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101.39 Another angle on angle?

Sometimes with complicated configurations it is helpful to use the *directed* angle between two lines. The idea is simple, but it seems to be less widely understood and appreciated than it should be, and the intention of this short note, with no claim to originality, is to try to improve this situation. Fuller treatment may be found in [1] and [2, Preface].

Using standard notation, we define the directed angle, $\measuredangle lm$, to mean the angle in the positive sense required by rotation and/or translation to bring line l into coincidence with line m. In a proof it would not normally be necessary to specify which sense is being taken as positive, but we will take anti-clockwise, unless otherwise stated. We shall continue to refer to a non-directed angle simply as an 'angle' (as 'sense-less angle' would be rather unfair) using the notation $\angle$.

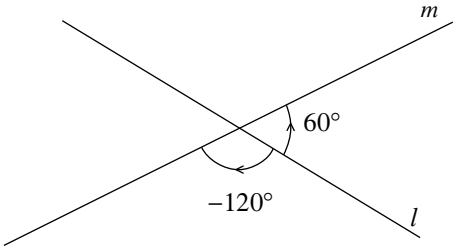


FIGURE 1

Figure 1 reminds us that as it stands this definition is ambiguous, so for an abstract definition we can define a directed angle to be a congruence class modulo 180°, and take any reference to a particular member of such a class to stand for the whole class.

So in Figure 1 we can take $\measuredangle lm$ to be $-120°$ or $60°$, or any $\theta°$ with $\theta \equiv 60 \pmod{180}$; and if we wish we may choose any interval of length 180, such as $0° \leqslant \theta° < 180°$, to contain the 'principal value' of $\measuredangle lm$.

Certain relations are immediately apparent: for lines l, m, n we have

$$\measuredangle lm + \measuredangle mn = \measuredangle ln, \tag{1}$$

$$\measuredangle ll = 0°, \tag{2}$$

$$\measuredangle ml = -\measuredangle lm. \tag{3}$$

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Also if T is a transformation and $T(x) \equiv x'$, then if T is a translation, rotation or enlargement,

$$(\measuredangle lm)' = \measuredangle l'm' \tag{4}$$

and if T is a reflection then

$$(\measuredangle lm)' = \measuredangle m'l'. \tag{5}$$

Next we consider angles defined by points A, B, C, …, and we define $\measuredangle ABC$ to be $\measuredangle lm$, where $l = \text{line}\,AB$ and $m = \text{line}\,BC$. Throughout this note XY means the line XY. Notice that from (1) we have

$$\measuredangle AXB + \measuredangle BXC + = \measuredangle AXC \tag{6}$$

and from (3) we have

$$\measuredangle ABC = -\measuredangle CBA. \tag{7}$$

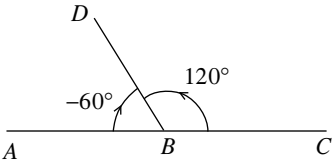


FIGURE 2

First we look at three simple situations:

(i) In Figure 2, consider $\measuredangle ABD$ and $\measuredangle CBD$: AB and BC are the same line, l say, so with BD as m, we have

$$\measuredangle ABD = \measuredangle CBD = \measuredangle lm = 120° \text{ (or } -60°).$$

Thus in general if $X_1X_2 \equiv 1$ and $Y_1Y_2 \equiv m$ and $l \cap m = A$ then regardless of the relative positions of X_i on l or Y_i on m,

$$\measuredangle X_1AY_1 = \measuredangle X_2AY_2. \tag{8}$$

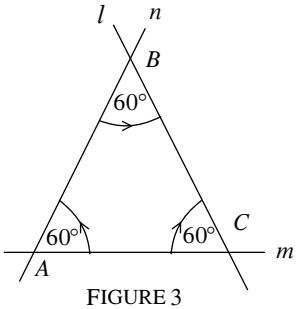


FIGURE 3

(ii) In Figure 3, using results (1) and (2) we get

$$\measuredangle BCA + \measuredangle CAB + \measuredangle ABC$$

$$= \measuredangle lm + \measuredangle mn + \measuredangle nl$$

$$= \measuredangle ln + \measuredangle nl = \measuredangle ll = 0°.$$

Since 60 + 60 + 60 = 180, this answer may seem perverse, but as each directed angle could 'naturally' be 60° or −120°, possible answers without reduction mod 180° are 180°, 0°, −180° or −360°, according as 60° is assigned to 3, 2, 1 or 0 angles.

Obviously summing the directed angles around any polygon gives the same result, i.e. 0°; while for any convex polygon, with the appropriate sense as positive, the *principal* value $\theta°$ (with $0 < \theta < 180$) of the directed angle at each vertex is equal to the exterior angle. Hence the sum of the *principal* values equals the sum of the exterior angles, which is 360°. Naturally $360 \equiv 0 \pmod{180}$.

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(iii) In Figure 4, consider the directed angles subtended at points of the circumference by the chord AB:

$$\measuredangle AXB = \measuredangle AY_1B = \measuredangle AY_2B.$$

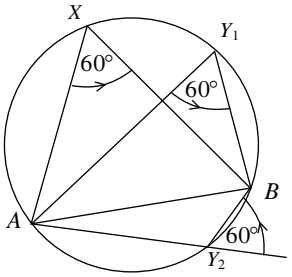


FIGURE 4

Conversely if $\measuredangle AXB = \measuredangle AYB \neq 0°$ then A, B, X, Y are concyclic whichever side of AB the points X and Y are on.

Thus if $ABCD$ is cyclic we may choose any of the six possible pairs of points, A and B say, and we have

$$\measuredangle ACB = \measuredangle ADB \neq 0° \qquad (9)$$

and conversely.

So the two theorems, that angles in the same segment are equal and that opposite angles of a cyclic quadrilateral are supplementary, become a single theorem with directed angles, which simplifies many proofs, as we now illustrate.

We conclude with proofs of two well-known results. The proofs depend heavily on statements (8) and (9), and after most steps we give the above number of the justifying statement. We give some diagrams, but readers may care to try to follow the proofs, or even to construct them, without reference to the diagram. As physical representations of geometrical results diagrams are usually both aesthetically pleasing and useful, but are not essential to the argument when using directed angles to establish results about points, lines and circles.

Theorem 1: If $ABCP$ is a cyclic quadrilateral, and points D, E, F are chosen on BC, CA, AB respectively, such that $\measuredangle PDB = \measuredangle PEC = \measuredangle PFA = \theta°$, then D, E, F are collinear.

Comment

Note that it is the fact that we are using directed angles that ensures that D, E, F are uniquely defined. With angles there would be two possible positions for each, unless $\theta = 90°$, in which case we have the Simson line theorem.

Proof: $\measuredangle BFP = \measuredangle BDP$, given.

Therefore $BFDP$ is cyclic, using (9), and hence by cyclic rotation over ABC, $CDEP$ is cyclic.

$$\therefore \measuredangle PDE = \measuredangle PCE \qquad \text{using (9)}$$

$$= \measuredangle PCA \qquad \text{using (8)}$$

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$$= \measuredangle PBA \qquad \text{using (9)}$$
$$= \measuredangle PBF \qquad \text{using (8)}$$
$$= \measuredangle PDF \qquad \text{using (9).}$$

Then

$$\measuredangle EDF = \measuredangle EDP + \measuredangle PDF \qquad \text{using (6)}$$
$$= -\measuredangle PDE + \measuredangle PDF \qquad \text{using (7)}$$
$$= 0°.$$

Thus D, E, F are collinear, essentially using (2).

Comments

If the above proof is worked through using angles, in Figure 5 it appears to work perfectly, but in Figure 6 to get $CDEP$ cyclic we need the exterior angle theorem. Thus the proof is diagram-dependent, and to use cyclic rotation to get $CDEP$ cyclic is not valid (nor necessary of course), since the proof method involved in the initial statement is not identical to the proof method in the derived statement. This problem does not arise with directed angles.

In Figure 5, P is on the major arc with respect to all three sides of triangle ABC, regarded as chords of the circle, and in Figure 6 P is on the major arc for two sides. A third possibility is just one side. Using angles each possibility requires its own diagram and modifications if necessary to the proof. And can we be sure that this completes the relevant possibilities, or is yet more exhaustive examination required to achieve rigour?

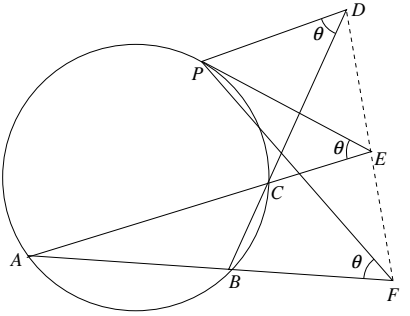


FIGURE 5

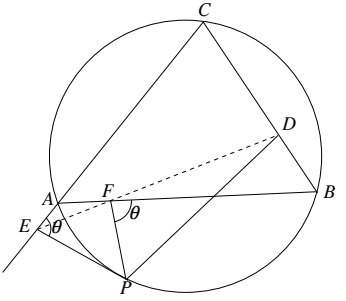


FIGURE 6

But with directed angles the basic system of results is independent of diagrams, and cyclic rotation is valid if we want to use it. We don't need to have a diagram at all, although just one may be useful in framing the argument.

Finally readers may agree that the deduction that $\measuredangle EDF = 0°$ to prove the collinearity of D, E, F is satisfying, but certainly it is not necessary:

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$\measuredangle PDE = \measuredangle PDF$ would be sufficient; but without the diagrams, $\angle PDE = \angle PDF$ would not.

Theorem 2: In any triangle ABC, let BA_1C be an equilateral triangle constructed outwards, so that A_1 and A are on opposite sides of BC, and using cyclic rotation define B_1 and C_1 similarly. Then AA_1, BB_1 and CC_1 are concurrent in F_1, say, and $\measuredangle AF_1B = \measuredangle BF_1B = \measuredangle CF_1A = 120°$.

(F_1 is the first isogonic point of triangle ABC, and if no angle of triangle ABC is greater than 120°, it is the point for which the sum of its distances from A, B, C is least, which was the property which interested Fermat. Our interest here is in angles, but for a beautiful treatment of all its main properties, due to J. E. Hofman, see [3].)

Proof: Let $BB_1 \cap CC_1 = F_1$ and let $\mathcal{R}$ be a rotation of 60° about A. Then $\mathcal{R}(A, C, C_1) = (A, B_1, B)$ and $\measuredangle C_1CA = \measuredangle BB_1A$ using (4) so $\measuredangle F_1CA = \measuredangle F_1B_1A$ using (7). Hence F_1AB_1C is cyclic using (9) and similarly F_1ABC_1 is cyclic.

Now we have

$$\begin{aligned}\measuredangle BF_1C &= \measuredangle B_1F_1C_1 \qquad \text{using (8)}\\ &= \measuredangle B_1F_1A + \measuredangle AF_1C_1 \qquad \text{using (6)}\\ &= \measuredangle B_1CA + \measuredangle ABC_1 \qquad \text{using (9)}\\ &= 60° + 60° = 120° \equiv -60° = \measuredangle BA_1C.\end{aligned}$$

Therefore BCF_1A_1 is cyclic using (8).

Then

$$\begin{aligned}\measuredangle BF_1A_1 &= \measuredangle BCA_1 \qquad \text{using (9)}\\ &= 60°\end{aligned}$$

and

$$\begin{aligned}\measuredangle AF_1B &= \measuredangle AC_1B \qquad \text{using (9)}\\ &= -60°.\end{aligned}$$

Therefore

$$\begin{aligned}\measuredangle AF_1A_1 &= \measuredangle AF_1B + \measuredangle BF_1A_1 \qquad \text{using (6)}\\ &= -60° + 60° = 0°\end{aligned}$$

whence A, F_1, A_1 are collinear using (2).

Thus AA_1, BB_1 and CC_1 are concurrent in F_1, and by cyclic rotation from $\measuredangle BF_1C$ above we have $120° = \measuredangle BF_1C = \measuredangle CF_1A = \measuredangle AF_1B$, as required.

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Comment

If $\angle BAC > 120°$ then A lies between A_1 and F_1, so $\angle BF_1C = 120°$ but $\angle AF_1B = \angle CF_1A = 60°$ as in Figure 8; but if all angles of triangle ABC are less than 120° the three angles at F_1 are all 120°, as in Figure 7. Thus a proof with angles requires two cases to be considered, and the proofs diverge quite early.

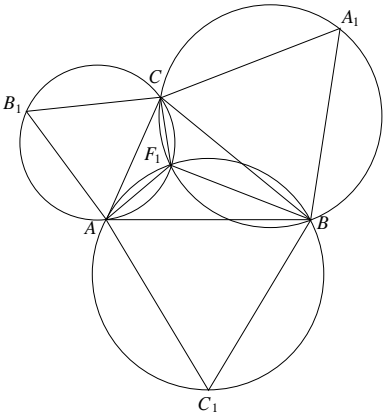


FIGURE 7

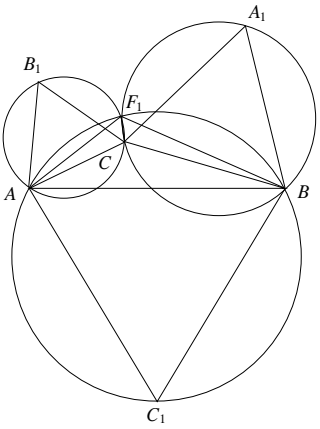


FIGURE 8

Theorem 2(a): As for Theorem 2, but the equilateral triangles now face 'inwards', so A_2, B_2, C_2 are reflections of A_1, B_1, C_1 respectively in BC, CA, AB. Then AA_2, BB_2, CC_2 are concurrent in F_2 (the second isogonic point) and $\measuredangle BF_2C = 120°$, etc with clockwise as positive.

Proof: With reference to Theorem 2 and in accordance with (5), the reflections in the sides of triangle ABC of the equilateral triangles on those sides reverse the signs of all the directed angles of the equilateral triangles, but the directed angles of triangle ABC are unchanged. So we now choose clockwise as positive (as is our right), and the directed angles of the equilateral triangles now have the same values as in Theorem 2, and we take $\mathcal{R}$ as a positive (i.e. clockwise) rotation. The values of the directed angles of triangle ABC have been reversed, but none of these is mentioned in the proof of Theorem 2. Hence the proof of Theorem 2(a) is now identical to that of Theorem 2 if we simply replace suffix 1 with suffix 2.

Acknowledgement

I am grateful to the referee for recommending the substantial extension of my original submission to include examples of directed angles in action, and for other helpful comments.

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10.1017/mag.2017.141 J. M. CHICK

101.40 Dandelin circles

The object of this Note is to show how the two-dimensional analogue of a familiar three-dimensional configuration reveals relations between the radii of the in- and e-circles of a triangle.

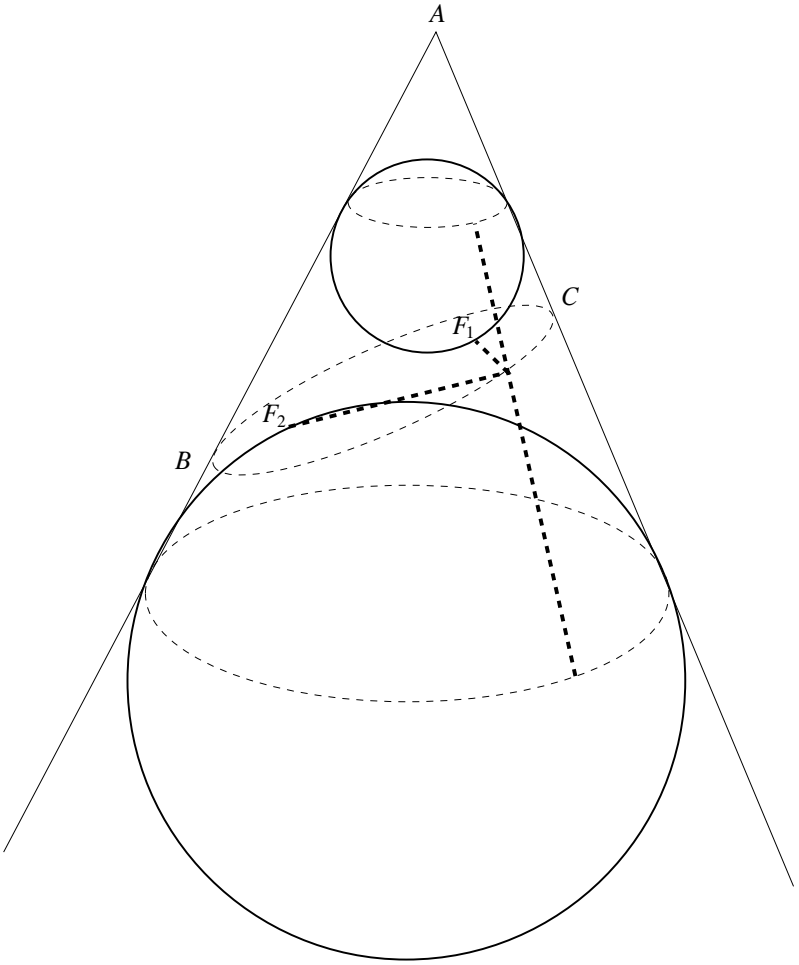


FIGURE 1

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