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ARTICLE

Hypergraphs with uniform Turán density equal to 8/27

Frederik Garbe[1,2], Daniel Il'kovič[2,3,4], Daniel Král'[2,3,5], Filip Kučerák[2,5] and Ander Lamaison[2,6]

[1]Czech Academy of Sciences, Institute of Computer Science, Prague, Czech Republic, [2]Faculty of Informatics, Masaryk University, Brno, Czech Republic, [3]Institute of Mathematics, Leipzig University, Leipzig, Germany, [4]Mathematics Institute and DIMAP, University of Warwick, Coventry, UK, [5]Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, and [6]Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea

Corresponding author: Frederik Garbe; Email: garbe@cs.cas.cz

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Abstract

In the 1980s, Erdős and Sós initiated the study of Turán problems with a uniformity condition on the distribution of edges: the uniform Turán density of a hypergraph H is the infimum over all d for which any sufficiently large hypergraph with the property that all its linear-size subhypergraphs have density at least d contains H. In particular, they asked to determine the uniform Turán densities of $K_4^{(3)-}$ and $K_4^{(3)}$. After more than 30 years, the former was solved in [Israel J. Math. 211 (2016), 349 – 366] and [J. Eur. Math. Soc. 20 (2018), 1139 – 1159], while the latter still remains open. Till today, there are known constructions of 3-uniform hypergraphs with uniform Turán density equal to 0, 1/27, 4/27, and 1/4 only. We extend this list by a fifth value: we prove an easy to verify sufficient condition for the uniform Turán density to be equal to 8/27 and identify hypergraphs satisfying this condition.

Keywords: hypergraphs; uniform Turán density

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1. Introduction

Turán problems, which ask for the minimum density threshold for the existence of a certain substructure, are one of the most fundamental kind of problems in extremal combinatorics. Formally, the Turán density of a graph G is the infimum over all d such that every sufficiently large n-vertex graph with $d\binom{n}{2}$ edges contains G as a subgraph. The classical theorems of Mantel [16] and Turán [27] determine the Turán density of complete graphs. For general graphs, Erdős and Stone [9] proved that the Turán density of any r-chromatic graph is equal to $\frac{r-2}{r-1}$, see also [7]. While the situation is well-understood in the graph setting, the analogous questions in the hypergraph setting are among the most challenging problems in extremal combinatorics. Erdős [5] offered \$1 000 for determining the Turán density of all complete k-uniform hypergraphs for $k \geq 3$ and \$500 for determining the Turán density of any single complete k-uniform hypergraph (with at least $k+1$ vertices). Even determining the Turán density of $K_4^{(3)}$, the complete 3-uniform hypergraph with 4 vertices, has resisted all attempts for its resolution since its formulation 80 years ago [27]; we refer to [3, 10, 18] for partial and related results, and also to the surveys by Keevash [13] and Sidorenko [26].

Most of the known and conjectured extremal constructions for Turán problems have large independent sets, that is, linear-size sets of vertices with no edges. This led Erdős and Sós [6, 8]

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to propose studying Turán problems with the additional requirement that the edges of the host hypergraph are distributed uniformly. Formally, the *uniform Turán density* of a k-uniform hypergraph H is the infimum over all d such that for every $\varepsilon > 0$, there exists n_0 such that the following holds: every k-uniform hypergraph H_0 with $n \geq n_0$ vertices such that any subset of $n' \geq \varepsilon n$ vertices of H_0 spans at least $d\binom{n'}{k}$ edges contains H as a subgraph. So, unlike in the case of Turán problems, uniform Turán problems require host hypergraphs to possess the required edge density on all linear-size vertex subsets. We remark that the notion is interesting for k-uniform hypergraphs with $k \geq 3$ only, since it can be shown that the uniform Turán density of every graph is equal to 0 (this follows e.g. from [25, Theorem 1]).

Erdős and Sós also asked to determine the uniform Turán density of the two smallest non-trivial 3-uniform hypergraphs: the complete 3-uniform hypergraph $K_4^{(3)}$ with four vertices and the 3-uniform hypergraph $K_4^{(3)-}$, which is the hypergraph $K_4^{(3)}$ with an edge removed. Determining the uniform Turán density of $K_4^{(3)}$ remains a challenging open problem though it is believed that a 35-year-old construction of Rödl [25] showing that the uniform Turán density of $K_4^{(3)}$ is at least $1/2$ is optimal [19]. On a positive note, the uniform Turán density of $K_4^{(3)-}$ was shown to be equal to $1/4$ by Glebov, Volec, and the third author [12] using arguments based on the flag algebra method of Razborov [17] and by Reiher, Rödl, and Schacht [23] using direct combinatorial arguments.

In addition to $K_4^{(3)-}$, there are very restricted families of 3-uniform hypergraphs whose uniform Turán density is known exactly. Reiher, Rödl, and Schacht [21] classified 3-uniform hypergraphs with uniform Turán density equal to 0. The first, third, and fifth authors [11] constructed a family of 3-uniform hypergraphs with uniform Turán density equal to $1/27$. Finally, Bucić, Cooper, Mohr, Munhá Correia, and the third author [1] determined the uniform Turán density of 3-uniform tight cycles of length at least five (the density is equal to $4/27$, if the length is not divisible by three, and it is 0 otherwise). Lastly, the approach from [23] was extended to a broader family of hypergraphs with uniform Turán density equal to $1/4$ [2, 15]. Hence there are only four values known to be the uniform Turán density of some hypergraph, and those are 0, $1/27$, $4/27$, and $1/4$. For further exposition, we refer the reader to the survey by Reiher [19] on uniform Turán densities of hypergraphs, which also includes results from, for example, [20, 22, 24] on stronger notions of uniform density.

Our main result is an easy to verify sufficient condition for a 3-uniform hypergraph H to have uniform Turán density equal to $8/27$. We then identify 3-uniform hypergraphs H that satisfy this condition, and so we add $8/27$ to the list of known uniform Turán densities. Since we mostly deal with 3-uniform hypergraphs, for the rest of the paper we drop the adjective 3-uniform when we talk about 3-uniform hypergraphs except for those places where we wish to emphasize that a considered hypergraph is 3-uniform in the interest of clarity (e.g. in the statements of lemmas and theorems).

To state our condition, we need the notion of a palette, which is inspired by the construction of Rödl from [25]; we also refer the reader to the survey [19] for further details on the notion. A *palette* is a set of ordered triples of colours. For example, the simplest palette is the palette $\Phi_0 = \{(\alpha, \beta, \gamma)\}$, which consists of a single triple of mutually distinct colours denoted by α, β, and γ. More complex examples of palettes are the palettes Φ_3 and Φ_8, which are defined below before the statement of Theorem 3. If Φ is a palette, we say that an n-vertex hypergraph H is Φ-*colourable* if there exists an ordering $v_1, \dots, v_n$ of the vertices of H and a colouring of pairs of the vertices of H with colours appearing in the palette Φ such that if the vertices v_i, v_j and v_k, $1 \leq i < j < k \leq n$, form an edge of H, then the triple (c_{ij}, c_{jk}, c_{ik}) is contained in Φ where c_{ij} is the colour of the pair v_i and v_j, c_{jk} is the colour of the pair v_j and v_k, and c_{ik} is the colour of the pair v_i and v_k.

Having defined the notions of a palette and a Φ-colourable hypergraph, we can state the above mentioned characterization of hypergraphs with the uniform Turán density equal to 0.

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Theorem 1 (Reiher, Rödl, and Schacht [21]). *The uniform Turán density of a* 3*-uniform hypergraph* H *is equal to* 0 *if and only if* H *is* Φ_0*-colourable.*

Palettes also provide a method for lower bound constructions, which we present next; also see the survey [19]. We remark that a recent result of the last author [14], which is discussed in the concluding section, asserts that colourability by palettes fully determines uniform Turán densities. To present the lower bound construction method using palettes, we need an additional definition: the *density* of a palette Φ, which consists of ordered triples of colours, is $|\Phi|/k^3$ where k is the number of colours appearing in the triples of Φ. For example, the density of the palette Φ_0 is $1/27$.

Fix a palette Φ with density d and an integer n. Let k be the number of colours appearing in the triples of Φ. We will construct an n-vertex hypergraph H_n with vertices $v_1, \ldots, v_n$ as follows. Colour each pair of vertices uniformly at random with one of the k colours appearing in the triples of Φ and include an edge formed by vertices v_i, v_j, and v_k, $1 \le i < j < k \le n$, if the triple (c_{ij}, c_{jk}, c_{ik}) is contained in Φ where c_{ij} is the colour of the pair v_i and v_j, c_{jk} is the colour of the pair v_j and v_k, and c_{ik} is the colour of the pair v_i and v_k. It is not hard to show that for every $\varepsilon > 0$ and $\delta > 0$, there exists an integer n such that the following holds with positive probability: every subset of $n' \ge \varepsilon n$ vertices of H_n spans at least $(d - \delta)\binom{n'}{3}$ edges. This construction yields that the uniform Turán density of any hypergraph that is not Φ-colourable must be at least the density of Φ. We state this conclusion as a proposition.

Proposition 2. *Let* Φ *be a palette. If a* 3*-uniform hypergraph* H *is not* Φ*-colourable, then the uniform Turán density of* H *is at least the density of* Φ.

Using Proposition 2 applied to the palette Φ_0, we derive from Theorem 1 that the uniform Turán density of any hypergraph H is either 0 or at least $1/27$ (recall that the construction of hypergraphs with uniform Turán density equal to $1/27$ was given in [11], so this jump value is the best possible).

To state our main result, we need the following two palettes, which we fix for the rest of the paper (the subscripts denote the number of triples contained in the palettes).

$$\Phi_3 = \{(\alpha^1, \beta^1, \omega), (\alpha^2, \omega, \gamma^2), (\omega, \beta^3, \gamma^3)\}$$
$$\Phi_8 = \{(x, y, z) \text{ such that } x \in \{\beta, \gamma\}, y \in \{\alpha, \gamma\} \text{ and } z \in \{\alpha, \beta\}\}$$

Note that the density of the palette Φ_8 is $8/27$.

We are now ready to state our main result, which provides a condition that guarantees that the uniform Turán density of a hypergraph is equal to $8/27$.

Theorem 3. *The uniform Turán density of every* Φ_3*-colourable* 3*-uniform hypergraph that is not* Φ_8*-colourable is* $8/27$.

Since the density of the palette Φ_8 is $8/27$, Theorem 3 follows from Proposition 2 and Theorem 10, which is proven in Section 3. In Section 4, we present a construction of hypergraphs that are Φ_3-colourable but not Φ_8-colourable, which establishes the existence of hypergraphs with uniform Turán density equal to $8/27$.

2. Preliminaries

In this section, we introduce the terminology related to partitioned hypergraphs. This framework, which is presented by Reiher in the survey [19], encapsulates hypergraph regularity arguments while avoiding some of the technicalities of a direct application of the Hypergraph Regularity Lemma.

An *N-partitioned hypergraph* H is a 3-uniform hypergraph such that its vertex set is partitioned to sets V_{ij}, $1 \le i < j \le N$, and for every edge e of H, there exist indices $1 \le i < j < k \le N$ such that one vertex of e is from V_{ij}, one from V_{jk} and one from V_{ik}. An (i, j, k)*-triad* is the set of all edges of

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H that have exactly one vertex in each of the sets V_{ij}, V_{jk} and V_{ik}; note that each edge of H belongs to exactly one triad. If v is a vertex of V_{ij}, we write $d_{ij\to k}(v)$ for the number of edges of the (i,j,k)-triad that contain v divided by $|V_{ik}|\cdot|V_{jk}|$; we use $d_{jk\to i}(v)$ and $d_{ik\to j}(v)$ analogously. Finally, the *density* of an (i,j,k)-triad is the number of edges forming the triad divided by $|V_{ij}|\cdot|V_{ik}|\cdot|V_{jk}|$, and the *density* of an N-partitioned hypergraph H is the minimum density of a triad of H.

An N-partitioned hypergraph H_0 with parts V_{ij}, $1\le i<j\le N$, *embeds* an n-vertex hypergraph H with vertices $v_1,\ldots,v_n$ if it is possible to choose distinct $1\le a_1,\ldots,a_n\le N$ corresponding to the vertices of H and vertices $w_{ij}\in V_{a_ia_j}$ for $1\le i<j\le n$ such that if vertices v_i, v_j, and v_k of H form an edge, then $\{w_{ij},w_{jk},w_{ik}\}$ is an edge in the (a_i,a_j,a_k)-triad of H_0. Reiher [19] provided a general statement that reduces proving an upper bound on the uniform Turán density of a hypergraph H to embedding H in partitioned hypergraphs.

Theorem 4 (Reiher [19, Theorem 3.3]). *Let H be a 3-uniform hypergraph and $d\in[0,1]$. Suppose that for every $\delta>0$ there exists N such that every N-partitioned hypergraph with density at least $d+\delta$ embeds H. Then, the uniform Turán density of H is at most d.*

We remark that an extension of this general statement yields that the uniform Turán density of a hypergraph H and any of its blowups are the same.

We next state four lemmas given in [1, Section 4], which we cast using our terminology; the lemmas were implicitly proven in [21] using a direct iterative approach, and alternative proofs based on Ramsey's Theorem can be found in [11]. The lemmas are instances of the following metastatement: if a sufficiently large set $I_0\subseteq[N]$ is given and every triad of an N-partitioned hypergraph H with indices from I_0 has a linear-size set of 'good' vertices (for instance such a property might be having a high degree in the triad), then there is a large set of indices $I\subseteq I_0$ such that there is a universal choice of vertices that are good with respect to all triads with indices from I.

The first of the lemmas, which we prove for illustrative purposes, is the instance of the metastatement in the case when good vertices come from the set V_{ik} of a (i,j,k)-triad.

Lemma 5 (Bucić, Cooper, Král', Mohr, and Munhá Correia [1, Lemma 4.3]). *For every $\varepsilon>0$ and n, there exists n_0 such that the following holds for every N-partitioned hypergraph H with $N\ge n_0$, every $I_0\subseteq\{1,\ldots,N\}$ with $|I_0|\ge n_0$, and every choice of subsets $W_{ijk}\subseteq V_{ik}$, $i<j<k$, $i,j,k\in I_0$ such that $|W_{ijk}|\ge\varepsilon|V_{ik}|$: there exist $I\subseteq I_0$ with $|I|\ge n$ and γ_{ik}, $i<k$, $i,k\in I$, such that $\gamma_{ik}\in W_{ijk}$ for all $j\in I$ such that $i<j<k$.*

Proof. Consider $\varepsilon>0$ and n, and let n_0 be the Ramsey number such that every 2-edge-coloured n-uniform complete hypergraph with n_0 vertices contains a monochromatic complete hypergraph with $K=\max\{n^2-n+1,\lceil n/\varepsilon\rceil+2\}$ vertices. Fix an N-partitioned hypergraph H with $N\ge n_0$, the set $I_0\subseteq\{1,\ldots,N\}$ such that $|I_0|\ge n_0$, and the subsets $W_{ijk}\subseteq V_{ik}$, $i<j<k$, $i,j,k\in I_0$ such that $|W_{ijk}|\ge\varepsilon|V_{ik}|$.

We now construct an auxiliary 2-edge-coloured n-uniform complete hypergraph with vertex set I_0 such that the edge formed by indices $i_1<\cdots<i_n$, $i_1,\ldots,i_n\in I_0$, is coloured blue if the sets W_{i_1,i_k,i_n}, $k=2,\ldots,n-1$, have a non-empty intersection, and it is coloured red otherwise, that is, the intersection of the sets W_{i_1,i_k,i_n}, $k=2,\ldots,n-1$, is the empty set. By the choice of n_0, there exists a K-element set $J\subseteq I_0$ such that all edges formed by the elements of J have the same colour. Let $i_1<\cdots<i_K$ be the elements of J.

First suppose that the common colour of the edges formed by the elements of J is red. By a simple averaging argument, the set V_{i_1,i_K} contains a vertex that is contained in at least $\frac{\varepsilon(K-2)|V_{i_1,i_K}|}{|V_{i_1,i_K}|}=\varepsilon(K-2)\ge n$ sets W_{i_1,i_k,i_K}, $k\in\{2,\ldots,K-1\}$; let x be any such vertex of V_{i_1,i_K}. However, any $n-2$ indices i_k, $k\in\{2,\ldots,K-1\}$, such that $x\in W_{i_1,i_k,i_K}$ together with i_1 and i_K form a blue edge. Hence, the common colour of the edges formed by the elements of J cannot be red.

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We next show that the set $I = \{i_1, i_{n+1}, i_{2n+1}, \dots, i_{(n-1)n+1}\}$ has the property from the statement of the lemma. To do so, we need to find elements γ_{i_a,i_b} for all $a, b \in \{1, n+1, \dots, (n-1)n+1\}$ such that $a < b$. Fix such i_a and i_b. Since the edge of the auxiliary complete hypergraph formed by the $(b-a)/n + 1 \le n$ elements $i_a, i_{a+n}, \dots, i_b$ and any $n - (b-a)/n - 1$ elements among $a_{a+1}, \dots, a_{a+n-1}$ is blue, the sets W_{i_a,i_k,i_b}, $k = a+n, \dots, b-n$, have a non-empty intersection, and we set γ_{i_a,i_b} to be any element contained in their intersection. □

The next two lemmas are the instance of the metastatement in the cases when good vertices come from the set V_{ij} and the set V_{jk} of a (i,j,k)-triad, respectively.

Lemma 6 (Bucić, Cooper, Král', Mohr, and Munhá Correia [1, Lemma 4.4]). *For every $\varepsilon > 0$ and n, there exists n_0 such that the following holds for every N-partitioned hypergraph H with $N \ge n_0$, every $I_0 \subseteq \{1, \dots, N\}$ with $|I_0| \ge n_0$, and every choice of subsets $W_{ijk} \subseteq V_{ij}$, $i < j < k$, $i, j, k \in I_0$ such that $|W_{ijk}| \ge \varepsilon |V_{ij}|$: there exist $I \subseteq I_0$ with $|I| \ge n$ and α_{ij}, $i < j$, $i, j \in I$, such that $\alpha_{ij} \in W_{ijk}$ for all $k \in I$ such that $k > j$.*

Lemma 7 (Bucić, Cooper, Král', Mohr, and Munhá Correia [1, Lemma 4.5]). *For every $\varepsilon > 0$ and n, there exists n_0 such that the following holds for every N-partitioned hypergraph H with $N \ge n_0$, every $I_0 \subseteq \{1, \dots, N\}$ with $|I_0| \ge n_0$, and every choice of subsets $W_{ijk} \subseteq V_{jk}$, $i < j < k$, $i, j, k \in I_0$ such that $|W_{ijk}| \ge \varepsilon |V_{jk}|$: there exist $I \subseteq I_0$ with $|I| \ge n$ and β_{jk}, $j < k$, $j, k \in I$, such that $\beta_{jk} \in W_{ijk}$ for all $i \in I$ such that $i < j$.*

The last lemma is an extension to the setting when good vertices come from the set V_{ij} of a (a,i,j)-triad, a (i,b,j)-triad and (i,j,c)-triad, that is, which vertices are good depends on all three triads.

We conclude this section with another lemma proven in [1, Section 4].

Lemma 8 (Bucić, Cooper, Král', Mohr, and Munhá Correia [1, Lemma 4.2]). *For every $\varepsilon > 0$ and n, there exists n_0 such that the following holds for every N-partitioned hypergraph H with $N \ge n_0$, every $I_0 \subseteq \{1, \dots, N\}$ with $|I_0| \ge n_0$, and every choice of subsets $W_{ij}^{abc} \subseteq V_{ij}$ where $a, b, c, i, j \in I_0$ and $a < i < b < j < c$ such that $|W_{ij}^{abc}| \ge \varepsilon |V_{ij}|$: there exist $I \subseteq I_0$ with $|I| \ge n$ and ω_{ij}, $i < j$, $i, j \in I$, such that $\omega_{ij} \in W_{ij}^{abc}$ for all $a, b, c \in I$ such that $a < i < b < j < c$.*

3. Embedding

This section is devoted to showing that every sufficiently large hypergraph with uniform density larger than $8/27$ contains any fixed Φ_3-colourable hypergraph H as a subhypergraph. We start with the following simple lemma, which can be found as [11, Lemma 6]; we include a short proof for completeness. Recall that $d_{ij\to k}(v)$ is the number of edges of a (i,j,k)-triad that contain v divided by $|V_{ik}| \cdot |V_{jk}|$, and $d_{jk\to i}(v)$ and $d_{ik\to j}(v)$ are used analogously.

Lemma 9. *For every $\varepsilon > 0$ and n, there exists n_0 such that the following holds for every N-partitioned hypergraph H with $N \ge n_0$: there exist $I \subseteq \{1, \dots, N\}$ with $|I| \ge n$ and reals a, b and c such that the following holds for all $i, j, k \in I$ such that $i < j < k$:*

$$a \le \frac{|\{v \in V_{ij} \text{ with } d_{ij\to k}(v) \ge \varepsilon\}|}{|V_{ij}|} < a + \varepsilon,$$

$$b \le \frac{|\{v \in V_{jk} \text{ with } d_{jk\to i}(v) \ge \varepsilon\}|}{|V_{jk}|} < b + \varepsilon, \text{ and}$$

$$c \le \frac{|\{v \in V_{ik} \text{ with } d_{ik\to j}(v) \ge \varepsilon\}|}{|V_{ik}|} < c + \varepsilon.$$

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Proof. Fix $\varepsilon \in (0,1)$ and n, and let $K = \lfloor \varepsilon^{-1} + 1 \rfloor^3$. Let n_0 be such that every K-edge-coloured complete 3-uniform hypergraph with n_0 vertices contains an n-vertex monochromatic complete hypergraph; such n_0 exists by Ramsey's Theorem. Let H be an N-partitioned hypergraph with $N \geq n_0$, and set

$$s_{ij\to k} = \frac{|\{v \in V_{ij} \text{ with } d_{ij\to k}(v) \geq \varepsilon\}|}{|V_{ij}|},$$
$$s_{jk\to i} = \frac{|\{v \in V_{jk} \text{ with } d_{jk\to i}(v) \geq \varepsilon\}|}{|V_{jk}|}, \text{ and}$$
$$s_{ik\to j} = \frac{|\{v \in V_{ik} \text{ with } d_{ik\to j}(v) \geq \varepsilon\}|}{|V_{ik}|}.$$

We next construct an auxiliary K-edge-coloured complete 3-uniform hypergraph H' with N vertices: the colour of the edge formed by the i-th, j-th and k-th vertex of H' is the triple $\big(\lfloor s_{ij\to k}/\varepsilon \rfloor, \lfloor s_{jk\to i}/\varepsilon \rfloor, \lfloor s_{ik\to j}/\varepsilon \rfloor\big)$. By the choice of N, there exist a subset $I \subseteq \{1,\dots,N\}$ with $|I| = n$ and a triple (a,b,c) such that all edges formed by the i-th, j-th and k-th vertex of H' have the colour (a,b,c) whenever $i,j,k \in I$. Since the set I and the reals $a\cdot\varepsilon$, $b\cdot\varepsilon$, and $c\cdot\varepsilon$ have the properties given in the statement of the lemma, the proof of the lemma is finished. □

We are now ready to prove the main result of this section.

Theorem 10. *Let H be a 3-uniform hypergraph. If H is Φ_3-colourable, then the uniform Turán density of H is at most $8/27$.*

Proof. Fix a Φ_3-colourable hypergraph H and let n be the number of vertices of H. By Theorem 4, it is enough to show that for every $\delta > 0$, there exists N_0 such that every N_0-partitioned hypergraph with density at least $8/27 + \delta$ embeds H. Fix $\delta \in (0,1)$. We next choose values $N_0,\dots,N_8$ such that $N_0 \gg N_1 \gg \cdots \gg N_7 \gg N_8$. To do so, set $\varepsilon = \delta/20$, $N_8 = n$, and the values of $N_0,\dots,N_7$ as follows.

- N_7 is the value of n_0 from Lemma 5 applied with ε and N_8.
- N_6 is the value of n_0 from Lemma 7 applied with ε and N_7.
- N_5 is the value of n_0 from Lemma 5 applied with ε and N_6.
- N_4 is the value of n_0 from Lemma 6 applied with ε and N_5.
- N_3 is the value of n_0 from Lemma 7 applied with ε and N_4.
- N_2 is the value of n_0 from Lemma 6 applied with ε and N_3.
- N_1 is the value of n_0 from Lemma 8 applied with ε and N_2.
- N_0 is the value of n_0 from Lemma 9 applied with 2ε and N_1.

For this value of N_0, we will show that every N_0-partitioned hypergraph with density at least $8/27 + \delta$ embeds H.

Fix an N_0-partitioned hypergraph H_0 with density at least $8/27 + \delta$. By Lemma 9, there exist $I_1 \subseteq \{1,\dots,N_0\}$ with $|I_1| \geq N_1$ and reals a, b, and c such that

$$a \leq \frac{|\{v \in V_{ij} \text{ with } d_{ij\to k}(v) \geq 2\varepsilon\}|}{|V_{ij}|} < a + 2\varepsilon,$$
$$b \leq \frac{|\{v \in V_{jk} \text{ with } d_{jk\to i}(v) \geq 2\varepsilon\}|}{|V_{jk}|} < b + 2\varepsilon, \text{ and}$$
$$c \leq \frac{|\{v \in V_{ik} \text{ with } d_{ik\to j}(v) \geq 2\varepsilon\}|}{|V_{ik}|} < c + 2\varepsilon,$$

for all $i,j,k \in I_1$ and $i < j < k$.

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We next show that $a+b+c$ is at least $2+3\varepsilon$, which we later show to imply that for every choice of $k<i<k'<j<k''$ from I_1 the set V_{ij} contains linearly many vertices v such that $d_{ij\to k''}(v)\geq 2\varepsilon$, $d_{ij\to k}(v)\geq 2\varepsilon$, and $d_{ij\to k'}(v)\geq 2\varepsilon$. Consider any of the triads, say the $(1,2,3)$-triad. Let A be the set of all vertices $v\in V_{12}$ with $d_{12\to 3}(v)\geq 2\varepsilon$, B the set of all vertices $v\in V_{23}$ with $d_{23\to 1}(v)\geq 2\varepsilon$, and C the set of all vertices $v\in V_{13}$ with $d_{13\to 2}(v)\geq 2\varepsilon$. Since each vertex of $V_{12}\setminus A$ is in at most $2\varepsilon\cdot|V_{23}|\cdot|V_{13}|$ edges of the $(1,2,3)$-triad, each vertex of $V_{23}\setminus B$ is in at most $2\varepsilon\cdot|V_{12}|\cdot|V_{13}|$ edges of the $(1,2,3)$-triad, and each vertex of $V_{13}\setminus C$ is in at most $2\varepsilon\cdot|V_{12}|\cdot|V_{23}|$ edges of the $(1,2,3)$-triad, the $(1,2,3)$-triad has at most

$$|A|\cdot|B|\cdot|C|+6\varepsilon\cdot|V_{12}|\cdot|V_{13}|\cdot|V_{23}|\leq\big(abc+12\varepsilon\big)\cdot|V_{12}|\cdot|V_{13}|\cdot|V_{23}|$$

edges. Since the density of the N_0-partitioned hypergraph H_0 is at least $8/27+\delta$, if follows that $abc+12\varepsilon$ is at least $8/27+\delta=8/27+20\varepsilon$ and so $abc\geq 8/27+8\varepsilon$. Using the Inequality of Arithmetic and Geometric Means, we obtain that

$$a+b+c\geq 3\sqrt[3]{abc}\geq 3\sqrt[3]{8/27+8\varepsilon}\geq 3(2/3+\varepsilon)=2+3\varepsilon.$$

Our next step is to apply Lemma 8 to identify vertices ω_{ij} such that $d_{ij\to k''}(\omega_{ij})\geq 2\varepsilon$, $d_{ij\to k}(\omega_{ij})\geq 2\varepsilon$, and $d_{ij\to k'}(\omega_{ij})\geq 2\varepsilon$ for every choice of $k<i<k'<j<k''$ from a suitable index set I_2; the vertices ω_{ij} will correspond to the colour ω from the palette Φ_3. To apply the lemma, we define sets $W_{ij}^{kk'k''}\subseteq V_{ij}$ for $i,j,k,k',k''\in I_1$ such that $k<i<k'<j<k''$. Consider such $k<i<k'<j<k''$. Let A be the set of all $v\in V_{ij}$ with $d_{ij\to k''}(v)\geq 2\varepsilon$, B the set of all vertices $v\in V_{ij}$ with $d_{ij\to k}(v)\geq 2\varepsilon$, and C the set of all vertices $v\in V_{ij}$ with $d_{ij\to k'}(v)\geq 2\varepsilon$. Let m_d, $d\in\{0,1,2,3\}$, be the number of vertices of V_{ij} contained in exactly d sets among A, B, and C. Observe that

$$(a+b+c)|V_{ij}|\leq|A|+|B|+|C|=m_1+2m_2+3m_3\leq 2(m_1+m_2)+3m_3\leq 2|V_{ij}|+3m_3,$$

which implies that there are at least $\frac{a+b+c-2}{3}|V_{ij}|\geq\varepsilon|V_{ij}|$ vertices of V_{ij} contained in $A\cap B\cap C$. We set $W_{ij}^{kk'k''}=A\cap B\cap C$. Lemma 8 implies that there exist $I_2\subseteq I_1$ with $|I_2|\geq N_2$ and $\omega_{ij}\in V_{ij}$, $i,j\in I_2$, and $i<j$, such that $\omega_{ij}\in W_{ij}^{kk'k''}$ for all $i,j,k,k',k''\in I_2$ such that $k<i<k'<j<k''$. In particular, it holds that $d_{ij\to k}(\omega_{ij})\geq 2\varepsilon$ for all $i,j\in I_2$, $i<j$, and $k\in I_2\setminus\{i,j\}$ (regardless whether $k<i<j$, $i<k<j$, or $i<j<k$).

Our next goal is to identify vertices that will correspond to the colours α^1 and β^1 of the palette Φ_3. To identify those corresponding to the colour α^1, we will apply Lemma 6. Consider $i,j,k\in I_2$ such that $i<j<k$ and let W_{ijk} be the set of all vertices $v\in V_{ij}$ contained together with ω_{ik} in at least $\varepsilon|V_{jk}|$ edges of the (i,j,k)-triad. Since the number of edges containing ω_{ik} in the triad (i,j,k)-triad is at most

$$|W_{ijk}|\cdot|V_{jk}|+\varepsilon|V_{ij}\setminus W_{ijk}|\cdot|V_{jk}|\leq|W_{ijk}|\cdot|V_{jk}|+\varepsilon|V_{ij}|\cdot|V_{jk}|$$

and $d_{ik\to j}(\omega_{ik})\geq 2\varepsilon$, we obtain that W_{ijk} contains at least $\varepsilon|V_{ij}|$ vertices. Lemma 6 yields that there exist $I_3\subseteq I_2$ with $|I_3|\geq N_3$ and $\alpha_{ij}^1\in V_{ij}$, $i,j\in I_3$, and $i<j$, such that α_{ij}^1 and ω_{ik} are contained together in at least $\varepsilon|V_{jk}|$ edges of the (i,j,k)-triad for all $i,j,k\in I_3$ such that $i<j<k$.

We next apply Lemma 7 to identify the vertices corresponding to the colour β^1. For $i,j,k\in I_3$ such that $i<j<k$, we set W_{ijk} to be the set of all vertices $v\in V_{jk}$ that form together with α_{ij}^1 and ω_{ik} an edge of the (i,j,k)-triad; note that W_{ijk} contains at least $\varepsilon|V_{jk}|$ vertices. Hence, Lemma 7 implies that there exist $I_4\subseteq I_3$ with $|I_4|\geq N_4$ and $\beta_{jk}^1\in V_{jk}$, $j,k\in I_4$, and $j<k$, such that α_{ij}^1, β_{jk}^1, and ω_{ik} form an edge of the (i,j,k)-triad for all $i,j,k\in I_4$ such that $i<j<k$. In particular, the vertices α_{ij}^1, β_{jk}^1, and ω_{ik} of the (i,j,k)-triad indeed correspond to the colours α^1, β^1, and ω of the palette Φ_3.

In the completely analogous way, we apply Lemmas 6 and 5 (in this order) to obtain $I_6\subseteq I_4$ with $|I_6|\geq N_6$, α_{ij}^2, $i,j\in I_6$ and $i<j$, and γ_{ik}^2, $i,k\in I_6$, and $i<k$, such that α_{ij}^2, ω_{jk}, and γ_{ik}^2 form an edge of the (i,j,k)-triad for all $i,j,k\in I_6$ such that $i<j<k$, and so the vertices α_{ij}^2 and β_{ij}^2 correspond

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to the colours α^2 and β^2 of the palette Φ_3. Then, we apply Lemmas 7 and 5 to obtain $I_8 \subseteq I_6$ with $|I_8| \geq N_8$, β^3_{jk}, $j, k \in I_6$, and $j < k$, and γ^3_{ik}, $i, k \in I_6$ and $i < k$, such that ω_{ij}, β^3_{jk}, and γ^3_{ik} form an edge of the (i, j, k)-triad for all $i, j, k \in I_8$ such that $i < j < k$; again, the vertices α^3_{ij} and β^3_{ij} correspond to the colours α^3 and β^3 of the palette Φ_3.

We now argue that the N_0-partitioned hypergraph H_0 embeds the hypergraph H. By the assumption of the theorem, the hypergraph H is Φ_3-colourable. Let $v_1, \ldots, v_n$ be the vertices of H listed in the order such that there exists a choice of $c_{ij} \in \{\omega, \alpha^1, \beta^1, \alpha^2, \gamma^2, \beta^3, \gamma^3\}$, $i, j \in \{1, \ldots, n\}$ such that if vertices v_i, v_j, and v_k, $1 \leq i < j < k \leq n$, form an edge of H, then $(c_{ij}, c_{jk}, c_{ik}) \in \Phi_3$. Let a_i, $i \in \{1, \ldots, n\}$, be the i-th smallest index contained in I_8 (note that I_8 has at least $N_8 = n$ elements). For $1 \leq i < j \leq n$, set w_{ij} among the vertices of V_{ij} as follows:

$$w_{ij} = \begin{cases} \omega_{a_i a_j} & \text{if } c_{ij} = \omega, \\ \alpha^1_{a_i a_j} & \text{if } c_{ij} = \alpha^1, \\ \beta^1_{a_i a_j} & \text{if } c_{ij} = \beta^1, \\ \alpha^2_{a_i a_j} & \text{if } c_{ij} = \alpha^2, \\ \gamma^2_{a_i a_j} & \text{if } c_{ij} = \gamma^2, \\ \beta^3_{a_i a_j} & \text{if } c_{ij} = \beta^3, \text{ and} \\ \gamma^3_{a_i a_j} & \text{if } c_{ij} = \gamma^3. \end{cases}$$

The choice of the vertices ω_{ij}, α^1_{ij}, β^1_{ij}, α^2_{ij}, γ^2_{ij}, β^3_{ij}, and γ^3_{ij}, $i, j \in I_8$ and $i < j$, yields that if vertices v_i, v_j, and v_k, $1 \leq i < j < k \leq n$, form an edge of H, then the vertices w_{ij}, w_{jk}, and w_{ik} form an edge of the (a_i, a_j, a_k)-triad. We conclude that the N_0-partitioned hypergraph H_0 embeds H and the proof of theorem is finished. □

4. Construction

In this section, we identify examples of 3-uniform hypergraphs that are Φ_3-colourable but not Φ_8-colourable. The key step in the construction is the following proposition. Recall that a hypergraph is *linear* if any two edges have at most one vertex in common.

Proposition 11. *Suppose that there exists an n-vertex m-edge 5-uniform linear hypergraph such that $n! < (10/9)^m$. Then, there exists an n-vertex $(3m)$-edge 3-uniform hypergraph that is Φ_3-colourable but not Φ_8-colourable.*

Let us first discuss which hypergraphs satisfy the assumption of Proposition 11. Observe that any edge-maximal 5-uniform linear hypergraph with n vertices has at least

$$\frac{\binom{n}{5}}{10\binom{n-2}{3}} = \frac{n(n-1)}{200} = \Omega(n^2)$$

edges. It follows that if n is sufficiently large, then any edge-maximal 5-uniform linear hypergraph with n vertices satisfies the assumption of Proposition 11. In particular, there exist constants $c < C$ such that for any n and m with $Cn \log n < m < cn^2$, there exists an n-vertex m-edge 5-uniform linear hypergraph satisfying the assumption of Proposition 11.

An explicit small example of a 5-uniform hypergraph satisfying the assumption can be obtained from the 5-dimensional affine space $\mathbb{F}_5^5$ of order 5. Recall that the vertices of $\mathbb{F}_5^5$ are 5-dimensional vectors over the 5-element field $\mathbb{F}_5$ and any five-element set $\{a + x \cdot b,\ x \in \mathbb{F}_5\}$ where $a, b \in \mathbb{F}_5^5$ and $b \neq (0, 0, 0, 0, 0)$ is a line; note that each line corresponds to $5 \cdot 4 = 20$ distinct choices of a and b and every pair of points is contained together in exactly one line. The vertex set of the

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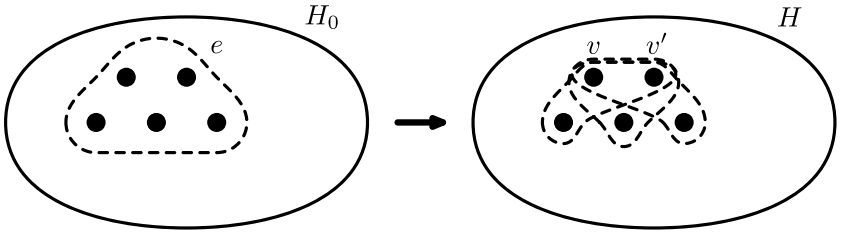


Figure 1. The illustration of the construction of the 3-uniform hypergraph H from the 5-uniform hypergraph H_0.

sought 5-uniform hypergraph is formed by the $n = 5^5 = 3\,125$ points of the space and each of the $m = \frac{n(n-1)}{20} = \frac{1}{10}\binom{n}{2} = 488\,125$ lines of the space forms an edge. It is easy to check that $n! < (10/9)^m$.

We now prove Proposition 11.

Proof of Proposition 11. Fix an n-vertex m-edge 5-uniform linear hypergraph H_0 such that $n! < (10/9)^m$. We construct the n-vertex $(3m)$-edge 3-uniform hypergraph H in a random way. The vertex set of H is the same as that of H_0. In each edge e of H_0 choose two vertices v and v' randomly (independently of the other edges) and include to H as edges the three triples containing v, v' and one of the three remaining vertices of e; see Fig. 1 for an illustration of the construction. We now verify that the hypergraph H is Φ_3-colourable and that it is not Φ_8-colourable with positive probability.

We start by verifying that the hypergraph H is Φ_3-colourable. Fix any order $v_1, \dots, v_n$ of the vertices of H (and so of H_0). If a pair of vertices v_i and v_j of H_0 is not contained in an edge of H_0, choose c_{ij} arbitrarily. Otherwise, consider any edge e of H_0, and let v_i and v_j, $i < j$, be the two vertices chosen in the construction of H. Set $c_{ij} = \omega$. For each of the three remaining vertices of e, say v_k, set $c_{ik} = \alpha^1$ and $c_{kj} = \beta^1$ if $i < k < j$, set $c_{ki} = \alpha^2$ and $c_{kj} = \gamma^2$ if $k < i$, and set $c_{ik} = \beta^3$ and $c_{jk} = \gamma^3$ if $k > j$. Since the hypergraph H is linear, the values of c_{ij}, $1 \le i < j \le n$, are well-defined. It is straightforward to verify that if $\{v_i, v_j, v_k\}$, $1 \le i < j < k \le n$, is an edge of H, then $(c_{ij}, c_{jk}, c_{ik}) \in \Phi_3$. Hence, the hypergraph H is Φ_3-colourable.

We next show that the hypergraph H is not Φ_8-colourable with positive probability. Fix any order $v_1, \dots, v_n$ of the vertices of H. Since the 5-uniform hypergraph H_0 is linear, the hypergraph H is not Φ_3-colourable with respect to the fixed order $v_1, \dots, v_n$ if and only if the 5-uniform hypergraph H_0 has an edge $\{v_a, v_b, v_c, v_d, v_e\}$, $1 \le a < b < c < d < e \le n$, such that the two chosen vertices in the construction of H are the vertices v_b and v_d. Indeed, if H_0 has such an edge, then there is no choice of a colour c_{bd} such that $(c_{ab}, c_{bd}, c_{aad}) \in \Phi_8$, $(c_{bc}, c_{cd}, c_{bd}) \in \Phi_8$, and $(c_{bd}, c_{de}, c_{be}) \in \Phi_8$. And if H_0 has no such edge, we can choose the colours of the pairs of the vertices v_a, v_b, v_c, v_d, v_e within each edge of H_0 and this choice does not affect the choices within other edges of H_0 as the hypergraph H_0 is linear. Hence, for any fixed order of the vertices $v_1, \dots, v_n$, the probability that there exists an assignment of colours to the pairs of vertices such that $(c_{ij}, c_{jk}, c_{ik}) \in \Phi_8$ for every edge $\{v_i, v_j, v_k\}$, $1 \le i < j < k \le n$, of H_0 is equal to $(9/10)^m$. The union bound implies that the probability that there exists a vertex ordering that admits such an assignment of colours to the pairs of vertices is at most $n! \cdot (9/10)^m < 1$, that is, the hypergraph H is not Φ_8-colourable with probability at least $1 - n! \cdot (9/10)^m > 0$. □

5. Conclusion

We are aware of several follow-up results concerning the uniform Turán densities of 3-uniform hypergraphs, and we would like to briefly mention the one that we believe to be the most surprising. Proposition 2 asserts that if a hypergraph H is not Φ-colourable for a palette Φ with density d, then the uniform Turán density of H is at least d. Since all known extremal constructions for uniform Turán densities are based on palette colouring constructions, see [19], it is natural to ask whether this is a general phenomenon. In the subsequent work [14], the fifth author shows that for

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every $\delta > 0$ every sufficiently large partitioned hypergraph with density d contains a partitioned subhypergraph with density $d - \delta$ that can be associated with a palette colouring construction. It follows that *the uniform Turán density of any* 3*-uniform hypergraph* H *is equal to the supremum over all* d *such that* H *is not* Φ*-colourable for some palette* Φ *with density* d. This result brings a powerful tool for determining the uniform Turán densities of hypergraphs; for example, it was shown that every tight cycle of length at least five is Φ-colourable for every palette Φ with density larger than 4/27 by Cooper in 2018 [4], but the uniform Turán density of tight cycles was determined only 5 years later [1].

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