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A note on variational theorems in non-linear elastostatics

By R. W. OGDEN
University of Bath

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1. *Introduction.* In a recent paper Koiter (5) discussed a principle of stationary complementary energy for the finite deformation of elastic materials. The complementary energy functional he uses depends only on the components of the *nominal* stress, and not on the displacement field. He, incorrectly, attributes the principle to Zubov (14).

The principle was in fact stated by Levinson (6) and (in part) earlier by Hill ((2), pp. 68–9). In 1953, however, Reissner (8) constructed a stationary principle involving independent variations in the components of nominal stress and displacement. This clearly includes the Hill–Levinson principle as a particular case when the displacement variations are set equal to zero and self-equilibrated stress fields are considered. Further discussion of variational principles in the context of finite elastic deformations can be found in the book by Washizu ((13), Chapter 5) and the recent paper by Nemat-Nasser (7), who also considers discontinuities.

Fraeijs de Veubeke (1) introduced a variant of Reissner's principle in which he considers independent variations in the rotational part of the deformation. In essence, it differs from the principles mentioned above in that it involves the rotational balance equations. Unfortunately, the paper is difficult to read.

The present note is written firstly with the intention of clarifying the above work by the use of more appropriate functionals which display a 'symmetry' between deformation gradient and nominal stress, and, secondly, to highlight the interconnexions of the various principles for finite elastic deformations. For a more general discussion of complementary variational principles the reader is referred to a paper by Sewell (9).

2. *Finite elasticity.* We consider an elastic body occupying volume V in its undeformed configuration in which a typical material point has position $\mathbf{X}$ with coordinates X_μ, $\mu = 1, 2, 3$. The deformation is defined by the transformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ or $x_i = x_i(X_\mu)$, $i = 1, 2, 3$. Cartesian coordinates are used throughout. In symbolic notation the deformation gradient is written $\boldsymbol{\alpha}$, having components

$$\alpha_{i\mu} = x_{i,\mu} \equiv \partial x_i / \partial X_\mu.$$

Suppose that the material possesses a strain–energy function $W(\boldsymbol{\alpha})$ per unit volume in V, and let V be bounded by surface Σ.

The nominal stress $\mathbf{s}$ is defined by

$$\mathbf{s} = \frac{\partial W}{\partial \boldsymbol{\alpha}}, \quad s_{\mu i} = \frac{\partial W}{\partial \alpha_{i\mu}}, \tag{1}$$

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and the equilibrium equations are

$$s_{\mu i,\,\mu} = 0, \tag{2}$$

in the absence of body forces. It is a simple matter to incorporate conservative body forces in the ensuing analysis if required.

Suppose that Σ consists of two disjoint parts Σ_x and Σ_s on which the following conditions are prescribed:

$$\mathbf{x} = \mathbf{x}_0 \quad \text{on} \quad \Sigma_x, \tag{3}$$

$$\mathbf{t} = \mathbf{t}_0 \quad \text{on} \quad \Sigma_s, \tag{4}$$

where $\mathbf{x}_0$ and $\mathbf{t}_0$ are given functions. The vector $\mathbf{t}$ is the surface traction per unit area of Σ, and given by

$$t_i = n_\mu s_{\mu i} \quad (i = 1, 2, 3), \tag{5}$$

$\mathbf{n}$ being the unit normal to Σ.

Equations (1) to (5) define a boundary-value problem for any given strain–energy function W. The complementary energy W_c is defined by

$$W + W_c = \mathbf{s}\boldsymbol{\alpha}, \tag{6}$$

where $\mathbf{s}\boldsymbol{\alpha}$ denotes the inner product $s_{\mu i}\alpha_{i\mu}$. A similar juxtaposition is used to denote the scalar product of two vectors. It is required that W satisfies the principle of frame indifference, i.e. that W is indifferent to superimposed rotations of the body in its deformed configuration, and so W must depend on $\boldsymbol{\alpha}$ only through the combination $\alpha_{i\mu}\alpha_{i\lambda}$, or some alternative frame-indifferent deformation measure. Correspondingly, W_c depends on $\mathbf{s}$ through $s_{\mu i}s_{\lambda i}$ or can be regarded as a function of some frame-indifferent stress measure.

When W_c exists we can write

$$\boldsymbol{\alpha} = \frac{\partial W_c}{\partial \mathbf{s}}, \tag{7}$$

and exploit the advantages of the $(\mathbf{s}, \boldsymbol{\alpha})$ variables as in (5).

The existence of W_c requires a one–one relationship between $\mathbf{s}$ and $\boldsymbol{\alpha}$ (as pointed out by Hill ((2), p. 67)). A necessary and sufficient condition for this is that the determinant of the Jacobian matrix $\partial\mathbf{s}/\partial\boldsymbol{\alpha}$ is non-zero. Suppose there is some domain $\mathscr{D}$ in the space of invertible second-order tensors $\boldsymbol{\alpha}$ where $\mathbf{s}(\boldsymbol{\alpha})$ is invertible, and let $\mathscr{D}_c$ be the image space of $\mathscr{D}$. The function $W_c(\mathbf{s})$ therefore exists in $\mathscr{D}_c$, a unique $\boldsymbol{\alpha}$ in $\mathscr{D}$ being given by (7) except in the trivial case $\mathbf{s} = \mathbf{0}$. There may be domains outside $\mathscr{D}$ in each of which W_c exists, although the corresponding $\mathbf{s}$-domains will probably overlap $\mathscr{D}_c$.

Now the domain $\mathscr{D}$ must be bounded in view of the actual behaviour of elastic materials. Indeed, it can be shown (Hill (3))† that a primary eigenmode where $\det(\partial\mathbf{s}/\partial\boldsymbol{\alpha}) = 0$ is possible under dead loading, local inversion is no longer valid, and W_c is not then definable. The boundary of $\mathscr{D}$ defines the stability limit for the material and therefore $\partial\mathbf{s}/\partial\boldsymbol{\alpha}$ is positive definite within $\mathscr{D}$, as is $\partial^2 W_c/\partial\mathbf{s}^2$.

Truesdell and Noll ((11), p. 328, footnote 1) state that the inversion $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}(\mathbf{s})$ (and hence the existence of W_c) is in contradiction to the requirements of frame indifference.

† I am grateful to the referee for bringing this reference to my attention.

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This is not so in $\mathscr{D}$ unless $\mathbf{s} = \mathbf{0}$. In any case this problem can be overcome by a change to frame indifferent variables, as in section 5 of this paper, for example.

The validity of the complementary variational theorem discussed in section 4 is therefore restricted to situations where a unique W_c exists (not necessarily just in $\mathscr{D}_c$, since separate branches of the inversion may be considered).

From equations (1) to (7) the important identity

$$\int_V W(\boldsymbol{\alpha})\,dV - \int_{\Sigma_s} \mathbf{x}\mathbf{t}_0\,d\Sigma = \int_{\Sigma_x} \mathbf{t}\mathbf{x}_0\,d\Sigma - \int_V W_c(\mathbf{s})\,dV \tag{8}$$

is easily established for any solution pair $(\mathbf{x}, \mathbf{s})$ of the boundary-value problem. Note, in particular, that the left-hand side of (8) depends only on $\mathbf{x}$ and its derivatives while the right-hand side depends only on $\mathbf{s}$ for any given form of W or W_c. Thus, the left-hand side of (8) can be defined for any differentiable function $\mathbf{x}$ satisfying (3) whether or not it be associated with an $\mathbf{s}$ satisfying (1), (2), (4) and (5). On the other hand, the right side of (8) may be defined for any self-equilibrated nominal 'stress' $\mathbf{s}$ satisfying (4) irrespective of it being associated with an $\mathbf{x}$ consistent with (3) and (7). Equality in (8) holds *only* when $(\mathbf{x}, \mathbf{s})$ satisfy all the requirements (1) to (7).

3. *The first variational theorem.* This is based upon the energy functional

$$E(\mathbf{x}) = \int_V W(\boldsymbol{\alpha})\,dV - \int_{\Sigma_s} \mathbf{x}\mathbf{t}_0\,d\Sigma, \tag{9}$$

defined for all differentiable functions $\mathbf{x}(\mathbf{X})$ and $\boldsymbol{\alpha} \equiv x_{i,\mu}$.

An $\mathbf{s}$ can be defined formally by means of (1) though it will not necessarily satisfy (2) or be consistent with (4). It is also assumed that W depends on $\boldsymbol{\alpha}$ only through $\alpha_{i\mu}\alpha_{i\kappa}$.

Let variations in quantities be indicated by a superposed dot so that from (9)

$$\dot{E} = -\int_V s_{\mu i,\mu}\dot{x}_i\,dV + \int_{\Sigma_s} (\mathbf{t} - \mathbf{t}_0)\,\dot{\mathbf{x}}\,d\Sigma$$

with the help of (1) and the divergence theorem.

Thus, the pair $(\mathbf{x}, \mathbf{s})$ is a solution of the boundary-value problem if and only if $\dot{E} = 0$ for all variations $\dot{\mathbf{x}}$ consistent with $\dot{\mathbf{x}} = 0$ on Σ_x. Details of this principle with full reference to earlier work can be found in Truesdell and Noll ((11), section 88). It should be noted that the equations of rotational balance are a consequence of frame indifference here.

4. *The second variational theorem.* We next consider the functional

$$E_c(\mathbf{s}) = \int_{\Sigma_x} \mathbf{t}\mathbf{x}_0\,d\Sigma - \int_V W_c(\mathbf{s})\,dV \tag{10}$$

for all self-equilibrated nominal 'stress' fields $\mathbf{s}$ consistent with the boundary conditions on Σ_s. For any given constitutive law $W_c(\mathbf{s})$ (consistent with the principle of frame indifference) the quantity $\boldsymbol{\alpha}$ may be defined formally by (7), but $\boldsymbol{\alpha}$ is not necessarily derivable from a differentiable deformation function $\mathbf{x}$ which satisfies the boundary condition on Σ_x. From (10) with the help of (7) the variation in E_c can be written as

$$\dot{E}_c = \int_{\Sigma_x} \dot{\mathbf{t}}\mathbf{x}_0\,d\Sigma - \int_V \boldsymbol{\alpha}\dot{\mathbf{s}}\,dV \tag{11}$$

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for all variations $\dot{\mathbf{s}}$ consistent with $\dot{\mathbf{t}} = 0$ on Σ_s. Use of the identity

$$\int_V x_{i,\mu}\dot{s}_{\mu i}\,dV = \int_{\Sigma_x} \mathbf{x}\dot{\mathbf{t}}\,d\Sigma$$

in (11) enables us to write

$$\dot{E}_c = \int_V (x_{i,\mu} - \alpha_{i\mu})\dot{s}_{\mu i}\,dV + \int_{\Sigma_x} (\mathbf{x}_0 - \mathbf{x})\dot{\mathbf{t}}\,d\Sigma \tag{12}$$

for self-equilibrated variations $\dot{\mathbf{s}}$, and all differentiable functions $\mathbf{x}$.

If $(\mathbf{x}, \mathbf{s})$ is a solution of the boundary-value problem then clearly $\dot{E}_c = 0$ for all variations $\dot{\mathbf{s}}$ consistent with the given restrictions, but the converse cannot be deduced immediately from (12) since the $\dot{\mathbf{s}}$ are not entirely arbitrary.

Since $\mathbf{s}$ is self-equilibrated we can write $s_{\mu i} = \epsilon_{\mu\kappa\lambda}\psi_{\lambda i,\kappa}$, where $\epsilon_{\mu\kappa\lambda}$ is the alternating symbol and the $\psi_{\lambda i}$ are stress functions. Correspondingly

$$\dot{s}_{\mu i} = \epsilon_{\mu\kappa\lambda}\dot{\psi}_{\lambda i,\kappa}, \quad \dot{t}_i = n_\mu \epsilon_{\mu\kappa\lambda}\dot{\psi}_{\lambda i,\kappa}, \tag{13}$$

and we consider arbitrary independent variations $\dot{\psi}_{\kappa i}$ in V and on Σ, and $\dot{\mathbf{t}}$ on Σ_x with $\dot{\mathbf{t}} = 0$ on Σ_s.

Substitution of (13) into (11) and use of the divergence theorem leads to

$$\dot{E}_c = \epsilon_{\mu\kappa\lambda}\int_V \alpha_{i\mu,\kappa}\dot{\psi}_{\lambda i}\,dV - \epsilon_{\mu\kappa\lambda}\int_\Sigma n_\kappa(\alpha_{i\mu} - x_{i,\mu})\dot{\psi}_{\lambda i}\,d\Sigma \tag{14}$$

for all differentiable functions $\mathbf{x}$ satisfying $\mathbf{x} = \mathbf{x}_0$ on Σ_x. It follows immediately from the first term in (14) that

$$\epsilon_{\mu\kappa\lambda}\alpha_{i\mu,\kappa} = 0 \quad \text{in } V$$

on setting $\dot{E}_c = 0$ for all variations $\dot{\psi}_{\lambda i}$ in V.

Thus $\boldsymbol{\alpha}$ is a deformation gradient and we can write $\alpha_{i\mu} = x_{i,\mu}$. Use of this in (11) leads to

$$\dot{E}_c = \int_{\Sigma_x} (\mathbf{x}_0 - \mathbf{x})\,\dot{\mathbf{t}}\,d\Sigma$$

and then the pair $(\mathbf{x}, \mathbf{s})$ is a solution of the boundary-value problem if and only if $\dot{E}_c = 0$ for all stress-function variations $\dot{\psi}_{\lambda i}$ such that $\dot{\mathbf{t}} = 0$ on Σ_s.

Another interpretation of (12) is as follows: the constraint term

$$-\int_V s_{\mu i,\mu}x_i\,dV \equiv 0,$$

where the x_i are Lagrange multipliers, may be added to (10). The variation of the resulting expression is again given by (12) but the $\dot{s}_{\mu i}$ may now be regarded as completely arbitrary, and the theorem follows (as in (14)).

Levinson (6) and Zubov (14) use the functional (10), but Koiter (5) prefers to work in terms of the displacement rather than the final position $\mathbf{x}$, and he defines W_c by $W + W_c = \mathbf{s}(\boldsymbol{\alpha} - \boldsymbol{\delta})$ where $\boldsymbol{\delta}$ is the identity. In taking the variation of his functional Koiter assumes that $\partial W_c/\partial \mathbf{s}$ is expressible as the gradient of a displacement function rather than deriving the result as one element in the variational theorem.

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If we wished to consider independent variations in $\mathbf{x}$ and $\mathbf{s}$ it would be appropriate to use the functional $E \pm E_c$ for all pairs $(\mathbf{x}, \mathbf{s})$ of differentiable functions $\mathbf{x}$ and nominal 'stress' fields $\mathbf{s}$. We have

$$E - E_c = \int_V W(x_{i,\mu})\, dV - \int_{\Sigma_s} \mathbf{x}\mathbf{t}_0\, d\Sigma + \int_V W_c(\mathbf{s})\, dV - \int_{\Sigma_x} \mathbf{t}\mathbf{x}_0\, d\Sigma, \tag{15}$$

which is different from Reissner's (8) functional, but is preferred because of its connexion with equation (8) and the two variational principles discussed above. Moreover, the present approach displays the 'symmetry' between the variational principles of place and nominal stress, a symmetry which is evident in the classical linear theory (Sokolnikoff (10)); then $W_c = W$ for an actual solution pair provided $\boldsymbol{\alpha}$ is replaced by $\boldsymbol{\alpha} - \boldsymbol{\delta}$.

The variation of $E - E_c$ is given by

$$\dot{E} - \dot{E}_c = \int_V \left(\frac{\partial W}{\partial x_{i,\mu}}\right)_{,\mu} \dot{x}_i\, dV + \int_V (\alpha_{i\mu} - x_{i,\mu})\, \dot{s}_{\mu i}\, dV + \int_{\Sigma_s} \left(n_\mu \frac{\partial W}{\partial x_{i,\mu}} - t_{0i}\right) \dot{x}_i\, d\Sigma + \int_{\Sigma_x} (\mathbf{x} - \mathbf{x}_0)\, \dot{\mathbf{t}}\, d\Sigma, \tag{16}$$

for all variations $\dot{\mathbf{x}}$ and $\dot{\mathbf{s}}$ with $\dot{\mathbf{x}} = 0$ on Σ_x, $\dot{\mathbf{t}} = 0$ on Σ_s and $\dot{s}_{\mu i,\mu} = 0$ in V, which is precisely the same as the variation of Reissner's (8) functional except that Reissner uses the second Piola–Kirchhoff stress and the right Cauchy–Green deformation tensor rather than $\mathbf{s}$ and $\boldsymbol{\alpha}$. He does not require the latter restriction on $\dot{\mathbf{s}}$. Truesdell and Toupin ((12), section 232) express Reissner's functional in terms of $\mathbf{s}$ and $\boldsymbol{\alpha}$; see the remarks concerning frame indifference in section 2.

Koiter (5) notes the difficulty of obtaining the inverse $W_c(\mathbf{s}) = \mathbf{s}\boldsymbol{\alpha} - W(\boldsymbol{\alpha})$ for a given constitutive law. One way round the difficulty in applying the principle of stationary complementary energy is to define an elastic constitutive law initially in terms of W_c. For example, one could specify W_c as a function of the invariants of $s_{\mu i} s_{\lambda i}$ for isotropic elastic materials, rather than specifying W as a function of the invariants of $\alpha_{i\mu}\alpha_{i\lambda}$. For the 'semi-linear' materials discussed by Koiter the present formulation in terms of position (rather than displacement) yields the attractive connexion $W = W_c = \frac{1}{2}\mathbf{s}\boldsymbol{\alpha}$.

5. *Fraeijs de Veubeke's principle.* From the polar decomposition theorem we have

$$\boldsymbol{\alpha} = \mathbf{r}_\circ \mathbf{u} \tag{17}$$

where $\mathbf{r}$ is a proper orthogonal tensor and $\mathbf{u}$ (the right stretch tensor) is positive definite and symmetric. The notation $\mathbf{r}_\circ \mathbf{u}$ is used for the usual matrix product of $\mathbf{r}$ and $\mathbf{u}$. Thus, we can write

$$\dot{W} = \mathbf{s}\dot{\boldsymbol{\alpha}} = \boldsymbol{\tau}\dot{\mathbf{u}} + (\boldsymbol{\alpha}_\circ \mathbf{s})\,\boldsymbol{\omega}, \tag{18}$$

where $\boldsymbol{\omega} = \dot{\mathbf{r}}_\circ \mathbf{r}'$ is antisymmetric, $\mathbf{r}'$ being the transpose of $\mathbf{r}$, and $\boldsymbol{\tau}$ is the symmetric part of $\mathbf{s}_\circ \mathbf{r}$. Under conditions of rotational balance $\boldsymbol{\alpha}_\circ \mathbf{s}$ is symmetric and the final term in (18) is identically zero. (This is equivalent to W being frame indifferent.) It follows that $\dot{W} = \boldsymbol{\tau}\dot{\mathbf{u}}$ and we may define $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{u}}, \tag{19}$$

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recalling that W depends only on $\boldsymbol{\alpha}'_{\circ}\boldsymbol{\alpha} = \mathbf{u}^2$. In fact, $\boldsymbol{\tau}$ is the symmetrized Biot stress discussed by Hill (4). In view of the symmetry of $\mathbf{u}$ we now have $\mathbf{s}\boldsymbol{\alpha} = \boldsymbol{\tau}\mathbf{u}$ so that (6) can be replaced by

$$W(\mathbf{u}) + W_c(\boldsymbol{\tau}) = \boldsymbol{\tau}\mathbf{u}, \tag{20}$$

where W_c is the Legendre dual of W relative to variables $\mathbf{u}$ and $\boldsymbol{\tau}$, and

$$\mathbf{u} = \frac{\partial W_c}{\partial \boldsymbol{\tau}}. \tag{21}$$

However, the domain of existence of the inversion may be limited on physical grounds.

To recover the essential features of Fraeijs de Veubeke's (1) principle we consider the functional (15) once again, this time expressed in the form

$$E - E_c = \int_V W(x_{i,\mu})\,dV - \int_{\Sigma_s} \mathbf{x}\mathbf{t}_0\,d\Sigma + \int_V W_c(\boldsymbol{\tau})\,dV - \int_{\Sigma_x} \mathbf{t}\mathbf{x}_0\,d\Sigma. \tag{22}$$

We consider differentiable functions $\mathbf{x}$, second-order tensor fields $\mathbf{s}$ *and* proper orthogonal tensors $\mathbf{r}$ together with $\boldsymbol{\tau}$ defined as the symmetric part of $\mathbf{s}_{\circ}\mathbf{r}$. It is also convenient to define $\mathbf{u}$ by $\mathbf{u} = \partial W_c/\partial\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha} = \mathbf{r}_{\circ}\mathbf{u}$. We admit independent variations in $\mathbf{x}$, $\mathbf{r}$ and $\mathbf{s}$ (or equivalently $\boldsymbol{\tau}$) consistent with $\dot{\mathbf{x}} = 0$ on Σ_x, $\dot{\mathbf{t}} = 0$ on Σ_s and $\dot{s}_{\mu i,\mu} = 0$ in V.

We note the identity

$$\mathbf{u}\dot{\boldsymbol{\tau}} = \boldsymbol{\alpha}\dot{\mathbf{s}} + (\boldsymbol{\alpha}_{\circ}\mathbf{s})\boldsymbol{\omega}, \tag{23}$$

and that in (22) $W(x_{i,\mu})$ and $W_c(\boldsymbol{\tau})$ are *not* necessarily related by (20). Moreover, there is no restriction now put on the way in which W depends on $x_{i,\mu}$.

After some manipulation the variation of (22) becomes

$$\begin{aligned}\dot{E} - \dot{E}_c = &-\int_V \left(\frac{\partial W}{\partial x_{i,\mu}}\right)_{,\mu}\dot{x}_i\,dV + \int_{\Sigma_s}\left(n_\mu\frac{\partial W}{\partial x_{i,\mu}} - t_{0i}\right)\dot{x}_i\,d\Sigma \\ &+ \int_{\Sigma_x}(\mathbf{x}-\mathbf{x}_0)\,\dot{\mathbf{t}}\,d\Sigma + \int_V (\alpha_{i\mu} - x_{i,\mu})\,\dot{s}_{\mu i}\,dV + \int_V (\boldsymbol{\alpha}_{\circ}\mathbf{s})\boldsymbol{\omega}\,dV. \end{aligned}\tag{24}$$

It follows that the pair $(\mathbf{x}, \mathbf{s})$ provides a solution of the boundary-value problem if and only if $\dot{E} - \dot{E}_c = 0$ for all variations $\dot{\mathbf{x}}$, $\dot{\mathbf{r}}$, $\dot{\mathbf{s}}$ consistent with $\dot{\mathbf{x}} = 0$ on Σ_x, $\dot{\mathbf{t}} = 0$ on Σ_s and $\dot{s}_{\mu i,\mu} = 0$ in V.

When $\boldsymbol{\omega}$ is set equal to zero in (24) and W allowed to depend only on $x_{i,\mu}x_{i,\kappa}$ we recover Reissner's principle as discussed in section 4.

Fraeijs de Veubeke uses the functional

$$\int_V \{s_{\mu i}x_{i,\mu} - W_c(\boldsymbol{\tau})\}\,dV - \int_{\Sigma_s}\mathbf{x}\mathbf{t}_0\,d\Sigma$$

((1), equation (80)), expressed in different notation, but the variation of this does not include the term

$$\int_{\Sigma_x}(\mathbf{x}-\mathbf{x}_0)\,\dot{\mathbf{t}}\,d\Sigma.$$

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His complementary functional ((1), equation (81)) is essentially E_c, and his use of stress functions and their variations is equivalent to that in the second variational theorem of section 4.

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Note added in proof.

A paper by Dill (15) has recently been brought to my attention by Dr M. J. Sewell. Dill rejects the use of the $(\mathbf{s}, \boldsymbol{\alpha})$ variables on the grounds of non-uniqueness at $\mathbf{s} = \mathbf{0}$. Moreover, his counterexample ((15), equation 6·7) is invalid since it allows the principal stretches to be negative. His other objections are clarified in section 2 by the classification into domains (such as $\mathscr{D}$) in each of which the inversion is unique.

It should be emphasised here that whichever pair of stress and deformation variables are employed the question of non-uniqueness is likely to arise, either because of the onset of instability or from some other physical consideration.

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