_\sigma(n/M)}{n^\sigma}\right\}$$
$$\geqslant\frac{1}{\log Y}\Re\left\{\hat f_\sigma(1-\sigma)M^{1-\sigma}\left(1-\frac{\hat f_\sigma(1-\sigma-\mathrm{i}t)M^{-\mathrm{i}t}}{\hat f_\sigma(1-\sigma)}\right)\right.$$
$$+\frac{\zeta'}{\zeta}(\sigma+\mathrm{i}t)-\frac{\zeta'}{\zeta}(\sigma)+O\left(\tilde\phi_{1,\sigma}(\sigma)\left(\frac{M}{Y}\right)^{2\sigma-1}\right)$$
$$\left.+O(M^{1-\sigma}\exp(-\sqrt{\log M}))\right\}.$$

We thus obtain

$$(20)\gg\frac{\tilde\phi_{1,\sigma}(\sigma)\min(t^2/(1-\sigma)^2,1)}{\log Y}$$

by applying Lemma 7.2. □

Recall that we denote the logarithmic derivatives of $\zeta(s,y)$ by

$$\psi(s,y)=\log\zeta(s,y),\qquad \psi_j(s,y)=(-1)^j\frac{d^j}{ds^j}\psi(s,y),\quad j=0,1,2,\dots,$$

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and note that

$$\psi_j(s,\, y) = \sum_{n:p|n\Rightarrow p<y} \frac{\Lambda(n)\log^{j-1}(n)}{n^s}.$$

In step C of our proof we will show that

$$\sum_{n\leqslant N} r_\sigma(n) \sim \Psi(N,\, (1+o(1))\kappa(\sigma)\log q)$$

with $\kappa(\sigma)^{1-\sigma} = (1-\sigma)\hat{f}_\sigma(1-\sigma)$ as in the theorem by making the comparison $\phi_{j,\sigma}(s) \approx \psi_j(s,\, \kappa M)$ for s near σ.

The following is an analogy of Lemma 7.5 for $\psi_j(\sigma,\, \kappa M)$.

LEMMA 7.7. *Assume that σ satisfies $\frac{1}{2} < \sigma < 1$ and $(\sigma - \frac{1}{2})\log M \to \infty$, $(1-\sigma)\log M \to \infty$ as $q \to \infty$. Then for $\kappa(\sigma)^{1-\sigma} = (1-\sigma)\hat{f}_\sigma(1-\sigma)$ as above,*

$$\psi_j(\sigma,\, \kappa M) \sim \psi_0(\sigma,\, \kappa M)\log^j M.$$

Proof. In analogy with (18), introduce the non-multiplicative approximation

$$\tilde{\psi}_j(s,\, \kappa M) = \sum_{n\leqslant \kappa M} \frac{\Lambda(n)\log^{j-1} n}{n^s}.$$

In the range $1/2 + 1/\log M \leqslant \sigma \leqslant 1 - 1/\log M$, the uniform bound

$$|\psi_j(\sigma,\, \kappa M) - \tilde{\psi}_j(\sigma,\, \kappa M)| \ll M^{\frac{1}{2}-\sigma}\log^j M$$

is straightforward to establish along the lines of Lemma 7.3.

Since $\psi_j(\sigma,\, \kappa M) \sim \tilde{\psi}_j(\sigma,\, \kappa M)$, the proof is completed by

$$\tilde{\psi}_j(\sigma,\, \kappa M) = \sum_{n\leqslant \kappa M} \frac{\Lambda(n)\log^{j-1} n}{n^\sigma} \sim \frac{(\kappa M)^{1-\sigma}\log^{j-1} M}{1-\sigma}. \qquad \square$$

The following essential lemma proves $\phi_{j,\sigma}(\sigma) \approx \psi_j(\sigma,\, \kappa M)$, $j = 1, 2$.

LEMMA 7.8. *Continue to set $\kappa(\sigma)^{1-\sigma} = (1-\sigma)\hat{f}_\sigma(1-\sigma)$. We have, uniformly in $1/2 + 2/\log M < \sigma < 1$:*

(1) $\phi_{1,\sigma}(\sigma) - \psi_1(\sigma,\, \kappa M) = O(M^{1-\sigma}\exp(-\sqrt{\log M}))$;

(2) $\phi_{0,\sigma}(\sigma) - \psi_0(\sigma,\, \kappa M) = O(\phi_{1,\sigma}(\sigma)/((2\sigma-1)\log^2 M))$.

Proof. It suffices to check this for $\tilde{\phi}$ and $\tilde{\psi}$. To prove the first statement, write

$$\tilde{\phi}_{1,\sigma}(\sigma) - \tilde{\psi}_1(\sigma,\, \kappa M) = \frac{1}{2\pi \mathrm{i}}\int_{(\frac{1}{2})} \left(-\frac{\zeta'}{\zeta}(s+\sigma)\right) M^s \left[\hat{f}_\sigma(s) - \frac{\kappa^s}{s}\right] ds$$

and note that the two poles, at $s = 1-\sigma$ and $s = 0$, are nullified by the difference. Shift contours.

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For the second, recall that

$$\phi_1(\sigma) \sim \hat{f}_\sigma(1-\sigma)M^{1-\sigma} \quad \text{and} \quad \hat{f}_\sigma(1-\sigma) = \frac{c_\sigma^{-\sigma}}{2(1-\sigma)} \quad \text{with } c_\sigma \gg \frac{1}{2\sigma-1}.$$

We have

$$\begin{aligned}
&\tilde{\phi}_{0,\sigma}(\sigma) - \tilde{\psi}_0(\sigma, \kappa M) \\
&\quad = \sum_{n \leqslant \kappa M} \frac{\Lambda(n)(f_\sigma(n/M)-1)}{n^\sigma \log n} + \sum_{n \geqslant \kappa M} \frac{\Lambda(n) f_\sigma(n/M)}{n^\sigma \log n} \\
&\quad = \frac{1}{\log M}\left[\sum_{n \leqslant \kappa M} \frac{\Lambda(n)(f_\sigma(n/M)-1)}{n^\sigma} + \sum_{n > \kappa M} \frac{\Lambda(n) f_\sigma(n/M)}{n^\sigma}\right] \\
&\qquad + O\left(\sum_{n \leqslant \kappa M} \frac{\Lambda(n)|1-f_\sigma(n/M)|}{n^\sigma} |\log^{-1} n - \log^{-1} M|\right) \\
&\qquad + O\left(\sum_{n \geqslant \kappa M} \frac{\Lambda(n) f_\sigma(n/M)}{n^\sigma} |\log^{-1} n - \log^{-1} M|\right).
\end{aligned}$$

The bracketed term is

$$\frac{1}{\log M}[\tilde{\phi}_{1,\sigma}(\sigma) - \tilde{\psi}_1(\sigma, \kappa M)] = O(M^{1-\sigma} \exp(-\sqrt{\log M})),$$

which is permissible since

$$\phi_{1,\sigma}(\sigma) \sim \psi_1(\sigma, \kappa M) \sim \frac{(\kappa M)^{1-\sigma} - 1}{1-\sigma}.$$

In the first error term, we use $|1 - f_\sigma(n/M)| \ll (n/c_\sigma M)^{\sigma/2}$ for $n < c_\sigma M$, and bound trivially $|1 - f_\sigma(n/M)| < 1$ for $n > c_\sigma M$ to obtain

$$\ll \frac{(c_\sigma M)^{1-\sigma}}{1-\sigma/2} \frac{|\log c_\sigma| + 1}{\log^2 M} + \frac{(\kappa M)^{1-\sigma}}{1-\sigma} \frac{|\log \kappa| + 1}{\log^2 M}.$$

In the second error term we bound $f_\sigma(n/M) \ll (c_\sigma M/n)^\sigma$ to obtain

$$\ll \frac{(\kappa M)^{1-\sigma}}{2\sigma - 1} \left(\frac{c_\sigma}{\kappa}\right)^\sigma \left[\frac{|\log \kappa| + 1}{\log^2 M} + \frac{1}{(2\sigma-1)\log^2 M}\right].$$

Since $1/(2\sigma-1)(c_\sigma/\kappa)^\sigma$ is bounded as $\sigma \downarrow \frac{1}{2}$ and $|\log c_\sigma|, |\log \kappa| \ll 1/(2\sigma-1)$, these errors are also permissible. □

We now finish the proof of the main case of Theorem 3.4 for $\Delta(N, q)$.

Step B. In the following proposition we evaluate $\sum_{n \leqslant N} r_\sigma(n)$.

PROPOSITION 7.9. *Let* $\frac{1}{2} < \sigma < 1$ *solve* $\log N = \phi_{1,\sigma}(\sigma)$. *Then*

$$\sum_{n \leqslant N} r_\sigma(n) \sim \frac{N^\sigma e^{\phi_{0,\sigma}(\sigma)}}{\sigma \sqrt{2\pi \phi_{2,\sigma}(\sigma)}}.$$

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Proof. Choose $T = \log N \log^2 M$, $\delta = 1 - \sigma$ and apply Lemma 5.5 and the second bound of Lemma 7.6 to deduce that

$$\sum_{n\leqslant N} r(n) = N^{\sigma} \mathrm{e}^{\phi_{0,\sigma}(\sigma)} \Bigg\{ \frac{1}{2\pi} \int_{-(1-\sigma)}^{1-\sigma} \exp(\mathrm{i}t \log N + \phi_{0,\sigma}(\sigma + \mathrm{i}t) - \phi_{0,\sigma}(\sigma)) \frac{dt}{\sigma + \mathrm{i}t} + O\Bigg(\frac{1}{\log M \sqrt{\log N}}\Bigg) + O\Bigg(\exp\Bigg(\frac{-c \log N}{\log M + 3 \log_2 M/(2\sigma - 1)}\Bigg) \log\log N\Bigg)\Bigg\}.$$

Since $\phi_{2,\sigma}(\sigma) \sim \log N \log M$, the errors are genuinely smaller than the main term. Now $|\phi_{3,\sigma}(\sigma + \mathrm{i}t)| \leqslant \phi_{3,\sigma}(\sigma) \sim \log^2 M \log N$ holds for all t. Splitting the integral accordingly at $|t| = \log^{-2/3} M \log^{-1/3} N$, and $|t| = 1/\log Y$ with Y as in Lemma 7.6 we obtain the main term by Taylor expanding ϕ_0 on the interval around 0, and error terms in the remaining part of the integral. □

Step C. The following proposition compares $\sum_{n\leqslant N} r_\sigma(n)$ to $\Psi(N, \kappa(\sigma)M)$, allowing us to complete the proof of the theorem.

PROPOSITION 7.10. *Let $\kappa(\sigma)$ be defined by $\kappa^{1-\sigma} = (1-\sigma)\hat{f}_\sigma(1-\sigma)$. As before, let $\log N = \phi_{1,\sigma}(\sigma)$. We have*

$$\left|\log\left(\frac{\sum_{n\leqslant N} r_\sigma(n)}{\Psi(N, \kappa(\sigma)M)}\right)\right| = O\left(\frac{\log N}{(2\sigma - 1)\log^2 M}\right).$$

Proof (Deduction of theorem). By Corollary 5.6, so long as $|\theta| < 1$ and $|\theta| \log_3 q \to \infty$ as $q \to \infty$,

$$\log \frac{\Psi(N, \mathrm{e}^{\theta}\kappa(\sigma)M)}{\Psi(N, \kappa(\sigma)M)} \sim \frac{\theta}{\log M} \frac{\log N}{\log M} \log \frac{\log N}{\log M}.$$

Therefore, since $(2\sigma - 1)\log(\log N/\log M) \to \infty$, Proposition 7.10 establishes that there is some $\kappa' = (1 + o(1))\kappa$ for which $\Delta(N, q) \geqslant \Psi(N, \kappa' M)$. □

Proof of Proposition 7.10. Define $\tilde{N}$ by $\log \tilde{N} = \psi_1(\sigma, \kappa M)$. Then

$$\left|\log\left(\frac{\sum_{n\leqslant N} r_\sigma(n)}{\Psi(N, \kappa M)}\right)\right| \leqslant \left|\log\left(\frac{\sum_{n\leqslant N} r(n)}{\Psi(\tilde{N}, \kappa M)}\right)\right| + \left|\log\left(\frac{\Psi(\tilde{N}, \kappa M)}{\Psi(N, \kappa M)}\right)\right| = I + II.$$

We first consider I. By Proposition 7.9 applied to $\sum_{n\leqslant N} r_\sigma(n)$ and Theorem 5.4 applied to $\Psi(\tilde{N}, \kappa M)$, we have

$$I = |\sigma(\phi_{1,\sigma}(\sigma) - \psi_1(\sigma, \kappa M)) + \phi_{0,\sigma}(\sigma) - \psi_0(\sigma, \kappa M) - \tfrac{1}{2}[\log \phi_{2,\sigma}(\sigma) - \log \psi_2(\sigma, \kappa M)]| + o(1).$$

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Since $\phi_{2,\sigma}(\sigma) \sim \psi_2(\sigma, \kappa M) \sim \log N \log M$, the bounds of Lemma 7.8 give

$$I = O\left(\frac{\log N \log(\log N/\log M)}{(2\sigma - 1)\log^2 M}\right).$$

For II, let α solve $\psi_1(\alpha, \kappa M) = \log N$ so that, by Theorem 5.4,

$$II = \sigma \log \tilde{N} - \alpha \log N + \psi_0(\sigma, \kappa M) - \psi_0(\alpha, \kappa M) - \tfrac{1}{2}[\log \psi_2(\sigma, \kappa M) - \log \psi_2(\alpha, \kappa M)] + o(1).$$

Now by the mean value theorem,

$$|\log \tilde{N} - \log N| = |\psi_1(\sigma, \kappa M) - \psi_1(\alpha, \kappa M)| = |\sigma - \alpha| \cdot \psi_2(\gamma, \kappa M),$$
$$|\psi_0(\sigma, \kappa M) - \psi_0(\alpha, \kappa M)| = |\sigma - \alpha| \cdot \psi_1(\gamma', \kappa M),$$

for some γ, γ' between α and σ. Moreover, $\log N \sim \log \tilde{N}$ so

$$\psi_2(\gamma, \kappa M) \sim \psi_2(\alpha, \kappa M) \sim \psi_2(\sigma, \kappa M) \sim \log N \log M$$

and therefore

$$|\sigma - \alpha| \ll \frac{|\phi_1(\sigma) - \psi_1(\sigma, \kappa M)|}{\log N \log M} \ll \exp(-\sqrt{\log M}).$$

Combining these estimates completes the proof, since

$$II \ll \sigma |\log \tilde{N} - \log N| + |\sigma - \alpha| \log N + o(1) = O\left(\frac{\log N}{\exp(-\sqrt{\log M})}\right). \quad \square$$

7.3. *Dual case* $\Delta(q/N, q)$. In this section we take $x = (1/\log q)\sqrt{q/N}$, $M = (1/2 - \epsilon)\log q$ and define the completely multiplicative function $r_\sigma(n)$ by $r_\sigma(p) = f_\sigma(p/M)$.

A calculation exactly analogous to the one in part (A) above proves that for ϵ tending sufficiently slowly to 0,

$$\sum_{n \leqslant 2x/N} r_\sigma(n)^2 = (1 + o(1)) \prod_p (1 - r_\sigma(p)^2)^{-1},$$

so that, by the second part of Proposition 5.2,

$$\Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N^2} \sum_{n \leqslant N/2} n r_\sigma(n).$$

The evaluation of $\sum_{n \leqslant N/2} n r_\sigma(n)$ by a Perron integral yields

$$\frac{N}{2} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{N}{2}\right)^s R_\sigma(s) \frac{ds}{s+1};$$

the only difference between this integral and the one evaluated in Proposition 7.9 is the factor of 2^{-s} and the fact that $1/s$ has been replaced by $1/(s+1)$ in

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the denominator. Exactly the same method as there yields, for σ solving $\phi_{1,\sigma}(\sigma) = \log N - \log 2$,

$$\sum_{n \leqslant N/2} n r_\sigma(n) \sim \frac{(N/2)^{1+\sigma} \mathrm{e}^{\phi_{0,\sigma}(\sigma)}}{(1+\sigma)\sqrt{2\pi \phi_{2,\sigma}(\sigma)}}.$$

It then follows that for $\kappa'(\sigma)$ solving ${\kappa'}^{1-\sigma} = (1-\sigma)\hat{f}(1-\sigma)$,

$$\Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N} \Psi(N, (1+o(1))\kappa'(\sigma)M).$$

Since $M = (1-o(1))(\log q/2)$, therefore,

$$\Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N} \Psi\left(N, \left(\frac{1}{2} + o(1)\right)\kappa(\sigma) \log q\right).$$

References

1. D. A. Burgess, The distribution of quadratic residues and non-residues. *Mathematika* **4** (1957), 106–112.
2. D. W. Farmer, S. M. Gonek and C. P. Hughes, The maximum size of L-functions. *J. Reine Angew. Math.* **609** (2007), 215–236.
3. S. W. Graham and C. J. Ringrose, Lower bounds for least quadratic nonresidues. In *Analytic Number Theory (Allerton Park, IL, 1989)* (*Progress in Mathematics* **85**), Birkhäuser (Boston, 1990), 269–309.
4. A. Granville and K. Soundararajan, Large character sums. *J. Amer. Math. Soc.* **14**(2) (2001), 365–397 (electronic).
5. A. Granville and K. Soundararajan, Large character sums: pretentious characters and the Pólya–Vinogradov theorem. *J. Amer. Math. Soc.* **20**(2) (2007), 357–384 (electronic).
6. A. Hildebrand and G. Tenenbaum, Integers without large prime factors. *J. Théor. Nombres Bordeaux* **5**(2) (1993), 411–484.
7. B. Hough, The resonance method for large character sums, arXiv:1109.1786.
8. J. E. Littlewood, On the class-number of the corpus $\sqrt{-k}$. *Proc. Lond. Math. Soc.* **27** (1927), 358–372.
9. D. Milicevic, Large values of eigenfunctions on arithmetic hyperbolic manifolds. *PhD Thesis*, Princeton University, ProQuest LLC, Ann Arbor, MI, 2006.
10. H. L. Montgomery and R. C. Vaughan, Exponential sums with multiplicative coefficients. *Invent. Math.* **43**(1) (1977), 69–82.
11. N. Ng, Extreme values of $\zeta'(\rho)$. *J. Lond. Math. Soc.* (2) **78**(2) (2008), 273–289.
12. K. Soundararajan, Extreme values of zeta and L-functions. *Math. Ann.* **342**(2) (2008), 467–486.

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