$v = x - y$ then delivers

$$8\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{du\,dv}{\cosh u\,\cosh v} = \left(\int_{-\infty}^{\infty}\frac{du}{\cosh u}\right)^2 = \pi^2,$$

where the last integral is evaluated by the substitution $w = e^u$.

Reference

1. Robin Chapman, evaluating $\zeta(2)$, available at:
http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf

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100.06 Using linear transformations to solve quadratic equations

A quadratic equation $x^2 - bx + c = 0$ with real coefficients and $c \neq 0$ can be rewritten in two ways,

$$x = b - \frac{c}{x} \qquad \text{or} \qquad x = \frac{c}{b - x}$$

and then each can be expressed as a recursive relation ($n \geqslant 1$):

$$x_{n+1} = f(x_n) = b - \frac{c}{x_n} \quad (1), \qquad x_{n+1} = g(x_n) = \frac{c}{b - x_n} \quad (2).$$

Initial values for (1) and (2) will generate infinite sequences $\{\mu_n\}_f = \{\mu_1, \mu_2, \mu_3, \ldots\}$ and $\{\omega_n\}_g = \{\omega_1, \omega_2, \omega_3, \ldots\}$, respectively. A straightforward calculation shows that f and g are inverse functions of one another, and so if $f(x_n) = x_{n+1}$ then $g(x_{n+1}) = x_n$. To avoid the problem of dividing by zero, we assume that $\frac{c}{0} = \infty$ and $\frac{c}{\infty} = 0$. One can also show that x_n can be represented by a continued fraction and this theory (see [1]) can be used to show the following.

Theorem 1: Assume that $x^2 - bx + c = 0$ has real roots r_1 and r_2 and that $c \neq 0$ (this implies that the roots are nonzero). Assume that $x_1 \in \mathbb{R}$ and $x_1 \neq r_1, r_2$. Then the following are true:

(a) If $0 < |r_1| < |r_2|$ then $\{x_n\}_f$ converges to r_2 and $\{x_n\}_g$ converges to r_1;

(b) If $r_1 = r_2$ then $\{x_n\}_f$ and $\{x_n\}_g$ both converge to the single root;

(c) If $r_1 = -r_2$, then neither $\{x_n\}_f$ nor $\{x_n\}_g$ converges.

An illustration of Theorem 1 using cobweb diagrams can be found in [2]. In this note, we shall show geometrically that the same result occurs if we begin with a complex number. Additionally, in the case where the roots

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of the quadratic equation are both complex, we prove the following.

Theorem 2: If $x^2 - bx + c = 0$ with $c \neq 0$ has complex roots then $\{z_n\}_f$ and $\{z_n\}_g$ converge only when z_1 is a root.

For Theorem 1, to get an idea of what happens in the case of z_1 being a complex number, we look at an example.

Example: For $x^2 - 7x + 10 = (x - 2)(x - 5) = 0$, we have for (1) $x_{n+1} = 7 - \dfrac{10}{x_n}$.

Table 1 shows values of the sequences when the initial values are $z_1 = 2 + 4i, 4 + i, 5 - i$ and $-8 - 3i$.

z_1	$2.0000 + 4.0000i$	$4.0000 + 1.0000i$	$5.0000 - 1.0000i$	$-8.0000 - 3.0000i$
z_2	$6.0000 + 2.0000i$	$4.6471 + 0.5882i$	$5.0769 - 0.3846i$	$8.0959 - 0.4110i$
z_3	$5.5000 + 0.5000i$	$4.8820 + 0.2681i$	$5.0415 - 0.1484i$	$5.7680 - 0.0625i$
z_4	$5.1967 + 0.1639i$	$4.9578 + 0.1121i$	$5.0182 - 0.0583i$	$5.2665 - 0.0004i$
z_8	$5.0049 + 0.0037i$	$4.9990 + 0.0029i$	$5.0005 - 0.0015i$	$5.0063 - 0.0004i$
z_{15}	$5.0000 + 0.0000i$	$5.0000 + 0.0000i$	$5.0000 + 0.0000i$	$5.0000 + 0.0000i$

TABLE 1

It appears as though $\{z_n\}_f$ approaches 5, the larger of the roots. We note that in column two, the real parts of $\{z_n\}_f$ approach 5 from the left while in the other columns, they (eventually) approach 5 from the right.

In complex analysis, functions of the form $F(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ (where α, β, γ and δ are complex numbers) are called *linear fractional* (or *Möbius*) *transformations* (LFT) [3]. They are examples of conformal mappings, that is, functions from the extended complex plane (that includes ∞) to the extended complex plane. These mappings preserve angle measure and rotational direction and map circles to circles (lines being considered circles of infinite radius). The composition of conformal mappings is once again conformal. Equation (1) can be put into this form by replacing x with the complex variable z.

$$f(z) = b - \frac{c}{z} = \frac{bz - c}{z + 0}.$$

Since we are now in the extended complex plane, we assume $f(0) = \infty$ and $f(\infty) = b$ and so z need not be restricted.

We shall consider f as a composition of two simpler LFTs, namely,

$$\phi(z) = \frac{-c}{z} \qquad \text{and} \qquad \sigma(z) = b + z$$

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and then $f(z) = \sigma(\phi(z)) = \sigma\phi(z)$ where σ is a translation. The function ϕ, called an *inversion*, has the following useful property.

Lemma: A circle in the extended complex plane, with centre on the real axis that passes through the origin and real number t, is mapped by ϕ into the set of points given by $\frac{-c}{t} + wi$ for all real values of w. Geometrically, this says that this circle is mapped to a vertical line by ϕ.

Proof: We have that $\phi(t) = \frac{-c}{t}$, $\phi(0) = \frac{-c}{0} = \infty$, and

$$\phi\left(\frac{t}{2} + \frac{t}{2}i\right) = \frac{-c}{\frac{t}{2} + \frac{t}{2}i} = \frac{-c\left(\frac{t}{2} - \frac{t}{2}i\right)}{\frac{t^2}{4} + \frac{t^2}{4}} = -c\left(\frac{t}{2} - \frac{t}{2}i\right)\left(\frac{2}{t^2}\right) = \frac{-c}{t} + \frac{c}{t}i.$$

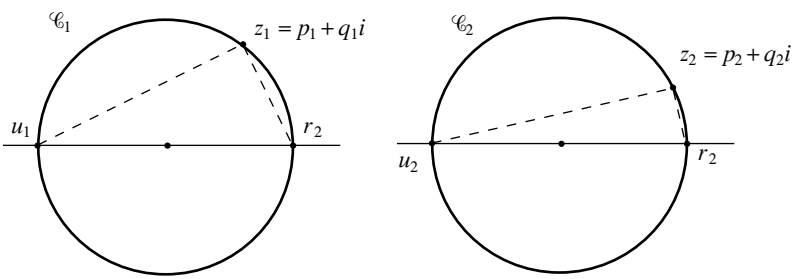
The three points, 0, t and $\frac{1}{2}t + \frac{1}{2}ti$ determine the circle and their images ∞, $\frac{-c}{t}$ and $\frac{-c}{t} + \frac{c}{t}i$, respectively, also determine a circle, in this case one of infinite radius which is the vertical line through $\frac{-c}{t}$.

We start with a complex number $z_1 = p_1 + q_1 i$ where $q_1 \neq 0$ and replace x with z in Theorem 1. To prove part (a) we need to show that $\{z_n\}_f \to r_2$.

Proof (Theorem 1, when $0 < r_1 \leqslant r_2$): Construct circle $\mathscr{C}_1$ whose centre is on the real axis and which contains z_1 and r_2. Since f is conformal we may assume $f(\mathscr{C}_1) = \mathscr{C}_2$ and, in general, $f(\mathscr{C}_k) = \mathscr{C}_{k+1}$. Let u_1 be the other real value where $\mathscr{C}_1$ intersects the real axis (Figure 1(a)). Assume $f(z_k) = z_{k+1}$ and $f(u_k) = u_{k+1}$ where the u_k always lie on the real axis. We now consider two cases:

Case 1: $u_1 \neq r_1$

If $r_1 = r_2$, then $u_1 \neq r_1$ and so this case covers the single root situation. Since the angle at z_1 is inscribed in a semicircle and f preserves angle measure, the angle at z_2 is inscribed in a semicircle, which implies that the



(a) (b)

FIGURE 1

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centre of $\mathscr{C}_2$ is also on the real axis (Figure 1(b)). This property holds even if $\mathscr{C}_k$ is a circle of infinite radius. Since Theorem 1(a) holds when starting with a real value, as $k \to \infty$, the u_k get closer and closer to r_2 which means that the radius of $\mathscr{C}_k$ approaches 0. Since z_k is always on $\mathscr{C}_k$ and r_2 always stays fixed, then z_k must approach r_2.

Case 2: $u_1 = r_1$

In this case $\mathscr{C}_1$ is a circle that contains both r_1 and r_2 and so $f(\mathscr{C}_1) = \mathscr{C}_1 = f(\mathscr{C}_k)$ since $f(r_i) = r_i$. If z_1 is on $\mathscr{C}_1$ then we know that $f(z_k)$ lies on $\mathscr{C}_1$ but it is not clear what happens to z_k as $k \to \infty$. Let Γ_1 be the circle that contains z_1 (on $\mathscr{C}_1$) and 0 and that has its centre on the real axis (Figure 2).

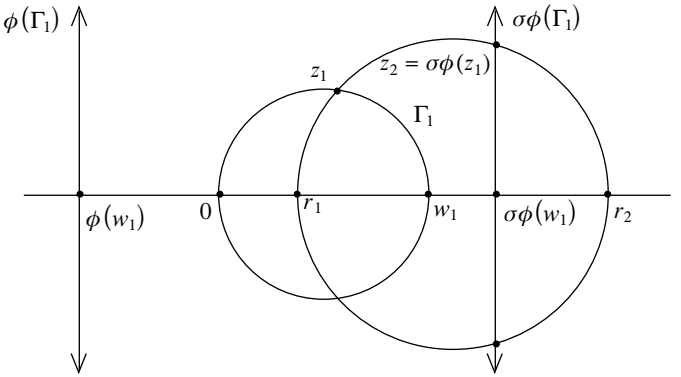


FIGURE 2

Assume Γ_1 also intersects the real axis at w_1. By the Lemma, $\phi(\Gamma_1)$ is the vertical line through $\phi(w_1)$ and since σ is a translation then $\sigma\phi(\Gamma_1)$ is the vertical line through $\sigma\phi(w_1)$. Now $r_1 < w_1 < r_2$ and so Theorem 1 implies that $f(w_1) = \sigma\phi(w_1) \in \mathbb{R}$. We claim that $\sigma\phi(w_1)$ is closer to r_2 than w_1.

We have

$$f(w_1) = b + \frac{-c}{w_1} = r_1 + r_2 - \frac{r_1 r_2}{w_1}$$

and so

$$r_2 - f(w_1) = \frac{r_1}{w_1}(r_2 - w_1).$$

Since $r_1 < w_1 < r_2$ and $f(w_1) = \sigma\phi(w_1)$, it follows that $0 < r_2 - \sigma\phi(w_1) < r_2 - w_1$. Now, $\sigma\phi(z_1) = z_2$ is the point of intersection of $\sigma\phi(\Gamma_1)$ and $\mathscr{C}_1$. We now let Γ_2 be the circle that contains z_2 (on $\mathscr{C}_1$) and 0 and that has its centre on the real axis. Assume Γ_2 intersects the real axis at w_2 and now w_2 will be closer to r_2 than w_1. Repeat the process to find successive values of w_k. As $k \to \infty$, $w_k \to r_2$ and $z_k \to r_2$ along $\mathscr{C}_1$.

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The proofs for $r_2 \leqslant r_1 < 0$ and part (c) follow the same idea as in (a).

Proof: (Theorem 2)

Assume that the solutions to $x^2 - bx + c = 0$ are $p \pm qi$ where $q \neq 0$. This immediately implies that $b = 2p$ and $c = p^2 + q^2$ and so

$$f(z) = b - \frac{c}{z} = \frac{(2p)z - (p^2+q^2)}{1z+0} \quad \text{and} \quad g(z) = \frac{c}{b-z} = \frac{p^2+q^2}{2p-z}.$$

Since $f(p \pm qi) = \dfrac{(2p)(p \pm qi) - (p^2+q^2)}{1(p \pm qi) + 0} = p \pm qi$ each root is a fixed point of f.

The LFTs F_1 and F_2 are *conjugate* if there exists an LFT H such that $F_2 = HF_1H^{-1}$. Under these circumstances, F_1 and F_2 share many properties and it is useful to find H so that F_2 is in a form that is easily analyzed. We shall first apply this idea to $f(z)$. Let $h(z) = \dfrac{z-p-qi}{z-p+qi}$. Then $h^{-1}(z) = \dfrac{(p-qi)z - p - qi}{z-1}$ and

$$\begin{aligned} hfh^{-1}(z) &= hf\left(\frac{(p-qi)z-p-qi}{z-1}\right) \\ &= h\left(\frac{-(p-qi)^2 z + (p+qi)^2}{(-p+qi)z+p+qi}\right) = \frac{p-qi}{p+qi}z = u(z). \end{aligned}$$

The mapping u has two fixed points, namely $z = 0$ and $z = \infty$, and h maps $p+qi$ to 0 and $p-qi$ to ∞. We begin with $z_1 \neq p \pm qi$ and assume that $w_1 = h(z_1)$. Let $z_{n+1} = f(z_n)$ and $w_{n+1} = u(w_n)$. Now $|p-qi| = |p+qi|$ therefore $\dfrac{|p-qi|}{|p+qi|} = 1$ and so $|w_{n+1}| = |w_n|$. This implies that the w_i lie on a circle centred at the origin. Now

$$\frac{p-qi}{p+qi} = \frac{p^2-q^2}{p^2+q^2} - \frac{2pq}{p^2+q^2}\,i$$

and so if angle θ is such that $\cos\theta = \dfrac{p^2-q^2}{p^2+q^2}$ and $\sin\theta = \dfrac{-2pq}{p^2+q^2}$ then $u(z) = (\cos\theta + i\sin\theta)z$. By De Moivre's Theorem, the $\{w_i\}$ will be periodic if $m\theta$ is a multiple of π, and will not approach any value if $m\theta$ is not a multiple of π. Thus $\{w_i\}$ has no limit except at its fixed points 0 and ∞. Because f and u are conjugate, $\{z_i\}$ will have no limit except at its fixed points, namely, $z_1 = p \pm qi$.

For $g(z)$, we use the same h function and then

$$hgh^{-1}(z) = \left(\frac{p+qi}{p-qi}\right)z$$

yields the same result.

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100.07 Yet another proof of the Steiner-Lehmus theorem

In order to prove the Steiner-Lehmus theorem (Theorem 2), we need a preliminary result.

Theorem 1: In a triangle ABC, with sides $a = BC$, $b = CA$ and $c = AB$, let CR be the bisector of the internal angle at C, where the point R is on the side AB. If $g = CM = CN$ is the length of the orthogonal projection of CR along the line CA, or along the line CB, then $g + \sqrt{g^2 + c^2} = a + b$.

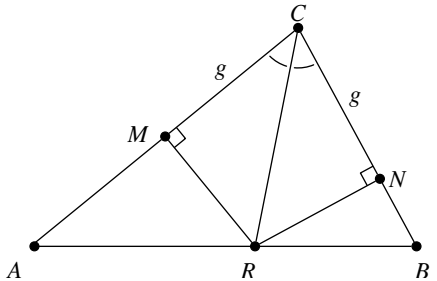


FIGURE 1

Proof:* We draw the perpendicular IL from the incentre I to the line CB:

Now twice the area of triangle ABC is $(a + b + c)IL$ and is also $(a + b)RN$. Futhermore $CN : CL = RN : IL$ and $CL = s - c$ (where s is the semiperimeter), so we get

$$g = CN = \frac{RN}{IL} CL = \frac{a + b + c}{a + b} \frac{a + b - c}{2} = \frac{a + b}{2} - \frac{1}{2} \frac{c^2}{a + b}.$$

* The author has by now collected eight proofs of Theorem 1, ranging from bumbling to inspired. The present proof, the most elegant of the eight, was contributed to the collection by Boris Lavrič from the Faculty of Mathematics and Physics, University of Ljubljana.

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