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A note on the equal sum and product problem with a fixed coefficient

Maciej Zakarczemny

Abstract. Let $a \geq 1$. Denote by $N_a(n)$ the number of solutions of

$$x_1 + x_2 + \cdots + x_n = a x_1 x_2 \cdot \ldots \cdot x_n, \ x_1 \geq x_2 \geq \cdots \geq x_n \geq 1.$$

For fixed $a > 1$, we obtain the asymptotic behavior of $\sum_{a \leq n \leq x} N_a(n)$ as $x \to \infty$. We also obtain asymptotics for $\sum_{a \leq n \leq x} n N_a(n)$, $\sum_{a \leq n \leq x} \frac{1}{n} N_a(n)$, $\sum_{a \leq n \leq x} \sum_{d \mid n} N_a(d)$, and $\sum_{a \leq n \leq x} \sum_{d \mid n} N_a(d)\, \tau\left(\frac{n}{d}\right)$. Furthermore, we prove that for every $\varepsilon > 0$ and sufficiently large n, the inequality $N_a(n) < n^{1+\varepsilon}$ holds.

1 Introduction

The equal sum product problem concerns, among other questions, estimating the number of solutions $N_1(n)$ to the Diophantine equation

$$x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdot \ldots \cdot x_n, \text{ where } x_1 \geq x_2 \geq \cdots \geq x_n \geq 1. \tag{1.1}$$

The initial results can be found in [2–4, 6, 17]. Schinzel asked in [12, 13] whether $N_1(n)$ tends to infinity with n. For equation (1.1) and its solution $(x_1, x_2, \ldots, x_n)$, several useful facts are known:

1) $x_1 \cdot \ldots \cdot x_n \leq 2n, \ x_n \leq n$ (see [6]).
2) $x_{n-1} \leq 1 + \lfloor \sqrt{n-1} \rfloor$ (see [11]).
3) If $N_1(n) = 1$, then $n - 1$ is a Sophie Germain prime (see [8]).
4) $N_1(n) \leq \binom{n+1}{2}$ (see [2]).
5) Let $\varepsilon > 0$, with the exception of the set of natural density zero, we have $(\log n)^{\log 2 - \varepsilon} < N_1(n) < e^{2\sqrt{\log n}}$ (see [11]).
6) Let $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that $N_1(n) < n^{1+\varepsilon}$ for $n \geq C(\varepsilon)$ (see [11]).
7) Let $\varepsilon > 0$, there exist infinitely many n such that $N_1(n) > n^{1-\varepsilon}$ (see [11]).

Average asymptotic behavior was obtained by Weingartner (see [16]):

$$\sum_{2 \leq n \leq x} N_1(n) \sim \frac{x e^{2\sqrt{\log x}}}{2\sqrt{\pi} (\log x)^{3/4}}.$$

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Hurwitz (see [5]) studied the family of equations $x_1^2 + x_2^2 + \cdots + x_n^2 = ax_1x_2 \cdot \ldots \cdot x_n$, where $a, n \in \mathbb{Z}^+$ and $n \geq 3$. This case is different because of the quadratic form and the use of the Vieta jumping method.

An interesting generalization that preserves linearity but loses the symmetry of the sum is the equation of the form:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = ax_1x_2 \cdot \ldots \cdot x_n,$$

with fixed a and c_i, where $a, c_i, n \in \mathbb{Z}^+$, $n \geq 2$.

Let us now assume that $n \in \mathbb{Z}$, $n \geq 2$, $a > 1$. In this article, we provide the average asymptotic behavior $N_a(n)$, the number of integer solutions $(x_1, x_2, \ldots, x_n)$ to the equation

$$x_1 + x_2 + \cdots + x_n = ax_1x_2 \cdot \ldots \cdot x_n \tag{1.2}$$

such that $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$. A related equation was also investigated in [14, 15]. It is worth noting that if equation (1.2) has solutions, then $n \geq a$. In particular, we have $N_a(n) = 0$ for $n < a$. The article is organized as follows. First, in Section 2, we introduce several lemmas and remarks. Next, in Section 3, we establish the average asymptotic behavior of $N_a(n)$, the number of integer solutions $(x_1, \ldots, x_n)$ to equation (1.2), together with the resulting upper bound for $N_a(n)$. Finally, we derive asymptotics for the sums $\sum_{a \leq n \leq x} n\, N_a(n)$ and $\sum_{a \leq n \leq x} \frac{1}{n} N_a(n)$, and for the Dirichlet convolution sums $\sum_{a \leq n \leq x} \sum_{d \mid n} N_a(d)$ and $\sum_{a \leq n \leq x} \sum_{d \mid n} N_a(d)\, \tau\left(\frac{n}{d}\right)$.

2 Preliminaries

In this section, we discuss the necessary basic results. First, we note an observation that if $n < a$, then $N_a(n) = 0$. Thus, we may assume that $n \geq a$.

Lemma 2.1 *Let $(x_1, \ldots, x_n)$ with $x_i \in \mathbb{Z}^+$ be a solution of (1.2), where $1 < a \leq n$. Then:*

(1)

$$\prod_{i=1}^{n} x_i \leq \frac{n-1}{a-1}.$$

(2) *If $k = |\{i : x_i > 1\}| \geq 2$, then*

$$\prod_{i=1}^{n} x_i \leq \frac{n-2}{a-1}.$$

Proof For $i \leq k$, we have $x_i = y_i + 1$ with $y_i \geq 1$, and for $i > k$, we have $x_i = 1$. Denote

$$S = \sum_{i=1}^{k} y_i, \quad P = \prod_{i=1}^{k} (y_i + 1) = \prod_{i=1}^{n} x_i.$$

From equation (1.2), we have $aP = S + n$. Expanding the product gives $P \geq 1 + S$. Thus, $P \geq 1 + aP - n$, hence, for $a > 1$, we obtain $P \leq \frac{n-1}{a-1}$.

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Assume now that $k = |\{i : x_i > 1\}| \geq 2$.

Expanding the product gives $P \geq 1 + S + S_2$, where $S_2 = \sum_{1 \leq i < j \leq k} y_i y_j \geq y_1 y_2 \geq 1$. Hence, $n - (a-1)P = P - S \geq 1 + S_2 \geq 2$. Therefore, $P \leq \frac{n-2}{a-1}$. ∎

Remark 2.2 If $k \geq 2$, then equality $P = \frac{n-2}{a-1}$ holds exactly for $k = 2$, $y_1 = y_2 = 1$, that is, for the solution $(x_1, \ldots, x_n) = (2, 2, 1, \ldots, 1)$. Then $P = 4$, $S = 2$, and from $aP = S + n$, we get $n = 4a - 2$, which gives $P = (n-2)/(a-1)$.

Remark 2.3 It is also useful to know a bound for the product $\prod_{i=2}^{n} x_i$, that is, the product excluding the largest term x_1.

Assume $a > 1$ and let $(x_1, \ldots, x_n)$ be a solution of $x_1 + \cdots + x_n = a\, x_1 \ldots x_n$ with $x_1 \geq \cdots \geq x_n \geq 1$. Set $Q = \prod_{i=2}^{n} x_i$. If $k = |\{i : x_i > 1\}| = 1$, then $Q = 1$. If $k \geq 2$, then $P \leq \frac{n-2}{a-1}$ and $x_1 \geq 2$, hence $Q = \frac{P}{x_1} \leq \frac{P}{2} \leq \frac{n-2}{2(a-1)}$. The bound is sharp: for $n = 4a - 2$, the solution $(2, 2, 1, \ldots, 1)$ gives $Q = 2 = \frac{n-2}{2(a-1)}$.

3 Main results

We denote by a_n the number of distinct unordered factorizations of the positive integer n into the factors larger than 1. Factorizations that differ only in the order of factors are considered the same. Let $f(x) = \sum_{2 \leq n \leq x} a_n$. It means that

$$f(x) = \Big|\{(z_1, z_2, \ldots, z_k) : z_1 \geq z_2 \geq \cdots \geq z_k \geq 2,\ z_i \in \mathbb{Z},\ k \in \mathbb{Z}^+, z_1 z_2 \cdot \ldots \cdot z_k \leq x\}\Big|.$$

Oppenheim proved in [9] that

$$f(x) \sim \frac{x e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}. \tag{3.1}$$

We begin with a lemma whose analog for the case $a = 1$ was proved in [10].

Lemma 3.1 *Let $x > a > 1$. We have*

$$1 + f\left(\frac{x}{a}\right) < \sum_{a \leq n \leq x} N_a(n) \leq 1 + f\left(\frac{x + \frac{x-1}{a-1} e^{-\sqrt{\log x}}}{a}\right) + \frac{x-1}{a-1} f\left(\left(\log_2 \frac{x-1}{a-1}\right) e^{\sqrt{\log x}}\right).$$

Proof If $x_1 + \cdots + x_n = a x_1 \cdot \ldots \cdot x_n, n \geq a > 1$, where $x_1 \geq x_2 \geq \cdots \geq x_k \geq 2$, $x_{k+1} = \cdots = x_n = 1$, then $a x_1 \cdot \ldots \cdot x_k = x_1 + \cdots + x_k + n - k$, that is, $n = a x_1 \cdot \ldots \cdot x_k - (x_1 + \cdots + x_k) + k$. Note that $k = 0$ is possible only when $n = a$, and $N_a(a) = 1$. Therefore, $-1 + \sum_{a \leq n \leq x} N_a(n) = \sum_{a < n \leq x} N_a(n)$. Thus, we assume $k \geq 1$ and determine bounds for the sum $-1 + \sum_{a \leq n \leq x} N_a(n)$. We have, for $n > a$,

$$N_a(n) = \Big|\{(x_1, \ldots, x_k) : x_1 \geq \cdots \geq x_k \geq 2, k \geq 1,$$
$$a x_1 \cdot \ldots \cdot x_k - (x_1 + \cdots + x_k) + k = n\}\Big|.$$

Note that if $a x_1 \cdot \ldots \cdot x_k \leq x$ and $x_1 \geq x_2 \geq \cdots \geq x_k \geq 2$, $k \geq 1$, $a > 1$, then

$$a \leq a x_1 \cdot \ldots \cdot x_k - (x_1 + \cdots + x_k) + k \leq x.$$

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Hence,

$$\begin{aligned}-1+\sum_{a\le n\le x} N_a(n) &= \Big|\{(x_1,\dots,x_k): x_1\ge\dots\ge x_k\ge 2, k\ge 1,\\ &\qquad\qquad a\le ax_1\cdot\ldots\cdot x_k-(x_1+\dots+x_k)+k\le x\}\Big|\\ &\ge \Big|\{(x_1,\dots,x_k): x_1\ge x_2\ge\dots\ge x_k\ge 2, k\ge 1,\ ax_1\cdot\ldots\cdot x_k\le x\}\Big|\\ &= \left|\left\{(x_1,\dots,x_k): x_1\ge x_2\ge\dots\ge x_k\ge 2,\ k\ge 1,\ x_1\cdot\ldots\cdot x_k\le\frac{x}{a}\right\}\right|\\ &\ge f\Big(\frac{x}{a}\Big).\end{aligned}$$

We now prove the second inequality of the lemma.

If $x_1+\dots+x_k<\frac{x-1}{a-1}e^{-\sqrt{\log x}}$, then $ax_1\cdot\ldots\cdot x_k\le x+(x_1+\dots+x_k)-k<x+\frac{x-1}{a-1}e^{-\sqrt{\log x}}$.

Thus,

$$\begin{aligned}&\Big|\{(x_1,\dots,x_k): x_1\ge\dots\ge x_k\ge 2, k\ge 1,\\ &\qquad a\le a\,x_1\dots x_k-(x_1+\dots+x_k)+k\le x,\\ &\qquad x_1+\dots+x_k<\tfrac{x-1}{a-1}e^{-\sqrt{\log x}}\}\Big|\\ \le&\Big|\{(x_1,\dots,x_k): x_1\ge\dots\ge x_k\ge 2, k\ge 1,\\ &\qquad x_1\dots x_k\le\tfrac{x+\frac{x-1}{a-1}e^{-\sqrt{\log x}}}{a}\}\Big|\le f\left(\frac{x+\frac{x-1}{a-1}e^{-\sqrt{\log x}}}{a}\right).\end{aligned}$$

If $x_1+\dots+x_n=ax_1\cdot\ldots\cdot x_n$, $x_1\ge\dots\ge x_n\ge 1$, $x_i\in\mathbb{Z}$, then by Lemma 2.1, $x_1\cdot\ldots\cdot x_n\le\frac{n-1}{a-1}$. It follows that $x_1\le x_1\cdot\ldots\cdot x_k=x_1\cdot\ldots\cdot x_n\le\frac{n-1}{a-1}\le\frac{x-1}{a-1}$. Therefore, we have at most $\frac{x-1}{a-1}$ possibilities for x_1. Note also that $2^k\le x_1\cdot\ldots\cdot x_k\le\frac{x-1}{a-1}$ implies $k\le\log_2\frac{x-1}{a-1}$.

If $x_1+\dots+x_k\ge\frac{x-1}{a-1}e^{-\sqrt{\log x}}$, then $x_1\ge\frac{\frac{x-1}{a-1}e^{-\sqrt{\log x}}}{k}\ge\frac{\frac{x-1}{a-1}e^{-\sqrt{\log x}}}{\log_2\frac{x-1}{a-1}}$. Thus,

$$x_2\cdot\ldots\cdot x_k\le\frac{\frac{x-1}{a-1}}{x_1}\le(\log_2\tfrac{x-1}{a-1})e^{\sqrt{\log x}}. \tag{3.2}$$

Therefore, we have at most $f\left((\log_2\frac{x-1}{a-1})e^{\sqrt{\log x}}\right)$ possibilities for $(x_2,\dots,x_k)$. Hence,

$$\begin{aligned}&\Big|\{(x_1,\dots,x_k):\ x_1\ge\dots x_k\ge 2,\ x_i\in\mathbb{Z},\ k\ge 1,\\ &\qquad a\le ax_1\cdot\ldots\cdot x_k-(x_1+\dots+x_k)+k\le x,\\ &\qquad x_1+\dots+x_k\ge\tfrac{x-1}{a-1}e^{-\sqrt{\log x}}\}\Big|\\ \le&\tfrac{x-1}{a-1}f\left((\log_2\tfrac{x-1}{a-1})e^{\sqrt{\log x}}\right).\end{aligned}$$

Thus, $-1+\sum_{a\le n\le x}N_a(n)\le f\left(\frac{x+\frac{x-1}{a-1}e^{-\sqrt{\log x}}}{a}\right)+\frac{x-1}{a-1}f\left((\log_2\frac{x-1}{a-1})\,e^{\sqrt{\log x}}\right)$. ∎

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Theorem 3.2 *Let $a > 1$ be fixed. As $x \to \infty$,*

$$S_a(x) = \sum_{a \le n \le x} N_a(n) \sim f\left(\tfrac{x}{a}\right) \sim \tfrac{1}{a} \tfrac{x e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

Proof From Lemma 3.1, we have

$$f\left(\tfrac{x}{a}\right) < S_a(x) - 1 \le f\left(\frac{x + \frac{x-1}{a-1} e^{-\sqrt{\log x}}}{a}\right) + \tfrac{x-1}{a-1} f\left((\log_2 \tfrac{x-1}{a-1}) e^{\sqrt{\log x}}\right).$$

We have $x + \frac{x-1}{a-1} e^{-\sqrt{\log x}} \sim x$. Using Oppenheim's asymptotic formula, $f(u) \sim \frac{u\, e^{2\sqrt{\log u}}}{2\sqrt{\pi}\,(\log u)^{3/4}}$. We obtain, as $x \to \infty$,

$$f\left(\frac{x + \frac{x-1}{a-1} e^{-\sqrt{\log x}}}{a}\right) \sim f\left(\frac{x}{a}\right).$$

The second term on the right-hand side is negligible. Indeed, we denote

$$w(x) = (\log_2 \tfrac{x-1}{a-1}) e^{\sqrt{\log x}}.$$

Then $\log w(x) \sim \sqrt{\log x} + \log\log x$ and $\sqrt{\log w(x)} \sim (\log x)^{1/4}$. A direct application of the asymptotic for f gives $\lim_{x\to\infty} \frac{\frac{x-1}{a-1} f\left(w(x)\right)}{f\left(\frac{x}{a}\right)} = 0$. From these estimates and the two-sided inequality, we get $S_a(x) \sim f\left(\frac{x}{a}\right)$. Finally, by Oppenheim's asymptotic formula, $f\left(\frac{x}{a}\right) \sim \frac{1}{a} f(x)$, since $\lim_{x\to\infty} \frac{e^{2\sqrt{\log(x/a)}} (\log x)^{3/4}}{e^{2\sqrt{\log x}} (\log(x/a))^{3/4}} = 1$. ∎

Remark 3.3 Together with the result of Weingartner (see [16]), this implies that for $a \ge 1$, we have

$$S_a(x) = \sum_{a \le n \le x} N_a(n) \sim \tfrac{1}{a} \tfrac{x e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

Note that one can try to estimate the approximation error by using the error estimate for $f(x)$ (see [9]).

Corollary 3.4 *Fix $a > 1$. For every $\varepsilon > 0$, there exists an integer $C(\varepsilon, a)$ such that*

$$N_a(n) < n^{1+\varepsilon} \text{ for all } n \ge C(\varepsilon, a).$$

Proof By Theorem 3.2, as $x \to \infty$, we have $S_a(x) \sim \frac{x}{a} \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{3/4}}$. Hence, there exist $x_0(a)$ and $C_a > 0$, depending only on a, such that for all $x \ge x_0(a)$,

$$S_a(x) \le C_a \frac{x e^{2\sqrt{\log x}}}{(\log x)^{3/4}}. \tag{3.3}$$

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Fix $\varepsilon > 0$. Choose $C(\varepsilon, a) \geq x_0(a)$ so large that for all $x \geq C(\varepsilon, a)$, we have

$$C_a\, x\, \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}} \leq x^{1+\varepsilon}.$$

Since, for any integer x, we have $N_a(x) \leq S_a(x)$, it follows that $N_a(x) < x^{1+\varepsilon}$ for all $x \geq C(\varepsilon, a)$. ∎

To prove the next lemma, we use a standard method. From an upper bound for the average size of an arithmetic function, we show that very large values occur only on a set of natural density zero.

Lemma 3.5 *Let $a > 1$ be fixed and let $h : [2, \infty) \to (0, \infty)$ be nondecreasing. Assume*

$$\lim_{x\to\infty} h\left(\tfrac{x}{\log x}\right)^{-1} \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}} = 0. \tag{3.4}$$

Define $E_h(x) = \{n \in \mathbb{Z}_{\geq a} : n \leq x, N_a(n) \geq h(n)\}$. Then for all sufficiently large x,

$$\frac{|E_h(x)|}{x} \leq \frac{1}{\log x} + C_a\, h(\tfrac{x}{\log x})^{-1} \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}, \tag{3.5}$$

for some constant $C_a > 0$.

The set $\{n :\ N_a(n) \geq h(n)\}$ has natural density 0.

Proof Let, as before, $S_a(x) = \sum_{a\leq n\leq x} N_a(n)$. Since h is nondecreasing,

$$S_a(x) \geq \sum_{n\in E_h(x)\cap[\frac{x}{\log x},x]} N_a(n) \geq \sum_{n\in E_h(x)\cap[\frac{x}{\log x},x]} h(n) \geq \left(|E_h(x)| - \tfrac{x}{\log x}\right) h(\tfrac{x}{\log x}).$$

From (3.3), it follows that, for sufficiently large x,

$$\left(|E_h(x)| - \tfrac{x}{\log x}\right) h(\tfrac{x}{\log x}) \leq C_a\, \frac{x\, e^{2\sqrt{\log x}}}{(\log x)^{3/4}}.$$

Dividing by x gives (3.5). The limit (3.4) implies that $\lim_{x\to\infty} \frac{|E_h(x)|}{x} = 0$. ∎

Remark 3.6 The assumption (3.4) holds, for example, in each of the following cases:

(1) $h(x) = \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}} L(x)$ with $\lim_{x\to\infty} L(x) = \infty$ and $\lim_{x\to\infty} \frac{L(\frac{x}{\log x})}{L(x)} = 1$;

(2) $h(x) = \frac{e^{2\sqrt{\log x}}}{(\log x)^{\gamma}}$ with any $\gamma < 3/4$;

(3) $h(x) = e^{(2+\eta)\sqrt{\log x}}$ with any $\eta > 0$.

In particular, the set $\left\{n :\ N_a(n) \geq \frac{e^{2\sqrt{\log n}}}{(\log n)^{\gamma}}\right\}$ with any fixed $\gamma < 3/4$ has natural density 0.

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Corollary 3.7 *Except for a set of natural density* 0*, for any* $\gamma < 3/4$*, the inequality holds:*

$$N_a(n) \le \frac{e^{2\sqrt{\log n}}}{(\log n)^{\gamma}}.$$

Theorem 3.8 *Let* $a \in \mathbb{Z}^+$ *and* $L_a(x) = \sum_{a\le n\le x} n\, N_a(n)$.
Then for every $x \ge a$*,*

$$L_a(x) = xS_a(x) - \sum_{a\le n\le x-1} S_a(n). \tag{3.6}$$

Moreover, as $x \to \infty$*,*

$$L_a(x) \sim \frac{1}{4\sqrt{\pi}\, a} \cdot \frac{x^2\, e^{2\sqrt{\log x}}}{(\log x)^{3/4}}.$$

Proof The identity (3.6) follows directly from [7, Theorem 6.8] on partial summation:

$$\sum_{n=a}^{x} f(n)g(n) = F(x)g(x) - F(a-1)g(a) - \sum_{n=a}^{x-1} F(n)\big(g(n+1) - g(n)\big),$$

where $f(n) = N_a(n)\mathbb{1}_{n\ge a}$, $g(n) = n$, and $F(u) = S_a(u)$.

Denote $k(x) = \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}$. Then $(x+1)^2k(x+1) - x^2k(x) = 2xk(x)(1+\varepsilon(x))$, where $\varepsilon(x) = o(1)$. By the Stolz–Cesàro theorem and from Theorem 3.2, we have

$$\lim_{x\to\infty} \frac{\sum_{a\le n\le x-1} S_a(n)}{x^2k(x)} = \lim_{x\to\infty} \frac{\sum_{a\le n\le x} S_a(n) - \sum_{a\le n\le x-1} S_a(n)}{(x+1)^2k(x+1) - x^2k(x)}$$

$$= \lim_{x\to\infty} \frac{S_a(x)}{2xk(x)} \cdot \frac{1}{1+\varepsilon(x)} = \frac{1}{4\sqrt{\pi}a}.$$

Therefore,

$$\sum_{a\le n\le x-1} S_a(n) \sim \frac{1}{4\sqrt{\pi}\, a} \cdot \frac{x^2 e^{2\sqrt{\log x}}}{(\log x)^{3/4}}.$$

Since $xS_a(x) \sim \frac{1}{2\sqrt{\pi}\, a} \cdot \frac{x^2 e^{2\sqrt{\log x}}}{(\log x)^{3/4}}$, then from (3.6), we get $L_a(x) \sim \frac{1}{4\sqrt{\pi}\, a} \cdot \frac{x^2 e^{2\sqrt{\log x}}}{(\log x)^{3/4}}$. ∎

Theorem 3.9 *Let* $a \in \mathbb{Z}^+$ *and* $H_a(x) = \sum_{a\le n\le x} \frac{1}{n} N_a(n)$.
Then for every $x \ge a$*,*

$$H_a(x) = \frac{S_a(x)}{x} + \sum_{a\le n\le x-1} \frac{S_a(n)}{n(n+1)}. \tag{3.7}$$

Moreover, as $x \to \infty$*,*

$$H_a(x) \sim \frac{1}{a} \cdot \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{1/4}}. \tag{3.8}$$

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Proof The identity (3.7) follows directly from [7, Theorem 6.8] on partial summation:

$$\sum_{n=a}^{x} f(n)g(n) = F(x)g(x) - F(a-1)g(a) - \sum_{n=a}^{x-1} F(n)\big(g(n+1) - g(n)\big),$$

where $f(n) = N_a(n)\,\mathbb{1}_{n\geq a}$, $g(n) = \frac{1}{n}$, and $F(u) = S_a(u)$.

We obtain

$$H_a(x) - \frac{S_a(x)}{x} = \sum_{a\leq n\leq x-1} \frac{S_a(n)}{n(n+1)}.$$

By Theorem 3.2, we have $S_a(n) = O\left(\frac{ne^{2\sqrt{\log n}}}{(\log n)^{3/4}}\right)$. Thus,

$$\begin{aligned} H_a(x) &= \frac{S_a(x)}{x} + \sum_{\log x\leq n\leq x-1} \frac{S_a(n)}{n(n+1)} + O\left(\sum_{a\leq n\leq \log x} \frac{e^{2\sqrt{\log n}}}{n(\log n)^{3/4}}\right) \\ &= \frac{S_a(x)}{x} + \sum_{\log x\leq n\leq x-1} \frac{S_a(n)}{n(n+1)} + O\left(\sum_{a\leq n\leq \log x} \frac{1}{\sqrt{n}}\right) \\ &= \frac{S_a(x)}{x} + \sum_{\log x\leq n\leq x-1} \frac{S_a(n)}{n(n+1)} + O\left(\sqrt{\log x}\right) \\ &= \sum_{\log x\leq n\leq x-1} \frac{S_a(n)}{n(n+1)} + O\left(\frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}\right). \end{aligned}$$

When x tends to infinity, the lower summation index $\log x$ also tends to infinity. Therefore, we may apply the asymptotics of $S_a(x) \sim \frac{1}{a}\cdot\frac{x\,e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}$ to the sum

$$\sum_{\log x\leq n\leq x-1} \frac{S_a(n)}{n(n+1)} \sim \frac{1}{2\sqrt{\pi}a} \sum_{\log x\leq n\leq x-1} \frac{e^{2\sqrt{\log n}}}{(n+1)(\log n)^{3/4}}.$$

From [7, Theorem 6.3] for decreasing function, taking care of the proper estimation of the boundary terms of the sum, we obtain

$$\sum_{\log x\leq n\leq x-1} \frac{e^{2\sqrt{\log n}}}{(n+1)(\log n)^{3/4}} \sim \int_{\log x}^{x-1} \frac{e^{2\sqrt{\log t}}}{(t+1)(\log t)^{3/4}}\,dt \sim \frac{e^{2\sqrt{\log x}}}{(\log x)^{1/4}}.$$

Thus, $H_a(x) \sim \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}a(\log x)^{1/4}}$. ∎

Remark 3.10 Let us note that $H_a(e^x) = \sum_{a\leq n\leq e^x} \frac{1}{n} N_a(n) \sim \frac{1}{2\sqrt{\pi}a} x^{-1/4} e^{2\sqrt{x}}$. A curious observation is that the above asymptotic form implies that the function $H_a(e^x)$ satisfies Benford's law. For this purpose, it suffices to verify the sufficient condition with $h = 1$ (see [1, Definition]). The fact that $H_a(e^x)$ satisfies Benford's law

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means that the distribution of its leading significant digit is given by $\log_{10}\left(1+\frac{1}{d}\right)$, $d \in \{1,\dots,9\}$. In particular, the digit 1 appears as the leading significant digit about 30.1% of the time, while the digit 9 occurs with probability less than 5%.

Finally, we study the Dirichlet convolution of N_a, namely, $N_a * \mathbf{1}$, where $\mathbf{1}(n) \equiv 1$, and also the convolution $N_a * \tau$, where $\tau(n) = \sum_{d|n} 1$ is the divisor function.

Theorem 3.11 *Let $a, n \in \mathbb{Z}^+$, $n \geq 2$, and $D_a(n) = \sum_{d|n} N_a(d)$. Then, as $x \to \infty$, we have*

$$\sum_{a\leq n\leq x} D_a(n) \sim \frac{1}{a}\,\frac{x\,e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{1/4}}.$$

Proof Since $N_a(n) = 0$ for $n < a$, we get $D_a(n) = 0$ for $n < a$. We have

$$\sum_{a\leq n\leq x} D_a(n) = \sum_{n\leq x} D_a(n) = \sum_{n\leq x}\sum_{d|n} N_a(d) = \sum_{d\leq x} N_a(d) \sum_{\substack{n\leq x\\ d|n}} 1 = \sum_{d\leq x} N_a(d)\left\lfloor \frac{x}{d}\right\rfloor$$

$$= \sum_{a\leq d\leq x} N_a(d)\left\lfloor \frac{x}{d}\right\rfloor = x \sum_{a\leq d\leq x} \frac{N_a(d)}{d} - \sum_{a\leq d\leq x} N_a(d)\left\{\frac{x}{d}\right\}.$$

By Theorem 3.9, we have

$$x \sum_{a\leq d\leq x} \frac{N_a(d)}{d} = xH_a(x) \sim \frac{1}{a}\,\frac{x\,e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{1/4}}.$$

For the remainder, since $0 \leq \{x/d\} \leq 1$, and by Remark 3.3,

$$0 \leq \sum_{a\leq d\leq x} N_a(d)\left\{\frac{x}{d}\right\} \leq \sum_{a\leq d\leq x} N_a(d) = S_a(x) = O\left(\frac{1}{a}\,\frac{x\,e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{3/4}}\right).$$

■

Theorem 3.12 *Let $a, n \in \mathbb{Z}^+$, $n \geq 2$, and $T_a(n) = (N_a * \tau)(n) = \sum_{d|n} N_a(d)\,\tau\left(\frac{n}{d}\right)$. Then, as $x \to \infty$, we have $\sum_{a\leq n\leq x} T_a(n) = O\left(x\,e^{2\sqrt{\log x}}(\log x)^{3/4}\right)$.*

Proof As $N_a(n) = 0$ for $n < a$, it follows that $T_a(n) = 0$ for $n < a$. Thus,

$$\sum_{a\leq n\leq x} T_a(n) = \sum_{n\leq x} T_a(n) = \sum_{n\leq x}\sum_{d|n} N_a(d)\,\tau\left(\frac{n}{d}\right) = \sum_{d\leq x} N_a(d) \sum_{m\leq x/d} \tau(m).$$

We have $\sum_{m\leq y} \tau(m) = O(y\log y)$, hence,

$$\sum_{a\leq n\leq x} T_a(n) = O\left(\sum_{d\leq x} N_a(d)\left(\tfrac{x}{d}\log\tfrac{x}{d}\right)\right) = O\left(x\log x \sum_{d\leq x} \tfrac{N_a(d)}{d}\right)$$

$$= O\left(x\log x H_a(x)\right) = O\left(x\,e^{2\sqrt{\log x}}(\log x)^{3/4}\right).$$

■

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Faculty of Computer Science and Mathematics, Cracow University of Technology, Poland
e-mail: maciej.zakarczemny@pk.edu.pl

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