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ALGEBRAS WHICH REPRESENT THEIR LINEAR FUNCTIONALS

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1. This paper was originally intended to contain a generalization of a theorem of Banach on the extension of linear functionals. This generalized theorem now appears as a by-product of a study of a class of algebras which we believe to be of much greater interest than the theorem itself. Let X be a vector space over the real field $\mathfrak{F}$ and let $\pi(x)$ be a sub-additive, positive-homogeneous functional on X. Banach ((2), pp. 27–9) proves that any real linear† functional f on a subspace X_0 of X which satisfies $f(x) \leqslant \pi(x)$ on X_0 can be extended to a real linear functional F on X with $F(x) \leqslant \pi(x)$ on X. One of the essential differences between this theorem and the Hahn-Banach theorem is that π can take negative values. Suppose now that X is a vector space over the field of complex numbers, X_0 a complex subspace, and f a complex linear functional on X_0 such that $\Re\{f(x)\} \leqslant \pi(x)$ on X_0. Then it may easily be deduced from Banach's theorem that there exists a complex linear functional $F(x)$ on X such that $F(x) = f(x)$ on X_0 and $\Re\{F(x)\} \leqslant \pi(x)$ on X. The question then naturally arises: given a linear associative algebra $\mathfrak{H}$ (not necessarily of finite dimension) over $\mathfrak{F}$ and a real functional on $\mathfrak{H}$ analogous to the real part $\Re$ on the algebra of complex numbers, can we prove a similar extension theorem for $\mathfrak{H}$-linear functionals‡ on a vector space over $\mathfrak{H}$?

The situation envisaged here may arise whenever we consider an algebra of linear operators on a real vector space, and hence is by no means artificial. It may be well to say something of the differences between the extension theorems proved in the present paper and the generalizations of the Hahn-Banach theorem due to Nachbin (6) and others. These generalized Hahn-Banach theorems are concerned with the extension, with norm preservation, of transformations T from one vector space X into another Y, T being additive and real or complex homogeneous, and not, as in our situation, homogeneous with respect to Y. This difference is reflected in the content of our theorems being essentially algebraic rather than topological.

A real functional τ on $\mathfrak{H}$ which is suitably analogous to the real part of a complex number is here called a *trace*, since it has properties similar to those of the trace in an

† By linear we mean additive and homogeneous, not necessarily continuous.

‡ Additive transformations from X into $\mathfrak{H}$ which are homogeneous with respect to $\mathfrak{H}$, that is, $f(hx) = hf(x)$ for all $h \in \mathfrak{H}$.

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algebra of matrices. The property of τ in which we are chiefly interested is its capacity to represent real linear functionals on $\mathfrak{H}$ in the form $\tau(ha)$ with a fixed a in $\mathfrak{H}$. If $\mathfrak{H}$ is normed and τ represents all continuous real linear functionals (if $\mathfrak{H}$ has finite dimension, we interpret this as representing *all* real linear functionals), then τ is called an *effective* trace. The existence of an effective trace implies an interesting relation between $\mathfrak{H}$ and the space $\mathfrak{H}^*$ of all continuous linear functionals on $\mathfrak{H}$. It embodies a representation theorem for linear functionals and at the same time makes it possible to use the known properties of linear functionals to study the structure of the algebra.

The concept of trace is defined in §2 and its elementary properties are discussed in §3. In §4 we study the case when $\mathfrak{H}$ is of finite dimension and show that in this case the trace algebras with effective trace are precisely the Frobenius algebras over $\mathfrak{F}$. This connexion between a Frobenius algebra and the linear functionals over it does not appear to have been explicitly stated before. The generalized extension theorem is proved in §5 in the case when $\mathfrak{H}$ is a finite-dimensioned algebra with effective trace.

Our results on algebras of infinite dimension with effective trace are much less complete. In §7 we prove that if an algebra has both left and right effective traces then it possesses duality in the following sense. If S_l and S_r denote the left and right annihilators of a set S, then $(I_l)_r = I$ and $(J_r)_l = J$ for each closed right ideal I and each closed left ideal J. The structure of algebras with such duality has been determined by Kaplansky (3, 4). In §8 we give some examples of algebras of infinite dimension with effective trace and in §9 we prove an extension theorem.

2. We consider a linear associative algebra $\mathfrak{H}$, possibly of infinite dimension, over the real field $\mathfrak{F}$. Let $\mathfrak{K}$ be a subspace of the vector space $\mathfrak{H}$ over $\mathfrak{F}$. The set of elements ak, $a \in \mathfrak{H}$, $k \in \mathfrak{K}$, is denoted by $\mathfrak{H}.\mathfrak{K}$. Although admitting the elements of $\mathfrak{H}$ as left multipliers (since $a(bk) = (ab)k$), $\mathfrak{H}.\mathfrak{K}$ need not be a left ideal of $\mathfrak{H}$, as it need not be closed under addition.

A real functional $\tau(= \tau(g))$ defined on a set containing $\mathfrak{H}.\mathfrak{K}$ is called a *right trace over* $\mathfrak{K}$ *if*

$$\tau((\alpha a + \beta b)k) = \alpha\tau(ak) + \beta\tau(bk) \tag{2·1}$$

and

$$\tau(a(\alpha k + \beta k')) = \alpha\tau(ak) + \beta\tau(ak') \tag{2·2}$$

for all $\alpha, \beta \in \mathfrak{F}$; $a, b \in \mathfrak{H}$; $k, k' \in \mathfrak{K}$.

Thus $\tau(ak)$ is a bilinear form having the special property that $\tau(ak) = \tau(a'k')$ whenever $ak = a'k'$.

We denote by τ_k the functional $\tau_k(a) = \tau(ak)$, whenever this is defined for all $a \in \mathfrak{H}$. Let $\mathfrak{K}^{\circledast}$ be a given subspace of the space $\mathfrak{F}(\mathfrak{H})$ of all real linear functionals on $\mathfrak{H}$. A right trace τ over $\mathfrak{K}$ is said to *represent* $\mathfrak{K}^{\circledast}$ *over* $\mathfrak{K}$ if

(i) there is a $(1-1)$ correspondence $\phi \leftrightarrow k$ between $\mathfrak{K}^{\circledast}$ and $\mathfrak{K}$ such that $\phi = \tau_k$; (2·3)

(ii) $j \in \mathfrak{K}$ whenever $\tau_j \in \mathfrak{K}^{\circledast}$. (2·4)

A right trace τ is said to be *right non-singular* if $\tau_k = 0$ implies $k = 0$. Since the zero element is in every subspace and since $\tau(a0) = 0$, we obtain at once

LEMMA 1. *A right trace τ which represents a subspace $\mathfrak{K}^{\circledast}$ of $\mathfrak{F}(\mathfrak{H})$ over a subspace $\mathfrak{K}$ is right non-singular.*

The concepts of *left trace* and so on are to be defined in a similar way. If τ is both a right and left trace representing $\mathfrak{K}^{\circledast}$ (or over $\mathfrak{K}$), then we say that τ is a *two-sided trace*

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representing $\mathfrak{R}^*$ (or *over* $\mathfrak{R}$). It is of course possible for $\mathfrak{H}$ to have a left trace and a right trace, each representing $\mathfrak{R}^*$, but which are distinct.

3. We now study the structure of an algebra $\mathfrak{H}$ with a right trace τ representing all linear functionals on $\mathfrak{H}$; in other words, $\mathfrak{R}^*$ is taken to be $\mathfrak{F}(\mathfrak{H})$.

THEOREM 1. *Let $\mathfrak{H}$ have a right trace representing $\mathfrak{F}(\mathfrak{H})$ over $\mathfrak{R}$. Then $\mathfrak{H}$ has finite dimension and $\mathfrak{R} = \mathfrak{H}$.*

Proof. $\mathfrak{F}(\mathfrak{H})$ and $\mathfrak{R}$ are isomorphic vector spaces, and hence have the same dimension† over $\mathfrak{F}$. If the dimension $\aleph$ of $\mathfrak{H}$ is infinite then that of $\mathfrak{F}(\mathfrak{H})$ is $2^{\aleph}$ (Mackey (5)), so that $\mathfrak{F}(\mathfrak{H})$ cannot be isomorphic to any subspace of $\mathfrak{H}$. Since $\aleph$ is therefore finite, $\mathfrak{H}$ and $\mathfrak{F}(\mathfrak{H})$ have the same dimension. It follows at once that $\mathfrak{R} = \mathfrak{H}$.

LEMMA 2. *Let τ be a non-singular left trace over $\mathfrak{H}$. If $a_1, \ldots, a_r$ are linearly independent elements of $\mathfrak{H}$, there exists $b \in \mathfrak{H}$ such that $\tau(a_1 b), \ldots, \tau(a_r b)$ are any r given real numbers $\gamma_1, \ldots, \gamma_r$.*

Proof. When $r = 1$ we are given $a_1 \neq 0$. Hence c exists such that $\tau(a_1 c) \neq 0$, and we take $b = \dfrac{\gamma_1}{\tau(a_1 c)} c$.

Now assume the lemma to be true for all choices of $r-1$ elements from $a_1, \ldots, a_r$. For each j in $1, \ldots, r-1$ we choose b_j such that

$$\tau(a_j b_j) = 1, \quad \tau(a_i b_j) = 0 \; (i = 1, \ldots, r-1, \text{ but } i \neq j).$$

Let $\tau(a_r b_j) = \beta_j$ and $a = a_r - \beta_1 a_1 - \ldots - \beta_{r-1} a_{r-1}$. Now $a \neq 0$, since $a_1, \ldots, a_r$ are linearly independent. Hence there exists c with $\tau(ac) = 1$.

Let $\tau(a_i c) = \alpha_i$ for $i = 1, \ldots, r-1$ and let $h_r = c - \alpha_1 b_1 - \ldots - \alpha_{r-1} b_{r-1}$.

Then
$$\tau(a_i h_r) = \tau(a_i c) - \alpha_1 \tau(a_i b_1) - \ldots - \alpha_{r-1} \tau(a_i b_{r-1}),$$
$$= 0 \text{ for } i = 1, \ldots, r-1.$$

Also
$$\tau(a_r h_r) = \tau(a_r c) - \alpha_1 \beta_1 - \ldots - \alpha_{r-1} \beta_{r-1} = \tau(ac) = 1.$$

In a similar way we can construct $h_j (j = 1, \ldots, r-1)$ such that $\tau(a_j h_j) = 1, \tau(a_i h_j) = 0$ $(i = 1, \ldots, r$, but $i \neq j)$. The element $b = \gamma_1 h_1 + \ldots + \gamma_r h_r$ satisfies the conditions of the lemma.

THEOREM 2. *$\mathfrak{H}$ has a two-sided trace τ representing $\mathfrak{F}(\mathfrak{H})$ if and only if $\mathfrak{H}$ has finite dimension over $\mathfrak{F}$ and τ is a non-singular two-sided trace over $\mathfrak{H}$.*

Proof. We have already proved that when $\mathfrak{H}$ has a two-sided trace representing $\mathfrak{F}(\mathfrak{H})$ then the other conditions are satisfied.

If these conditions are satisfied with τ as a non-singular two-sided trace over $\mathfrak{H}$, then by the previous lemma there exists, for a given basis $a_1, \ldots, a_n$ of $\mathfrak{H}$ and a given $\phi(a) \in \mathfrak{F}(\mathfrak{H})$, an element $b \in \mathfrak{H}$ such that

$$\tau(a_i b) = \phi(a_i) \quad (i = 1, \ldots, n).$$

It follows at once that $\tau(ab) = \phi(a)$ for all $a \in \mathfrak{H}$. Moreover, b is unique for a given $\phi(a)$ since τ is non-singular.

† The dimension of a vector space is the cardinal number of any Hamel basis of the space. See Mackey ((5), pp. 157–9).

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THEOREM 3. *If $\mathfrak{H}$ has finite dimension then every non-singular right trace (left trace) over $\mathfrak{H}$ is a non-singular two-sided trace over $\mathfrak{H}$.*

Proof. Let $a_1, \ldots, a_n$ be a basis for $\mathfrak{H}$ and τ be a non-singular right trace over $\mathfrak{H}$. If $a = \sum_{i=1}^{n} \alpha_i a_i$ and $b = \sum_{j=1}^{n} \beta_j b_j$, then $\tau(ab) = \sum_{i,j=1}^{n} \tau_{ij}\alpha_i\beta_j$, where $\tau_{ij} = \tau(a_i a_j)$. We are given that if $\tau(ab) = 0$ for all a, then $b = 0$. This is equivalent to the assertion that when $\sum_{j=1}^{n} \tau_{ij}\beta_j = 0$ for all $i = 1, \ldots, n$, then $\beta_j = 0$ for all $j = 1, \ldots, n$. In other words the matrix (τ_{ij}) is non-singular. Similarly, to say that τ is a non-singular left trace over $\mathfrak{H}$ is to assert that the transposed matrix (τ_{ji}) is non-singular. Since (τ_{ij}) and (τ_{ji}) must be singular or non-singular together, our theorem is proved.

Theorems 2 and 3 reduce the problem of determining the algebras with right (or left, or two-sided) trace representing $\mathfrak{F}(\mathfrak{H})$ to that of determining the algebras of finite dimension over $\mathfrak{F}$ with a non-singular right trace. We now prove that these are precisely the Frobenius algebras over $\mathfrak{F}$.

4. We now require some properties of Frobenius algebras. Let A be a linear associative algebra of finite dimension n over $\mathfrak{F}$, and $a_1, \ldots, a_n$ be a basis of A such that

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k \quad (i, j = 1, \ldots, n). \tag{4·1}$$

For any real vector $(\lambda_1, \ldots, \lambda_n)$ the parastrophic matrix P_λ is defined by

$$P_\lambda = (p_{ij}) \quad \text{where} \quad p_{ij} = \sum_{k=1}^{n} \gamma_{ijk}\lambda_k. \tag{4·2}$$

A is a Frobenius algebra if P_λ is a non-singular matrix for some basis $a_1, \ldots, a_n$ and some vector $(\lambda_1, \ldots, \lambda_n)$. If the basis is changed there is a non-singular parastrophic matrix with respect to the new basis, but the vector $(\lambda_1, \ldots, \lambda_n)$ alters.

Another characterization of these algebras is as follows. Let L be a left ideal of A and L_r be the right annihilator of L in A, that is, the set of elements $b \in A$ such that $Lb = 0$. Clearly L_r is a right ideal of A. If R is any right ideal of A then the left annihilator R_l of R is defined analogously, and is a left ideal. Then A is a Frobenius algebra if and only if

$$(L : \mathfrak{F}) + (L_r : \mathfrak{F}) = (A : \mathfrak{F}) = (R : \mathfrak{F}) + (R_l : \mathfrak{F}), \tag{4·3}$$

for every left ideal L and every right ideal R.

Condition (4·3) implies that the following equations hold;

$$(L_r)_l = L, \quad (R_l)_r = R. \tag{4·4}$$

Finally, we should add that a Frobenius algebra always has a unit element (Nakayama (7)).

If $\mathfrak{H}$ has finite dimension over $\mathfrak{F}$, a trace representing $\mathfrak{F}(\mathfrak{H})$ will be called an *effective trace*. This agrees with the more general definition introduced in § 6.

THEOREM 4. *An algebra with finite dimension over $\mathfrak{F}$ has an effective right trace if and only if it is a Frobenius algebra.*

Proof. Let $\mathfrak{H}$ be a Frobenius algebra and $a_1, \ldots, a_n$ be a basis of $\mathfrak{H}$ over $\mathfrak{F}$, satisfying (4·1). If $(\lambda_1, \ldots, \lambda_n)$ is any vector such that the parastrophic matrix P_λ (see

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(4·2)) is non-singular, then define $\tau(a) = \sum_{i=1}^{n} \alpha_i \lambda_i$, where $a = \sum_{i=1}^{n} \alpha_i a_i$ is any element of $\mathfrak{H}$. Evidently τ is a right trace over $\mathfrak{H}$. If $b = \sum_{j=1}^{n} \beta_j a_j$, then $ab = \sum_{i,j,k=1}^{n} \alpha_i \beta_j \gamma_{ijk} a_k$ and $\tau(ab) = \sum_{i,j,k=1}^{n} \alpha_i \beta_j \gamma_{ijk} \lambda_k$. Suppose that $\tau(ab) = 0$ for all $a \in \mathfrak{H}$. Then $\sum_{j,k=1}^{n} \gamma_{ijk} \lambda_k \beta_j = 0$ for $i = 1, \ldots, n$. This means that the non-singular matrix P_λ annihilates $(\beta_1, \ldots, \beta_n)$ which must therefore be the zero vector. Hence $b = 0, \tau$ is a non-singular right trace over $\mathfrak{H}$ and hence is an effective right trace.

Conversely, let $\mathfrak{H}$ have finite dimension n over $\mathfrak{F}$ and let τ be an effective right trace. Then $\tau(ab) = \sum_{i,j=1}^{n} \tau_{ij} \alpha_i \beta_j$, where $a = \sum_{i=1}^{n} \alpha_i a_i$, $b = \sum_{j=1}^{n} \beta_j a_j$, and $\tau_{ij} = \tau(a_i a_j)$. In proving Theorem 3, we showed that the matrix (τ_{ij}) is non-singular. Let $\mathfrak{L}$ be a left ideal of $\mathfrak{H}$ and denote by $\mathfrak{L}'$ the set of elements $b \in \mathfrak{H}$ such that $\tau(ab) = 0$ for all $a \in \mathfrak{L}$. Since (τ_{ij}) is non-singular, we have

$$(\mathfrak{L} : \mathfrak{F}) + (\mathfrak{L}' : \mathfrak{F}) = (\mathfrak{H} : \mathfrak{F}).$$

However, as $\mathfrak{L}$ is a left ideal we have $\tau(hab) = 0$, where $h \in \mathfrak{H}$, $a \in \mathfrak{L}$ and $b \in \mathfrak{L}'$. As τ is right non-singular this means that $ab = 0$. Conversely, if $ab = 0$ then $\tau(ab) = 0$. Thus $\mathfrak{L}' = \mathfrak{L}_r$ and the first part of (4·3) is proved. Likewise the other part of (4·3) can be proved and hence $\mathfrak{H}$ is a Frobenius algebra.

COROLLARY. *An algebra of finite dimension with an effective right trace has a unit element.*

Finally, we may remark that when an algebra $\mathfrak{H}$ of finite dimension over $\mathfrak{F}$ has an effective trace the method used in proving the last theorem enables us to obtain every trace which is effective for $\mathfrak{H}$. To do this we put

$$\tau(a) = \sum_{i=1}^{n} \alpha_i \lambda_i \quad \text{where} \quad a = \sum_{i=1}^{n} \alpha_i a_i,$$

and $(\lambda_1, \ldots, \lambda_n)$ is *any* real vector such that P_λ is non-singular.

5. We now apply the theory of trace algebras to obtain a generalization of Banach's theorem ((2), pp. 27–9) on the extension of real linear functionals.

Let $\mathfrak{H}$ be a Frobenius algebra over $\mathfrak{F}$ with unit element e and X be a vector space over both $\mathfrak{F}$ and $\mathfrak{H}$. We assume in this connexion that

$$ex = x \quad \text{and} \quad \alpha(ax) = (\alpha a)x = a(\alpha x), \quad \text{whenever} \quad \alpha \in \mathfrak{F}, \quad a \in \mathfrak{H}, \quad x \in X.$$

A subspace of the space X over $\mathfrak{H}$ is said to be an $\mathfrak{H}$-*subspace of* X. A mapping $f(x)$ of an $\mathfrak{H}$-subspace X_0 into $\mathfrak{H}$ is an $\mathfrak{H}$-*linear functional on* X_0 if

$$\text{(i)} \quad f(x+y) = f(x) + f(y) \quad (x, y \in X_0)$$

and

$$\text{(ii)} \quad f(ax) = af(x) \quad (a \in \mathfrak{H}).$$

We take $\pi(x)$ to be a real-valued (but not necessarily positive) sub-additive and positive-homogeneous functional on X. In other words

$$\pi(x+y) \leqslant \pi(x) + \pi(y), \quad \pi(\alpha x) = \alpha \pi(x) \quad \text{if} \quad \alpha \geqslant 0.$$

THEOREM 5. *Let τ be an effective trace and $f(x)$ be an $\mathfrak{H}$-linear functional on an $\mathfrak{H}$-subspace X_0 of X, having the property*

$$\tau(f(x)) \leqslant \pi(x) \quad \text{if} \quad x \in X_0.$$

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Then there exists an $\mathfrak{H}$-linear functional $F(x)$ on X such that

$$(1)\ F(x) = f(x) \qquad (x \in X_0);$$

$$(2)\ \tau(F(x)) \leqslant \pi(x) \quad (x \in X).$$

Proof. By Banach's theorem applied to real linear functionals on X, there is a real linear functional $T(x)$ on X such that

$$\text{(i)}\ T(x) = \tau(f(x)) \quad (x \in X_0),$$

$$\text{(ii)}\ T(x) \leqslant \pi(x) \qquad (x \in X).$$

For fixed x, $T(ax)$ is a real linear functional on $\mathfrak{H}$ and hence can be represented in the form

$$T(ax) = \tau(ah_x) \quad (a \in \mathfrak{H}). \tag{5·1}$$

Clearly h_x is uniquely determined by x, and setting $a = e$ we have $T(x) = \tau(h_x)$. Moreover

$$\tau(a(h_{bx} - bh_x)) = T(a(bx)) - T((ab)x) = 0,$$

for all $a, b \in \mathfrak{H}$, and $x \in X$. As τ is non-singular we obtain

$$h_{bx} = bh_x \quad (b \in \mathfrak{H}, x \in X). \tag{5·2}$$

Also

$$\tau(a(h_{x+y} - h_x - h_y)) = T(a(x+y)) - T(ax) - T(ay) = 0,$$

so that

$$h_{x+y} = h_x + h_y \quad (x, y \in X). \tag{5·3}$$

Now define $F(x) = h_x$ for all $x \in X$. $F(x)$ is an $\mathfrak{H}$-linear functional by (5·2) and (5·3). Also, by (i) and (ii),

$$\tau(F(x)) = \tau(h_x) = T(x) \leqslant \pi(x).$$

Finally, if $x \in X_0$,

$$\tau(a(F(x) - f(x))) = \tau(F(ax) - f(ax)) = T(ax) - T(ax) = 0,$$

using (i) and (ii). Since τ is non-singular, we obtain $F(x) = f(x)$ for $x \in X_0$, and the theorem is proved.

6. We have proved (Theorem 1) that if the class $\mathfrak{R}^{\circledast}$ of linear functionals represented by τ consists of *all* real linear functionals on $\mathfrak{H}$, then the algebra $\mathfrak{H}$ must be of finite dimension. If $\mathfrak{H}$ has infinite dimension it is therefore necessary to restrict the class $\mathfrak{R}^{\circledast}$. The natural restriction is to suppose that $\mathfrak{H}$ (considered as a vector space) is normed and to take the class $\mathfrak{H}^*$ of all *continuous* real linear functionals on $\mathfrak{H}$ as $\mathfrak{R}^{\circledast}$. In the remainder of this paper we shall make this restriction, and it will therefore be convenient to redefine the concept of effectiveness.

Let $\mathfrak{H}$ be a linear associative algebra over the real field $\mathfrak{F}$, which is also a normed vector space over $\mathfrak{F}$. A right trace τ over $\mathfrak{R} \subset \mathfrak{H}$ (2·1, 2·2) will be called an *effective right trace* if the following holds: given any continuous linear functional ϕ, there exists a unique $k \in \mathfrak{R}$ such that $\phi(a) = \tau(ak)$ for all $a \in \mathfrak{H}$, and $\tau_c (= \tau(ac))$ is continuous if and only if $c \in \mathfrak{R}$. In other words, we require (2·3) and (2·4) to hold when $\mathfrak{R}^{\circledast}$ is the space of all continuous (i.e. bounded) real linear functionals on $\mathfrak{H}$. In these circumstances $\mathfrak{R}$ will be called an *effective subspace*. Effective left and two-sided traces are similarly defined.

We do not at present require that the product operation in $\mathfrak{H}$ be continuous nor that $\mathfrak{H}$ be complete, but it is to be expected that from our standpoint the most interesting algebras of infinite dimension will be those which are also Banach algebras. In §8 we bexhiit examples of algebras of infinite dimension having effective traces.

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7. THEOREM 6. *Let τ be an effective right trace with effective subspace $\mathfrak{K}$ and let the operation $x \to xb$ be continuous in $\mathfrak{H}$ for each $b \in \mathfrak{H}$. Then $\mathfrak{K}$ is a left ideal.*

Proof. For each $k \in \mathfrak{K}$ and $b \in \mathfrak{H}$ we have

$$\tau_{bk}(x) = \tau(x(bk)) = \tau((xb)\,k) = \tau_k(xb).$$

Since τ_k and the operation $x \to xb$ are both continuous linear operations, $\tau_{bk} \in \mathfrak{H}^*$. It follows from (2·4) that $bk \in \mathfrak{K}$. Since $\mathfrak{K}$ is, by hypothesis, a linear manifold the theorem is proved.

It is interesting to notice a partial converse of Theorem 6.

THEOREM 7. *If an algebra $\mathfrak{H}$ is complete and possesses an effective right trace with a left ideal $\mathfrak{K}$ as effective subspace, then the operation $x \to xb$ is continuous in $\mathfrak{H}$ for each $b \in \mathfrak{H}$.*

Proof. This depends on a theorem of Banach ((2), p. 42). Since $\mathfrak{K}$ is a left ideal, the functional $\phi(x) = \tau_k(xb) = \tau_{bk}(x)$ is continuous for every $k \in \mathfrak{K}$ and $b \in \mathfrak{H}$. Since all continuous linear functionals are of the form τ_k and since $b \to xb$ is a linear operation, the theorem is proved.

THEOREM 8. *If an algebra $\mathfrak{H}$ possesses an effective right trace then $(\mathfrak{J}_r)_l = \mathfrak{J}$ for every closed left ideal $\mathfrak{J}$ in $\mathfrak{H}$.*

Proof. Let $\mathfrak{J}$ be a closed left ideal in $\mathfrak{H}$, then evidently $(\mathfrak{J}_r)_l \supset \mathfrak{J}$. Suppose that $a_0 \notin \mathfrak{J}$, then by a theorem of Banach ((2), p. 57) there exists a continuous linear functional ϕ such that $\phi(a_0) = 1$, but $\phi(i) = 0$ for all $i \in \mathfrak{J}$. Let τ be an effective right trace, then there exists k in $\mathfrak{H}$ such that $\phi = \tau_k$. Thus $\tau(ik) = 0$ for all $i \in \mathfrak{J}$. Since $\mathfrak{J}$ is a left ideal it follows that $\tau(aik) = 0$ for all $a \in \mathfrak{H}$ and all $i \in \mathfrak{J}$. By Lemma 1 it follows that $ik = 0$ and hence $k \in \mathfrak{J}_r$. But $\tau(a_0 k) = \phi(a_0) = 1$ and therefore $a_0 k \neq 0$. Thus $a_0 \notin (\mathfrak{J}_r)_l$.

We say that an algebra has *duality* if $(\mathfrak{J}_r)_l = \mathfrak{J}$ and $(\mathfrak{R}_l)_r = \mathfrak{R}$ for every closed left ideal $\mathfrak{J}$ and closed right ideal $\mathfrak{R}$. From Theorem 8 we may at once deduce the

COROLLARY. *If an algebra $\mathfrak{H}$ possesses both a left and a right effective trace then it has duality.*†

8. *Example* 1. Let $\mathfrak{H}$ be the set of all sequences $a = \{\alpha_n\}$ of real numbers for which $\lim_{n\to\infty} \alpha_n = 0$. Let

$$\|a\| = \sup_{1 \leqslant n < \infty} |\alpha_n|, \quad a+b = \{\alpha_n + \beta_n\}, \quad \lambda a = \{\lambda \alpha_n\}, \quad ab = \{\alpha_n \beta_n\} \quad \text{and} \quad \tau(a) = \sum_{n=1}^{\infty} \alpha_n.$$

Let $\mathfrak{K} \subset \mathfrak{H}$ be the set of elements a such that $\sum_{n=1}^{\infty} |\alpha_n| < \infty$. It is clear that $\mathfrak{H}$ is a commutative Banach algebra over $\mathfrak{F}$ and that τ is a trace over $\mathfrak{K}$. Considered as a normed vector space, $\mathfrak{H}$ is a subspace of the Banach space of all convergent sequences. Hence if ϕ is any continuous real linear functional on $\mathfrak{H}$ we may conclude ((2), p. 66) that

$$\phi(a) = \sum_{n=1}^{\infty} \alpha_n \gamma_n \quad \text{with} \quad \sum_{n=1}^{\infty} |\gamma_n| < \infty.$$

It follows that $c = \{\gamma_n\} \in \mathfrak{K}$ and that

$$\phi(a) = \tau(ac) \quad \text{for all} \quad a \in \mathfrak{H}.$$

The uniqueness of the element c is readily established, for if $\tau(ab) = 0$ for all $a \in \mathfrak{H}$, then $\tau(bb) = \sum_{n=1}^{\infty} \beta_n^2 = 0$, and this implies $b = 0$. Also $\tau_k (= \tau(ak))$ is a continuous linear

† See Kaplansky (3, 4).

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functional if and only if $k \in \mathfrak{K}$. Thus τ is an effective two-sided trace with $\mathfrak{K}$ as its effective subspace.

Example 2. Let $\mathfrak{T}$ be the algebra (Stone (8), p. 66) of bounded transformations T, with finite norm $N(T)$, of a Hilbert space H. If $\{h_\alpha\}$ is any complete orthonormal sequence in H,

$$N(T) = \left\{ \sum_{\alpha=1}^{\infty} \| Th_\alpha \|^2 \right\}^{\frac{1}{2}}, \quad Q(T_1, T_2) = \sum_{\alpha=1}^{\infty} (T_1 h_\alpha, T_2 h_\alpha).\dagger$$

If $T \in \mathfrak{T}$ so does its adjoint T^*, and we may set

$$\tau(T_1 T_2) = \Re Q(T_2, T_1^*),$$

with $\Re$ denoting the real part, provided that we can show that $\tau(T_1 T_2)$ depends only on $T_1 T_2$ and not on its separate factors. This follows at once from the definition of the adjoint, for

$$Q(T_2, T_1^*) = \sum_{\alpha=1}^{\infty} (T_2 h_\alpha, T_1^* h_\alpha) = \sum_{\alpha=1}^{\infty} (T_1 T_2 h_\alpha, h_\alpha).$$

If $\phi(T)$ is a continuous real linear functional on $\mathfrak{T}$ then $\psi(T) = \phi(T) - i\phi(iT)$ is a continuous complex linear functional on $\mathfrak{T}$, and, since $\mathfrak{T}$ is a Hilbert space with bilinear form Q, can be represented in the form

$$\psi(T) = Q(T, T_0) \quad \text{with} \quad T_0 \in \mathfrak{T}.$$

Thus

$$\phi(T) = \tau(T_0^* T).$$

The uniqueness of T_0^* follows from the corresponding property of Q, and so τ is an effective left trace with $\mathfrak{T}$ as its effective subspace.

Example 3. Let $\mathfrak{A}$ be the Banach algebra of all bounded linear transformations T of a Hilbert space H into itself with the bound of T taken as its norm, i.e.

$$\| T \| = \sup_{\| h \| = 1} \| Th \|.$$

Let $\mathfrak{T}$ be the subalgebra of $\mathfrak{A}$ considered in Example 2, and let $\mathfrak{H} = \overline{\mathfrak{T}}$ be its closure with respect to $\| \ \|$. We exhibit an effective trace for the algebra $\mathfrak{H}$.

Since any element in H with norm 1 can be taken as the first element of an orthonormal sequence, we have

$$\| T \| \leqslant N(T),$$

that is, the topology in $\mathfrak{T}$ induced by $\| \ \|$ is coarser than that induced by N. Consequently, any complex linear functional continuous on $\mathfrak{H}$ with respect to $\| \ \|$ is also continuous on $\mathfrak{T}$ with respect to N. Thus there exists $T_0 \in \mathfrak{T}$ such that

$$\phi(T) = Q(T, T_0), \quad \text{all } T \in \mathfrak{T}. \tag{8·1}$$

Now let $\mathfrak{K} \subset \mathfrak{T}$ be the set of all $T_1 \in \mathfrak{T}$ such that there exists a constant M_1 with

$$\sum_{\alpha=1}^{\infty} | (Th_\alpha, T_1 h_\alpha) | \leqslant M_1 . \| T \|$$

for all $T \in \mathfrak{A}$ and all complete orthonormal systems $\{h_\alpha\}$ in H. If $T_1 \in \mathfrak{K}$ and $\{h_\alpha\}$ is a complete orthonormal sequence in H then clearly $Q(T, T_1; \{h_\alpha\}) = \sum_{\alpha=1}^{\infty} (Th_\alpha, T_1 h_\alpha)$ is a continuous linear functional on $\mathfrak{A}$. It follows from this continuity that $Q(T, T_1; \{h_\alpha\})$ is independent of the choice of $\{h_\alpha\}$ whenever $T_1 \in \mathfrak{K}$ and $T \in \mathfrak{H} (= \overline{\mathfrak{T}})$. For such T_1 and

† Values of $N(T)$ and $Q(T_1, T_2)$ are independent of the choice of complete orthonormal sequence.

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T we may therefore denote $Q(T, T_1; \{h_\alpha\})$ by $Q(T, T_1)$ without ambiguity. We prove that $T_0 \in \mathfrak{R}$, in fact we have

THEOREM 9. *Every continuous complex linear functional ϕ on the Banach algebra $\mathfrak{H}(= \overline{\mathfrak{T}})$ is uniquely representable in the form $\phi(T) = Q(T, T_0)$ with $T_0 \in \mathfrak{R}$.*

Proof. We show first that T_0 of (8·1) is in $\mathfrak{R}$. Let M be the bound of the functional ϕ, i.e.

$$|\phi(T)| \leqslant M . \|T\| \quad \text{for all} \quad T \in \mathfrak{H}. \tag{8·2}$$

We prove that given an arbitrary orthonormal sequence $\{h_\alpha\}$ in H

$$\sum_{\alpha=1}^{\infty} |(Th_\alpha, T_0 h_\alpha)| \leqslant M . \|T\| \quad \text{for all} \quad T \in \mathfrak{A}.$$

In fact, let $T \in \mathfrak{A}$ and set $\zeta_\alpha = (Th_\alpha, T_0 h_\alpha)$. We define another linear transformation T' by setting

$$T'h_\alpha = \eta_\alpha (Th_\alpha),$$

where

$$\eta_\alpha = \bar{\zeta}_\alpha / |\zeta_\alpha| \quad \text{if } \zeta_\alpha \neq 0,$$
$$= 1 \quad \text{if } \zeta_\alpha = 0.$$

If $x = \sum_{\alpha=1}^{\infty} \xi_\alpha h_\alpha$, then

$$T'x = \sum_{\alpha=1}^{\infty} \xi_\alpha T'h_\alpha = \sum_{\alpha=1}^{\infty} \xi_\alpha \eta_\alpha Th_\alpha = Tx' \quad \text{where} \quad x' = \sum_{\alpha=1}^{\infty} \xi_\alpha \eta_\alpha h_\alpha.$$

Since $|\eta_\alpha| = 1$, $\|x'\| = \|x\|$, and hence

$$\|T'x\| = \|Tx'\| \leqslant \|T\| . \|x'\| = \|T\| . \|x\|.$$

Thus

$$T' \in \mathfrak{A} \quad \text{and} \quad \|T'\| \leqslant \|T\|.$$

Let E_n for each n be the projection operator of H on to the linear manifold spanned by $h_1, \ldots, h_n$, i.e.

$$E_n \sum_{\alpha=1}^{\infty} \xi_\alpha h_\alpha = \sum_{\alpha=1}^{n} \xi_\alpha h_\alpha.$$

Then whenever $T'' \in \mathfrak{A}$ we have $T''E_n \in \mathfrak{T}$. For

$$N^2(T''E_n) = \sum_{\alpha=1}^{\infty} \|T''E_n h_\alpha\|^2 = \sum_{\alpha=1}^{n} \|T''h_\alpha\|^2 \leqslant n \|T''\|^2.$$

It follows that $T_n = T'E_n \in \mathfrak{T}$ and

$$\|T_n\| \leqslant \|T'\| . \|E_n\| \leqslant \|T\|.$$

Now $$Q(T_n, T_0) = \sum_{\alpha=1}^{\infty} (T'E_n h_\alpha, T_0 h_\alpha) = \sum_{\alpha=1}^{n} (T'h_\alpha, T_0 h_\alpha) = \sum_{\alpha=1}^{n} |(Th_\alpha, T_0 h_\alpha)|.$$

But, since $T_n \in \mathfrak{T}$,

$$|Q(T_n, T_0)| = |\phi(T_n)| \leqslant M . \|T_n\| \leqslant M . \|T\|,$$

from which it follows that

$$\sum_{\alpha=1}^{n} |(Th_\alpha, T_0 h_\alpha)| \leqslant M . \|T\|$$

for every n, and the theorem is proved. By restricting the functional to $\mathfrak{T}$ we see that the representing element T_0 is unique and independent of the choice of $\{h_\alpha\}$.

We prove next that if $T_0 \in \mathfrak{R}$ so does T_0^*. In fact, if $T_0 \in \mathfrak{R}$ and $\{h_\alpha\}$ is a complete orthonormal sequence, then

$$\theta(T) = \sum_{\alpha=1}^{\infty} (T_0 h_\alpha, T^* h_\alpha)$$

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is a continuous complex linear functional on $\mathfrak{H}$. Therefore by Theorem 9

$$\theta(T) = Q(T, T_1) = \sum_{\alpha=1}^{\infty} (Th_\alpha, T_1 h_\alpha) \quad \text{with} \quad T_1 \in \mathfrak{R}.$$

Now for all $T \in \mathfrak{T}$, we have

$$\sum_{\alpha=1}^{\infty} (Th_\alpha, T_1 h_\alpha) = \sum_{\alpha=1}^{\infty} (T_0 h_\alpha, T^* h_\alpha) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (T_0 h_\alpha, h_\beta)(h_\beta, T^* h_\alpha)$$
$$= \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (h_\alpha, T_0^* h_\beta)(Th_\beta, h_\alpha).$$

Since T and T_0^* are in $\mathfrak{T}$, the series on the right is absolutely convergent, and changing the order of summation we conclude that

$$\sum_{\alpha=1}^{\infty} (Th_\alpha, T_1 h_\alpha) = \sum_{\beta=1}^{\infty} (Th_\beta, T_0^* h_\beta).$$

Since this holds for all $T \in \mathfrak{T}$ it follows that $T_0^* = T_1 \in \mathfrak{R}$.

We now set $\tau(T_1 T_2) = \Re Q(T_2, T_1^*)$ as in Example 2. Then τ is a left trace over $\mathfrak{R}$. To show that it is effective for the algebra $\mathfrak{H}$ with $\mathfrak{R}$ as its effective subspace it is now enough to prove that if, for some $T_0 \in \mathfrak{H}$, $\tau(T_0 T)$ is a continuous linear functional on $\mathfrak{H}$ then $T_0 \in \mathfrak{R}$. Suppose that $T_0 \in \mathfrak{H}$ and that $Q(T, T_0)$ is continuous on $\mathfrak{H}$. By Theorem 9 there exists $T_1 \in \mathfrak{R}$ such that $Q(T, T_0) = Q(T, T_1)$. Thus for all $T \in \mathfrak{H}$,

$$\sum_{\alpha=1}^{\infty} (Th_\alpha, (T_0 - T_1) h_\alpha) = 0.$$

Since $(T_0 - T_1) \in \mathfrak{H}$, we may set $T = T_0 - T_1$ and it follows at once that $T_0 = T_1$.

Example 4. An interesting class of Banach algebras, which includes our Example 2, has been studied by Ambrose (1). An H^* −algebra is a Banach algebra, with or without unit element, which considered as a vector space is a Hilbert space of arbitrary dimension. In a *proper* H^*-algebra $\mathfrak{H}$ a unique element $a^* \in \mathfrak{H}$ corresponds to each $a \in \mathfrak{H}$ with the properties

$$(xy, z) = (y, x^* z), \quad (yx, z) = (y, zx^*) \tag{8·3}$$

for all $x, y, z \in \mathfrak{H}$.

Provided that (b, a^*) is determined by the product ab, a proper H^*-algebra has a left trace

$$\tau(ab) = \Re(b, a^*).$$

Just as in Example 2, it can be proved that τ is an effective left trace with $\mathfrak{H}$ as its effective subspace. It only remains to prove that (b, a^*) is a function of ab only.

A result given by Ambrose† establishes the existence in $\mathfrak{H}$ of a doubly orthogonal, maximal family $\mathscr{S} = \{e_\alpha\}$ of primitive self-adjoint idempotents. This has the property that for every element $a \in \mathfrak{H}$ there exists a sequence $\{e_n\}$ of elements of $\mathscr{S}$ such that $a = \sum_{i=1}^{\infty} a_i$, where $a_i \in \mathfrak{H} e_i$. By the continuity of the product, we see that

$$a e_k = \sum_{i=1}^{\infty} a_i e_k = a_k \quad \text{since} \quad e_i e_k = \delta_{ik} e_k.$$

Thus $a = \sum_{i=1}^{\infty} a e_i$. From (8·3) we have

$$(be_i, a^* e_j) = (b, a^* e_j e_i) = 0 \quad \text{if} \quad i \neq j,$$

and

$$(be_i, a^* e_i) = (abe_i, e_i).$$

† See (1), p. 378, Theorem (3·3).

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Hence $$(b, a^*) = \left(\sum_{i=1}^{\infty} be_i, \sum_{i=1}^{\infty} a^* e_i\right) = \sum_{i=1}^{\infty} (abe_i, e_i),$$

so that $\tau(ab)$ is a function of ab only.

Example 5. The next two examples are algebras of functions on compact and locally compact groups respectively. To keep our examples as concrete as possible, we consider real-valued functions of a real variable. It is essential in these examples to keep in mind the distinction between functions and classes of equivalent functions, and we shall therefore distinguish them by separate notations.

Let $1 \leqslant p \leqslant 2$ and let $\mathfrak{L}_p$ denote the class of all real-valued measurable functions $f(\xi)$, of a real variable ξ, with period 1, such that

$$\int_0^1 |f(\xi)|^p \, d\xi < \infty.$$

Two functions f, g are said to be *equivalent* if $f(\xi) = g(\xi)$ almost everywhere. Let $\mathfrak{H}_p$ be the algebra of equivalence classes $\{f\}$ of such functions, with the following definitions of the operations:

$$\{f\} + \{g\} = \{f(\xi) + g(\xi)\}, \quad \alpha\{f\} = \{\alpha f(\xi)\}, \quad \{f\}\{g\} = \{(f * g)\},$$

where $(f*g)$ is the convolution of f and g, that is,

$$(f*g)(\xi) = \int_0^1 f(\eta)\, g(\xi - \eta)\, d\eta.$$

Finally, let the norm in $\mathfrak{H}_p$ be

$$\|\{f\}\| = \left\{\int_0^1 |f(\xi)|^p \, d\xi\right\}^{1/p}.$$

It is known that the convolution of two functions in $\mathfrak{L}_p$ is in $\mathfrak{L}_p$ and that if $f \in \mathfrak{L}_p$ and $g \in \mathfrak{L}_q$, where $1/p + 1/q = 1$, then $(f*g)(\xi)$ is *continuous*. Also $(f*g)(\xi)$ depends only on the classes $\{f\}$, $\{g\}$ and not on the particular representatives f, g. We notice that, with our restriction on p, $\mathfrak{H}_q \subset \mathfrak{H}_p$ and define τ on $\mathfrak{H}_p \cdot \mathfrak{H}_q$ by setting

$$\tau(\{f\}\{g\}) = (f*g)(0) = \int_0^1 f(\eta)\, g(-\eta)\, d\eta.$$

We have to show that τ depends only on the product $\{f\}\{g\}$. Suppose then that $\{f\}\{g\} = \{f'\}\{g'\}$, then $(f*g)(\xi) = (f'*g')(\xi)$ almost everywhere. But $(f*g)$ and $(f'*g')$ are continuous functions and hence, being equal almost everywhere, are identical. Thus $(f*g)(0) = (f'*g')(0)$.

It is well known (Banach (2), p. 64) that every continuous real linear functional on the space $\mathfrak{H}_p$ is of the form

$$\phi(\{f\}) = \int_0^1 f(\xi)\, g(\xi)\, d\xi,$$

with $g \in \mathfrak{L}_q$, and that such functionals are continuous if and only if $g \in \mathfrak{L}_q$. Thus

$$\phi(\{f\}) = \tau(\{f\}\{g^*\}), \quad \text{where} \quad g^*(\xi) = g(-\xi).$$

$\mathfrak{H}_p$ is a commutative trace algebra with an effective trace τ with $\mathfrak{H}_q$ as its effective subspace.

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Example 6. Let $\mathfrak{H}$ be the algebra of classes $\{f\}$ of equivalent real-valued functions f such that $\int_{-\infty}^{\infty} |f(\xi)|\,d\xi < \infty$, with product defined by convolution

$$(f*g)(\xi) = \int_{-\infty}^{\infty} f(\eta)\,g(\xi-\eta)\,d\eta.$$

If we take $\int_{-\infty}^{\infty} |f(\xi)|\,d\xi$ as the norm in $\mathfrak{H}$ then the trace $\tau(\{f\}\{g\}) = (f*g)(0)$ is not effective, for the general linear functional, continuous with respect to this norm, is of the form $\int_{-\infty}^{\infty} f(\xi)\,g(\xi)\,d\xi$ with g an essentially bounded function and hence not necessarily a function in $\mathfrak{L}_1(-\infty,\infty)$. This difficulty can be avoided by taking a coarser topology in $\mathfrak{H}$ so that there are fewer continuous functionals. With this aim we take

$$\|\{f\}\| = \sup_{-\infty<n<\infty} \int_n^{n+1} |f(\xi)|\,d\xi.$$

Since $\|\{f\}\| \leqslant \|\{f\}\|_1 = \int_{-\infty}^{\infty} |f(\xi)|\,d\xi$, any linear functional $\phi(\{f\})$ continuous with respect to $\|\ \|$ is also continuous with respect to $\|\ \|_1$ and hence is of the form

$$\phi(\{f\}) = \int_{-\infty}^{\infty} f(\xi)\,g(\xi)\,d\xi,$$

with $g(\xi)$ essentially bounded in $(-\infty,\infty)$. We prove that also $g(\xi) \in \mathfrak{L}_1(-\infty,\infty)$. In fact, let

$$f_R(\xi) = \overline{\operatorname{sgn} g(\xi)}, \quad \text{when } |\xi| < R,$$
$$= 0, \quad \text{when } |\xi| \geqslant R.$$

(sgn z denotes, as usual, $z/|z|$ when $z \neq 0$ and 0 when $z = 0$.) Then

$$|f_R(\xi)| \leqslant 1 \quad \text{when} \quad |\xi| < R, \qquad |f_R(\xi)| = 0 \quad \text{when} \quad |\xi| \geqslant R$$

and

$$\|\{f_R\}\| \leqslant 1 \quad \text{for all } R.$$

Also

$$\phi(\{f_R\}) = \int_{-R}^{R} |g(\xi)|\,d\xi.$$

Since ϕ is a bounded functional,

$$\int_{-R}^{R} |g(\xi)|\,d\xi = |\phi(\{f_R\})| \leqslant M \,.\, \|\{f_R\}\| \leqslant M \quad \text{for all } R.$$

Now let $\tau(\{f\}\{g\}) = (f*g)(0)$ and let $\mathfrak{R}$ be the subspace of $g \in \mathfrak{H}$ for which $\tau(\{f\}\{g\})$ is a continuous linear functional on $\mathfrak{H}$. Since the integral is an additive and homogeneous functional whenever it exists, and since $\int_{-\infty}^{\infty} f(\xi)\,g(\xi)\,d\xi = 0$ for all $f \in \mathfrak{L}(-\infty,\infty)$ implies $g = 0$, it follows that τ is an effective trace with $\mathfrak{R}$ as its effective subspace.

9. We now consider an analogue of our extension theorem (Theorem 5) in which the algebra $\mathfrak{H}$ is taken to be of infinite dimension and to possess an effective trace. Suppose that an algebra $\mathfrak{H}$ which is also a normed vector space has an effective right trace τ and that X is a vector space over both $\mathfrak{H}$ and the real field $\mathfrak{F}$. Let π be a real-valued, subadditive and positive homogeneous functional on X, with the further property that $P(a) = \pi(ax)$ is a continuous functional on $\mathfrak{H}$ for each fixed $x \in X$. If we were to assume the existence of a unit element in $\mathfrak{H}$ then our extension theorem could be stated in the same terms as Theorem 5, but this excludes many interesting cases.

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THEOREM 10. *Let X_0 be both an $\mathfrak{F}$-subspace and an $\mathfrak{H}$-subspace of X and let $f(x)$ be an $\mathfrak{H}$-linear functional on X_0 such that $\tau(f(x))$ is a real linear functional on X_0 and*

$$\tau(f(x)) \leqslant \pi(x) \quad \text{if } x \in X_0.$$

Then there exists an $\mathfrak{H}$-linear functional $F(x)$ on X such that

(i) $F(x) = f(x)$ if $x \in X_0$,

(ii) $\tau(aF(x)) \leqslant \pi(ax)$ *if* $a \in \mathfrak{H}$ *and* $x \in X$.

In connexion with this theorem, it should be observed that $F(x)$ takes its values in the effective subspace of $\mathfrak{H}$, and that, if $\mathfrak{H}X = X$, then (ii) takes the simpler form

$$\tau(F(x)) \leqslant \pi(x) \quad \text{if} \quad x \in X.$$

Proof. Since $\tau(f(x))$ is a real linear functional on X_0 it follows from Banach's theorem that there exists a real linear functional $T(x)$ on X such that

(i) $T(x) = \tau(f(x))$ on X_0,

(ii) $T(x) \leqslant \pi(x)$ on X.

For each fixed $x \in X$, $\phi(a) = T(ax)$ is a real linear functional on $\mathfrak{H}$ which is continuous, since $\phi(a) \leqslant P(a)$ and $P(a)$ is a continuous functional. Thus for each $x \in X$, there exists $h_x \in \mathfrak{R}$ such that $\phi(a) = \tau(ah_x)$. By a similar argument we see that $f(x)$ takes its values in $\mathfrak{R}$. Clearly $\tau(ah_x) = T(ax) \leqslant \pi(ax)$ for all $a \in \mathfrak{H}$ and $x \in X$. We now prove that $F(x) = h_x$ is an $\mathfrak{H}$-linear functional on X. On the one hand

$$T(bax) = \tau(bah_x),$$

but also considering ax as an element of X,

$$T(bax) = \tau(bh_{ax}).$$

By the linearity properties of τ, it follows that

$$\tau(b(ah_x - h_{ax})) = 0 \quad \text{for all} \quad b \in \mathfrak{H},$$

and consequently, $ah_x = h_{ax}$.

Additivity is proved in a similar manner, in fact,

$$\tau(ah_{x+y}) = T(a(x+y)) = T(ax) + T(ay) = \tau(ah_x) + \tau(ah_y),$$

from which

$$\tau(a(h_{x+y} - h_x - h_y)) = 0.$$

It remains only to prove that $F(x) = f(x)$ on X_0.

Let $x \in X_0$; then, for all $a \in \mathfrak{H}$,

$$\tau(ah_x) = \tau(h_{ax}) = T(ax) = \tau(af(x)).$$

Since this holds for all a, the usual argument shows that $h_x = f(x)$.

COROLLARY. *If X, $\mathfrak{H}$, π satisfy the conditions of Theorem* 10, *there exists an $\mathfrak{H}$-linear functional $F(x)$ on X such that* $\tau(aF(x)) \leqslant \pi(ax)$ *for all $a \in \mathfrak{H}$ and all $x \in X$.*

Proof. Take X_0 to be the subspace consisting of the zero element in X, and apply Theorem 10 to extend the $\mathfrak{H}$-zero-valued functional on X_0. Since $\tau(0) = \pi(0) = 0$, all the conditions of the theorem are satisfied.

We remark, finally, that if $f(x)$ is an $\mathfrak{H}$-homogeneous functional on X taking its values in the effective subspace $\mathfrak{R}$, then $f(x)$ is also real homogeneous. For

$$\alpha a f(x) = f((\alpha a)x) = f(a(\alpha x)) = af(\alpha x).$$

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Since α, a commute we have, for all $a \in \mathfrak{H}$,

$$a(\alpha f(x) - f(\alpha x)) = 0.$$

Since $\alpha f(x) - f(\alpha x) \in \mathfrak{R}$, it follows that $\alpha f(x) = f(\alpha x)$.

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