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On a nonlinear heat equation with a time delay

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A nonlinear forward-backward heat equation with a regularization term was proposed by Barenblatt *et al.* [1, 2] to model the heat and mass exchange in stably stratified turbulent shear flow. It was proven to be well-posed in the case of given initial and Neumann boundary conditions. However, the solution was found to have an unphysical discontinuity with certain smooth initial functions. In this paper, a nonlinear heat equation with a time delay originally used by Barenblatt *et al.* [1, 2] to derive their model is investigated. The same type of initial-boundary value problem is shown to have a unique smooth global solution when the initial function is reasonably smooth. Numerical examples are used to demonstrate that its solution forms step-like profiles in finite times. A semi-discretization of the initial-boundary value problem is proved to have a unique asymptotically and globally stable equilibrium.

1 Introduction

Consider the one-dimensional diffusion process

$$u_t + q_x = 0, \quad q = -ku_x, \tag{1.1}$$

where u is the temperature (or concentration), q is the heat (or mass) flux, and k is the diffusivity. In a stably stratified turbulent shear flow, k is assumed to be dependent on the temperature gradient: $k = k_0(u_x)$, where k_0 is a continuous positive function. Since the diffusivity k decreases very quickly at large values of the temperature gradient, it is often assumed that $k_0(s) = \alpha/(\beta + s^2)$, where α and β are positive constants. The balance law (1.1) leads to the following nonlinear heat equation for the temperature:

$$u_t = (\phi(u_x))_x, \tag{1.2}$$

where the function ϕ is defined by $\phi(s) = sk_0(s)$. This equation is equivalent to

$$u_t = \phi'(u_x)u_{xx}$$

which is not of forward parabolic type at points where $\phi'(u_x) < 0$. It has been explained by Barenblatt *et al.* [1, 2] that initial and/or initial-boundary value problems of (1.2) may be ill-posed, though rigourous analysis is still unavailable at that time. The study of the Perona–Malik Paradox by Kichenassamy [8], however, shows that if there is a weak

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solution to (1.2) then it must be infinitely smooth initially in any region where its gradient is greater than a cut-off value, subject to the conditions that $k_0(s) > 0$ for all $s > 0$, both $k_0(s)$ and $\phi'(s)$ tends to zero as $s \to \infty$, $\phi'(s) > 0$ for $|s| < K$, and $\phi'(s) < 0$ for $|s| > K$. Therefore, if the initial condition is not infinitely differentiable, there is no weak solution at all.

To have a well-posed model for the heat and mass exchange in stably stratified turbulent shear flow, Barenblatt *et al.* [1, 2] proposed the following regularization:

$$u_t = (\phi(u_x))_x + \tau\psi(u_x)_{xt}, \tag{1.3}$$

where $\tau > 0$ is a small parameter, and the function ψ is defined by

$$\psi(s) = -\int_0^s sk_0'(s)\,\mathrm{d}s.$$

Their reason for adding the regularization term $\tau\psi(u_x)_{xt}$ is that the diffusivity coefficient k depends on $u_x(x, t-\tau)$ instead of $u_x(x,t)$:

$$k = k_0(u_x(x, t-\tau)), \tag{1.4}$$

where τ is regarded as the average relaxation time of the turbulent field. In comparison with the characteristic time of the whole field, τ is perhaps small but nevertheless positive. They obtained equation (1.3) from equation (1.1) by taking the following linear approximation:

$$k \approx k_0(u_x) - \tau k_0'(u_x)u_{xt}, \tag{1.5}$$

noting that $\psi(u_x)_t = -u_x k_0'(u_x)u_{xt}$. Interested readers may find detailed discussion on the justification of this kind of regularization in Barenblatt *et al.* [1, 2]. In passing, we mention that Driver [4, pp. 253–254] has several interesting examples of converting delay differential equation into ordinary differential equation via finite term Taylor expansions.

Barenblatt *et al.* [1, 2] studied the well-posedness and asymptotics of the initial-boundary value problem

$$\begin{cases} u_t = (\phi(u_x))_x + \tau\psi(u_x)_{xt}, \quad x \in (0,1), \quad t > 0, \\ u_x(0,t) = u_x(1,t) = 0, \quad t \geqslant 0, \\ u(x,t) = f(x), \quad x \in (0,1), \end{cases} \tag{1.6}$$

where f is a given initial function. They have proved among many other results that

(1) (1.6) has a weak solution that exists locally and is unique;

(2) the solution can become discontinuous in a finite time; and

(3) the discontinuity of the solution persists.

Despite the fact that (1.6) is a locally well-posed problem, the finite time discontinuity of its solution seems to be unphysical.

However, if the diffusivity coefficient k is chosen to have the form (1.4) without resorting to the approximation (1.5), the balance law (1.1) leads to the following nonlinear heat equation with time delay

$$u_t(x,t) = (k_0(u_x(x, t-\tau))u_x(x,t))_x \tag{1.7}$$

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which may be better than equation (1.3) for modelling of the heat and mass exchange in stably stratified turbulent shear flow. This leads us to consider the initial-boundary value problem

$$\begin{cases} u_t(x,t) = (k_0(u_x(x,t-\tau))u_x(x,t))_x, \quad x \in (0,1),\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, \quad t \geqslant 0, \\ u(x,t) = f(x,t), \quad x \in (0,1), -\tau \leqslant t \leqslant 0, \end{cases} \tag{1.8}$$

where τ is a positive constant time delay and f is a given initial function. It will be shown in § 2 that (1.8) has a unique smooth global solution when the initial function is sufficiently smooth. As a way of studying the asymptotic behaviour of the exact solution, a semi-discretization of (1.8) is investigated in § 3 and it is proved that this semi-discretization has a unique asymptotically and globally stable equilibrium. In § 4, numerical examples are used to demonstrate that, given smooth initial conditions, solutions of (1.8) may form step-like profiles in finite times. This shows that the corresponding exact solution can have extremely large gradients at certain points as time evolves, but yet remain differentiable.

2 Existence of global solutions

The so-called method-of-steps can be used to study the existence and uniqueness of solutions to the initial-boundary value problem (1.8) when $\tau > 0$, since (1.7) is a linear heat equation of u for $t \in (n\tau, (n+1)\tau]$ for any given nonnegative integer n once $u(x,t)$ is known for $t \in [-\tau, n\tau]$. In this section, the function k_0 is assumed to be positive and have continuous derivatives up to order 3. Under suitable assumptions on the function f, Theorems 7 and 9 in Friedman [5, pp. 65, 69] can be applied to the initial-boundary value problem

$$\begin{cases} v_t = (k_0(v(x,t-\tau)v(x,t))_{xx}, \quad x \in (0,1),\ 0 < t \leqslant \tau, \\ v(0,t) = v(1,t) = 0, \quad t \geqslant 0, \\ v(x,t) = f_x(x,t), \quad x \in (0,1), -\tau \leqslant t \leqslant 0. \end{cases}$$

If v is a solution of the preceding problem then

$$u(x,t) = w(t) + \int_0^x v(s,t)\,\mathrm{d}s$$

is a solution of (1.8) for $t \in [0,\tau]$, where

$$w(t) = \int_0^1 [f(x,0) - (1-x)v(x,t)]\,\mathrm{d}x$$

is obtained through the conservation law

$$\int_0^1 u(x,t)\,\mathrm{d}x = \int_0^1 f(x,0)\,\mathrm{d}x, \quad t > 0, \tag{2.1}$$

which is satisfied by the solution of (1.8). Using the method of steps, it is not difficult to derive from Theorems 7 and 9 of Friedman [5, pp. 65, 69] the following two theorems for the initial-boundary value problem (1.8).

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Theorem 1 *Suppose that* $\tau > 0$, $f \in C([0,1] \times [-\tau, 0])$, $f_x \in \overline{C}_{2+\alpha}([0,1] \times [-\tau, 0])$ *for some* $\alpha \in (0,1)$, *and* $f_x(0,t) = f_x(1,t) = 0$ *for* $t \in [-\tau, 0]$. *Then the initial-boundary value problem* (1.8) *has a unique solution in* $\overline{C}_{2+\alpha}([0,1] \times [-\tau, \infty))$.

Theorem 2 *Suppose that* $\tau > 0$, $f \in C([0,1] \times [-\tau, 0])$, $f_x \in C_{2+\alpha}([0,1] \times [-\tau, 0])$ *for some* $\alpha \in (0,1)$, *and* $f_x(0,t) = f_x(1,t) = 0$ *for* $t \in [-\tau, 0]$. *Then the initial-boundary value problem* (1.8) *has a unique solution in* $C_{2+\alpha}([0,1] \times [-\tau, \infty))$.

Readers are referred to Friedman [5, pp. 61–63] for a description of the notation $\overline{C}_{2+\alpha}$ and $C_{2+\alpha}$.

These two theorems imply that if the initial function f is sufficiently smooth then the initial-boundary value problem (1.8) has a sufficiently smooth solution. This is in contrast with the result obtained in Barenblatt *et al.* [1], where they have proved that the solution of (1.6) may become discontinuous in a finite time even with a smooth initial function. However, we are unable to obtain any result on the existence of weak solution when f does not satisfy those assumptions made in Theorems 1 or 2.

Besides the conservation law (2.1), the solution of (1.8) also obeys the identity

$$\frac{1}{2}\int_0^1 u^2(x,t)\,\mathrm{d}x + \int_0^t \int_0^1 k_0(u_x(x,s-\tau))\,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\int_0^1 f^2(x,0)\,\mathrm{d}x, \quad t > 0, \tag{2.2}$$

the maximum principles

$$\max_{0 \leqslant x \leqslant 1} u(x,t) = \max_{0 \leqslant x \leqslant 1} f(x,0), \qquad t > 0$$

and the following monotonicity properties:

(1) $f_x(x,0) \geqslant 0 \Longrightarrow u_x(x,t) \geqslant 0$ for $t > 0$;

(2) $f_x(x,0) \leqslant 0 \Longrightarrow u_x(x,t) \leqslant 0$ for $t > 0$.

These can be proved in the standard way (with the help of the method-of-steps). We omit the proof here.

3 Asymptotics of a semi-discretization

The only equilibrium of (1.8) is $\int_0^1 f(x,0)\,\mathrm{d}x$. It is unknown whether it is asymptotically stable or not. In this section we study the stability of the equilibrium of a semi-discretization of (1.8).

Let $\Delta x = 1/N$ for some positive integer N, and denote

$$\Delta u_n(x) = u_{n+1}(x) - u_n(x), \qquad n = 0, 1, \ldots, N-1,$$

where $u_n(t)$ is our approximation to $u(n\Delta x, t)$. A suitable semi-discretization of (1.8) based

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on a second-order difference formula with respect to x assumes the form[1]

$$u_n'(t) = \frac{1}{(\Delta x)^2}\left[k_0(\Delta u_n(t-\tau)/\Delta x)\Delta u_n(t) - k_0(\Delta u_{n-1}(t-\tau)/\Delta x)\Delta u_{n-1}(t)\right], \quad n = 1, 2, \ldots, N-1, \tag{3.1}$$

while the boundary conditions are discretized as

$$u_1(t) = u_0(t), \quad u_{N-1}(t) = u_N(t), \quad t \geqslant 0, \tag{3.2}$$

and the initial condition becomes

$$u_n(t) = f(n\Delta x, t), \quad -\tau \leqslant t \leqslant 0, \qquad n = 0, 1, \ldots, N. \tag{3.3}$$

It can be shown easily that the solution of (3.1)–(3.3) obeys the conservation law

$$\sum_{n=1}^{N-1} u_n(t) = \sum_{n=1}^{N-1} f(n\Delta x, 0) \tag{3.4}$$

which can be regarded as a discretized version of (2.1). Because of (3.4) and the boundary conditions (3.2), the only equilibrium of the semi-discretization (3.1)–(3.3) is

$$\frac{1}{N-1}\sum_{n=1}^{N-1} f(n\Delta x, 0)(1, 1, \ldots, 1)^T. \tag{3.5}$$

Theorem 3 *The equilibrium* (3.5) *of the semi-discretization* (3.1)–(3.3) *is uniformly and globally asymptotically stable.*

Proof Using a Liapunov function of the form

$$(u_1, u_2, \ldots, u_{N-1}) = \sum_{n=1}^{N-1} u_n^2.$$

it can be shown that the solution $(u_0(t), u_1(t), \ldots, u_N(t))^T$ of (3.1)–(3.3) satisfies the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}U(u_1(t), u_2(t), \ldots, u_{N-1}(t)) = -\frac{2}{(\Delta x)^2}\sum_{n=1}^{N-2} k_0\left(\Delta u_n(t-\tau)/\Delta x\right)(\Delta u_n(t))^2. \tag{3.6}$$

The positivity of the function k_0 implies that $U(u_1(t), u_2(t), \ldots, u_{N-1}(t))$ is uniformly bounded, which in turn implies that the solution is uniformly bounded. Therefore, $k_0(\Delta u_n(t-\tau)/\Delta x)$ has a positive lower bound. It can be shown from (3.6) that

$$\lim_{t\to\infty} \Delta u_n(t) = 0, \qquad n = 1, 2, \ldots, N-2, \tag{3.7}$$

for otherwise $U(u_1(t), u_2(t), \ldots, u_{N-1}(t))$ becomes negative when t is large enough. The uniform boundedness of $u_k(t)$, $1 \leqslant k \leqslant N-1$, implies that the ω-limit set of $u_k(t)$ is

[1] Another popular approach is to use the central difference formula $\Delta_0 u_n(t) = u_{n+1/2}(t) - u_{n-1/2}(t)$ to obtain $u_n'(t) = \frac{1}{(\Delta x)^2}\Delta_0[k_0(\Delta_0 u_n(t-\tau))\Delta_0 u_n(t)]$, which is a second-order semi-discretization as well. However, its long-time dynamical behaviour seems more difficult to analyse.

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bounded. Suppose that u^* is in the ω-limit set of the function $u_k(t)$, i.e. there exists a sequence $\{t_m\}_{m=0}^{\infty}$, $\lim_{m\to\infty} t_m = \infty$, such that

$$\lim_{m\to\infty} u_k(t_m) = u^*.$$

It follows from (3.7) and the conservation law (3.4) that

$$u^* = \frac{1}{N-1}\sum_{n=1}^{N-1} f(n\Delta x, 0).$$

Hence, the ω-limit set of the function $u_k(t)$ contains just u^*. A slight modification of the preceding argument tell us that the equilibrium (3.5) is asymptotically stable in the sense of definition 1.1 in Hale & Verduyn Lunel [6, p. 130]. It follows from Lemma 1.1 in Hale & Verduyn Lunel [6, p. 131] that it is uniformly asymptotically stable. It is also globally asymptotically stable because other solutions tend to it regardless of the initial conditions. □

Because of the extral diffusivity induced by the semi-discretization, it is difficult to say whether or not the equilibrium $\int_0^1 f(x,0)\,\mathrm{d}x$ of the initial-boundary value problem (1.8) is uniformly asymptotically stable. However, it does tell us what the expected long-time asymptotic behaviour of the numerical solution of (1.8) is.

Theorem 3 holds when $\tau = 0$. However, the exact solution of (1.8) in the case $\tau > 0$ may be very different from that of (1.8) in the case $\tau = 0$ (supposing solutions of the late problem exists). As a comparison example, let us consider the initial value problem

$$y'(t) = y(t-\tau)y(t), \quad t > 0, \qquad y(t) = y_0(t), \quad t \in [-\tau, 0] \tag{3.8}$$

where $\tau \geqslant 0$ is a constant. When $\tau = 0$, the solution of (3.8) blows up in a finite time if $y_0 := y_0(0) > 0$, since the solution is $y(t) = y_0/(1 - y_0 t)$. However, when $\tau > 0$, it has a unique solution that exists globally regardless how small τ is.

To make a comparison study, we consider the asymptotics of a semi-discretization of the initial value problem (1.6). Assume that there exists a constant $\beta_0 > 0$ such that

$$\psi'(y) = -yk_0'(y) \geqslant 0, \quad \phi'(y) = k_0(y) + yk_0'(y) \geqslant 0, \qquad y^2 \leqslant \beta_0. \tag{3.9}$$

In the case of $k_0(y) = \alpha/(\beta + y^2)$ with $\alpha, \beta > 0$, we have $\phi'(y) \geqslant 0$ when $y^2 \leqslant \beta$, but $\psi'(y) \geqslant 0$ for any real y. Rewriting the first equation of (1.6) as

$$\frac{\mathrm{d}}{\mathrm{d}t}[u - \tau\psi(u_x)_x] = (\phi(u_x))_x,$$

a second-order semi-discretization can be formulated in the following way:

$$\begin{aligned} &\left\{u_n(t) - \frac{\tau}{\Delta x}\left[\psi(\Delta u_n(t)/\Delta x) - \psi(\Delta u_{n-1}(t)/\Delta x)\right]\right\}' \\ &= \frac{1}{\Delta x}\left[\phi(\Delta u_n(t)/\Delta x) - \phi(\Delta u_{n-1}(t)/\Delta x)\right], \quad n = 1, 2, \ldots, N-1, \end{aligned} \tag{3.10}$$

while the boundary and initial conditions of (1.6) are discretized as

$$u_1(t) = u_0(t), \quad u_{N-1}(t) = u_N(t), \quad t \geqslant 0 \tag{3.11}$$

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and

$$u_n(0) = f(n\Delta x), \quad n = 0, 1, \ldots, N. \tag{3.12}$$

Since the solution of (3.10)–(3.12) obeys the conservation law

$$\sum_{n=1}^{N-1} u_n(t) = \sum_{n=1}^{N-1} f(n\Delta x), \tag{3.13}$$

the only equilibrium of the semi-discretization (3.10)–(3.12) is

$$\frac{1}{N-1} \sum_{n=1}^{N-1} f(n\Delta x)(1, 1, \ldots, 1)^T. \tag{3.14}$$

Theorem 4 *The equilibrium* (3.14) *of the semi-discretization* (3.10)–(3.12) *is uniformly asymptotically stable.*

Proof Because (3.10) is an implicit system of ordinary differential equations, we use a Liapunov function of the form

$$V(v_1, v_2, \ldots, v_{N-1}) = \sum_{n=1}^{N-1} \left\{ v_n - \frac{\tau}{\Delta x} \left[\psi(\Delta v_n/\Delta x) - \psi(\Delta v_{n-1}/\Delta x) \right] \right\}^2,$$

where $v_0 = v_1$ and $v_N = v_{N-1}$. Let $v_n(t) = u_n(t) - u^*$, $n = 0, 1, \ldots, N$, where $(u_0(t), u_1(t), \ldots, u_N(t))^T$ is the solution of (3.10)–(3.12). Then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} V(v_1(t), v_2(t), \ldots, v_{N-1}(t)) \\ &= -\frac{2}{(\Delta x)^2} \sum_{n=1}^{N-2} k_0 \left(\Delta v_n(t)/\Delta x \right) (\Delta v_n(t))^2 \\ &\quad -\frac{2\tau}{(\Delta x)^2} \sum_{n=1}^{N-1} \left[\psi(\Delta v_n(t)/\Delta x) - \psi(\Delta v_{n-1}(t)/\Delta x) \right] \\ &\qquad \cdot \left[\phi(\Delta v_n(t)/\Delta x) - \phi(\Delta v_{n-1}(t)/\Delta x) \right]. \end{aligned} \tag{3.15}$$

Suppose that the initial function f satisfies

$$\sum_{n=1}^{N-1} (f(n\Delta x) - u^*)^2 < \beta_0 (\Delta x)^2 / 2. \tag{3.16}$$

Then it can be shown from (3.15) that $V(v_1(t), v_2(t), \ldots, v_{N-1}(t))$ decreases monotonically and that

$$\sum_{n=1}^{N-1} (u_n(t) - u^*)^2 < \beta_0 (\Delta x)^2 / 2, \qquad t \geqslant 0. \tag{3.17}$$

The proof of this is based on two observations. First, if

$$\sum_{n=1}^{N-1} v_n^2 < \beta_0 (\Delta x)^2 / 2,$$

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then

$$(\Delta v_n/\Delta x)^2 < \beta_0, \qquad n = 0, 1, \ldots, N-1 \tag{3.18}$$

which, in tandem with (3.9), implies that

$$\left[\psi(\Delta v_n/\Delta x) - \psi(\Delta v_{n-1}/\Delta x)\right] \cdot \left[\phi(\Delta v_n/\Delta x) - \phi(\Delta v_{n-1}/\Delta x)\right] \geqslant 0, \quad n = 1, 2, \ldots, N-1.$$

Secondly,

$$\sum_{n=1}^{N-1} v_n^2 \leqslant V(v_1, v_2, \ldots, v_{N-1}) \tag{3.19}$$

if (3.18) holds. This is because the left-hand side of (3.19) is $\|Av\|_2^2$, where $\|\cdot\|_2$ denotes the Euclidean norm, $v = (v_1, v_2, \ldots, v_{N-1})^T$ and $A = (a_{ij})$ is a $(N-1) \times (N-1)$ symmetric tridiagonal matrix:

$$a_{1,1} = 1 + \frac{\tau}{(\Delta x)^2}\xi(\Delta v_1/\Delta x) \quad a_{1,2} = -\frac{\tau}{(\Delta x)^2}\xi(\Delta v_1/\Delta x)$$

$$a_{i,i-1} = -\frac{\tau}{(\Delta x)^2}\xi(\Delta v_{i-1}/\Delta x), \quad a_{i,i} = 1 + \frac{\tau}{(\Delta x)^2}\left[\xi(\Delta v_{i-1}/\Delta x) + \xi(\Delta v_i/\Delta x)\right]$$

$$a_{i,i+1} = -\frac{\tau}{(\Delta x)^2}\xi(\Delta v_i/\Delta x), \qquad i = 2, 3, \ldots, N-2,$$

$$a_{N-1,N-2} = -\frac{\tau}{(\Delta x)^2}\xi(\Delta v_{N-2}/\Delta x), \quad a_{N-1,N-1} = 1 + \frac{\tau}{(\Delta x)^2}\xi(\Delta v_{N-2}/\Delta x).$$

where the function ξ is defined by

$$\xi(y) = \begin{cases} \psi(y)/y, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Since $\xi(y) \geqslant 0$ when $y^2 \leqslant \beta_0$, it follows from Gersgorin's theorem that the smallest eigenvalue of A is no less than one if (3.18) holds. Hence, the inequality (3.19) holds if (3.18) does.

The inequality (3.17) implies that the equilibrium (3.14) is stable. Similar to the proof of Theorem 3, it can be shown that (3.14) is uniformly asymptotically stable. □

Because of assumption (3.16) made in the proof of Theorem 4, the result obtained for the semi-discretization (3.10)–(3.12) of (1.6) is much weaker than that of the semi-discretization (3.1)–(3.3) of (1.8). In the latter case, there is no restriction on the initial function f.

4 Numerical simulation

In the previous section, Theorem 3 indicates that the solution of a semi-discretization of (1.8) tend to a constant equilibrium as t tends to infinity regardless of the initial conditions, while Theorem 4 tells us that solutions of a semi-discretization of (1.6) tend to a constant equilibrium as t tends to infinity with a certain range of initial conditions. However, before these solutions converge to their equilibria, step-like profiles may form in finite times. This is found to be true by numerical computation in Posmentier [9] for the case (1.8) with $\tau = 0$, and in Barenblatt *et al.* [2] for the case (1.6). In this section, we

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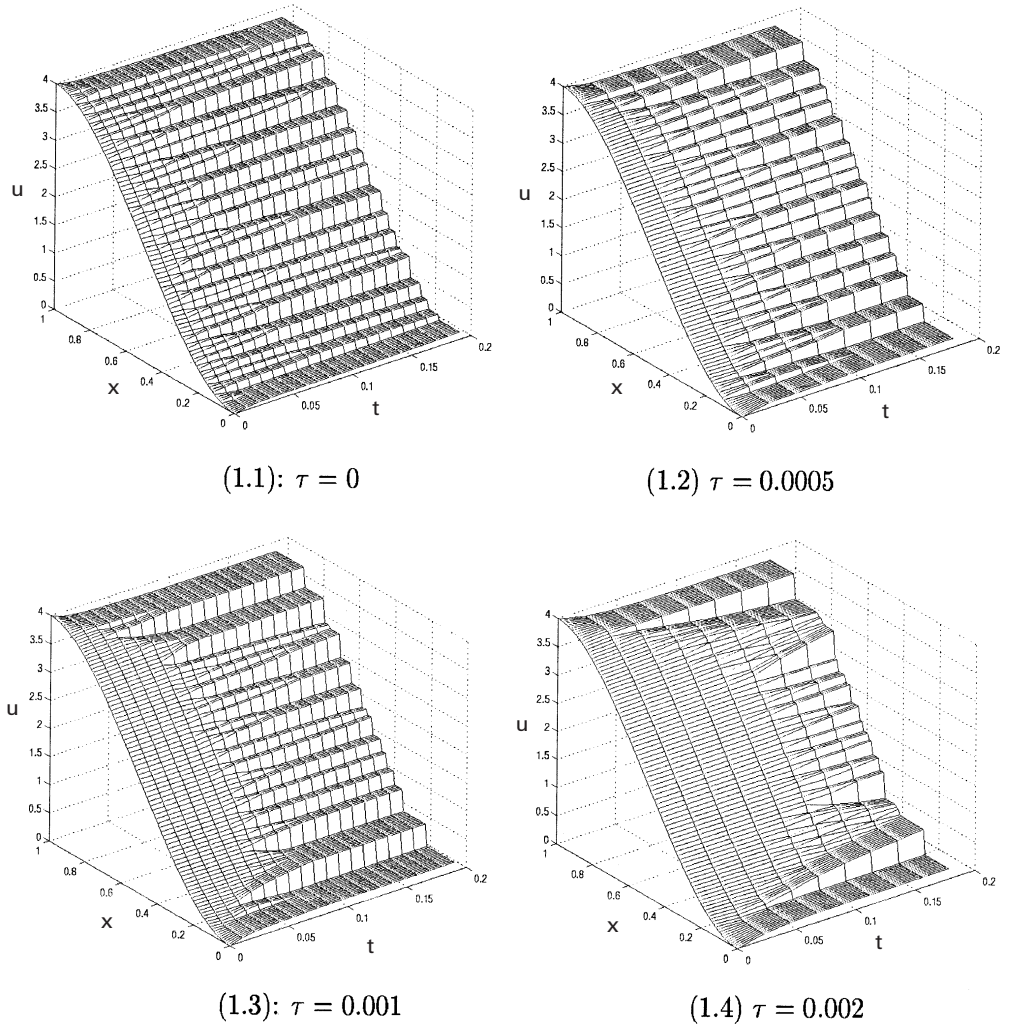


FIGURE 1. Numerical solutions of (1.8): $\Delta x = 10^{-2}$, $\Delta t = \tau/10$.

use numerical examples to show that the same phenomenon occurs in the case of (1.8) with $\tau > 0$. We also demonstrate how the size of τ can affect the start of the generation of step-like profiles.

Let $\Delta t > 0$ be the temporal step size. In the case of (1.8) with $\tau > 0$ we choose $\tau = m_0 \Delta t$ with m_0 being a positive integer. The symbol u_m^n denotes our approximation of a function $u(x,t)$ at $(x,t) = (n\Delta x, m\Delta)$. Let Δ_x and Υ_t the forward difference and averaging operator in the x and t directions respectively, i.e.

$$\Delta_x u_n^m = u_{n+1}^m - u_n^m, \qquad \Upsilon_t u_n^m = \frac{1}{2}(u_n^{m+1} + u_n^m).$$

A suitable numerical method for solving equation (1.7) can be obtained by applying the

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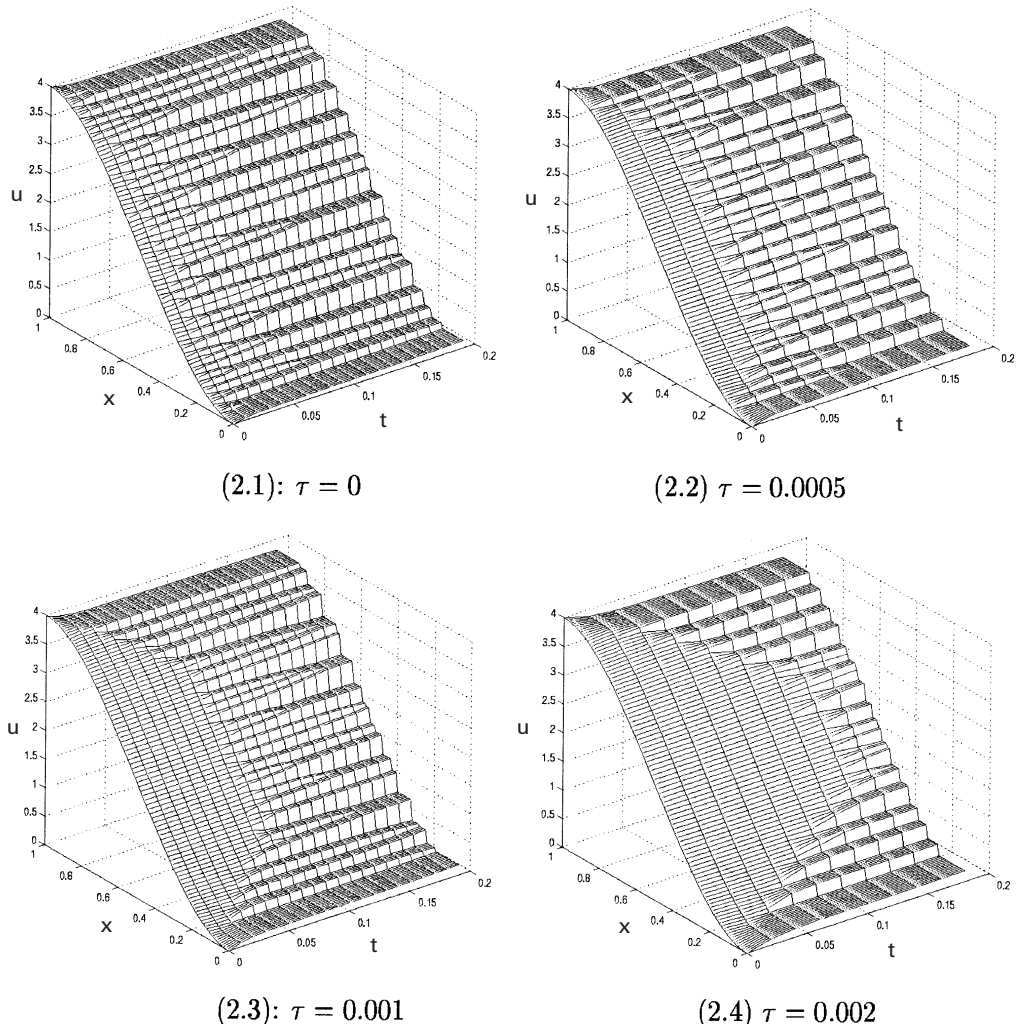


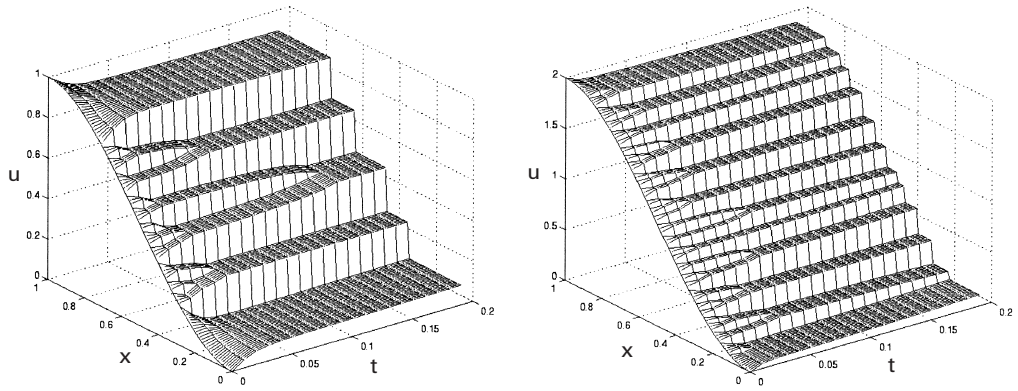
FIGURE 2. Numerical solutions of (1.6): $\Delta x = 10^{-2}$, $\Delta t = \tau/10$.

implicit midpoint rule to the semi-discretization (3.1).[2] This yields the following difference equation:

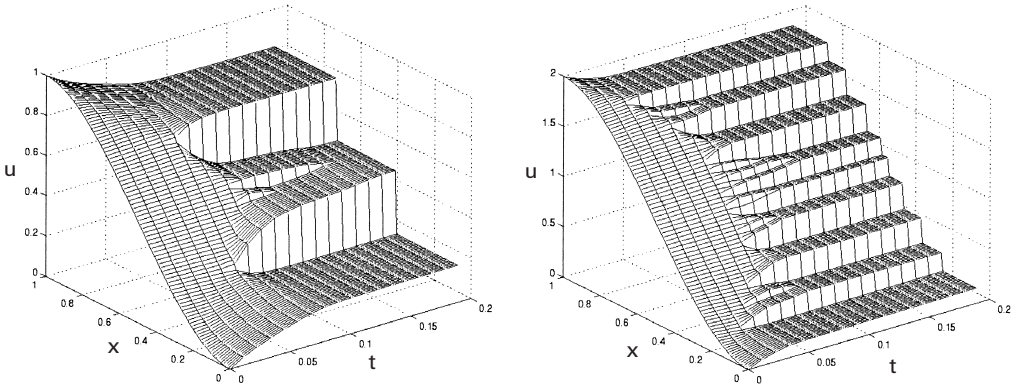
$$u_n^{m+1} - u_n^m = \frac{\Delta t}{(\Delta x)^2}\left[k_0\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_n^{m-m_0}\right)\Upsilon_t \Delta_x u_n^m - k_0\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_{n-1}^{m-m_0}\right)\Upsilon_t \Delta_x u_{n-1}^m\right]. \tag{4.1}$$

[2] The Crank–Nicolson method, based on the trapezoidal rule, can be used to solve equation (1.7). However, we prefer using the method derived from the implicit midpoint rule. We believe that the implicit midpoint rule is better at preserving long-time quantitative behaviour of the continuous equation in the current situation. In the case of Hamiltonian systems, it is well known that the implicit midpoint rule is symplectic but the trapezoidal rule is not.

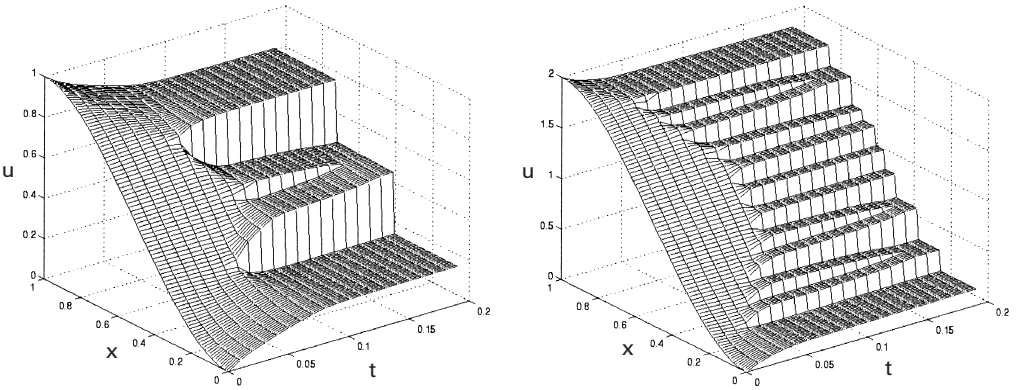
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(3.1): Numerical solutions of (8) with $\tau = 0$



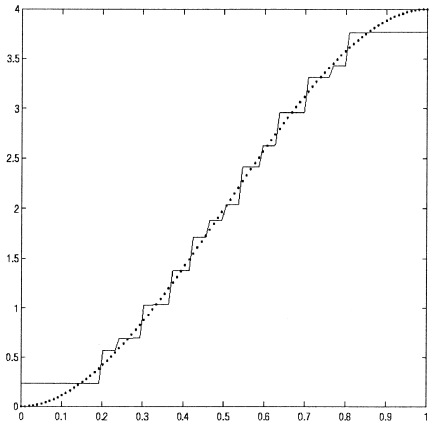
(3.2): Numerical solutions of (8) with $\tau = 0.001$



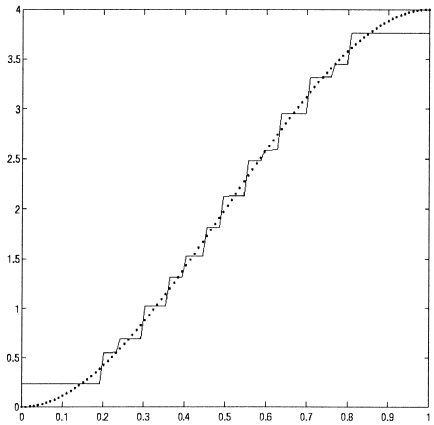
(3.3): Numerical solutions of (6) with $\tau = 0.001$

FIGURE 3. $\Delta t = 10^{-4}$, $\Delta x = 5 \times 10^{-3}$, $A = 1$ (on the left), $A = 2$ (on the right).

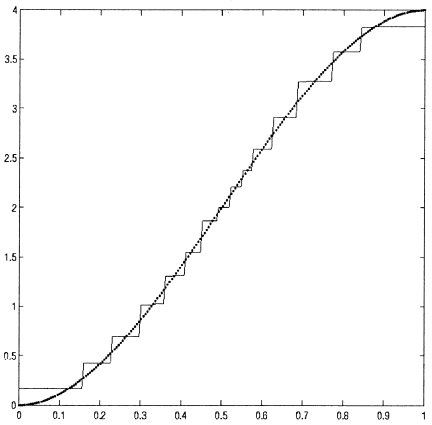
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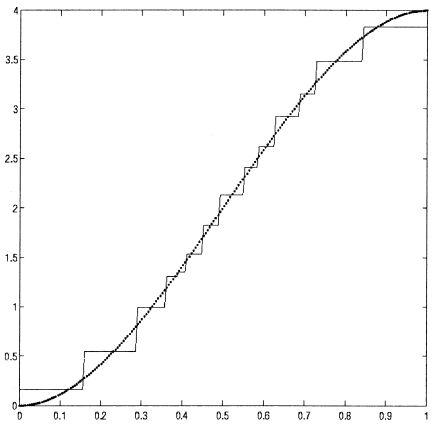
(4.1): $\Delta t = 2 \times 10^{-4}$, $\Delta x = 10^{-2}$



(4.2): $\Delta t = 10^{-4}$, $\Delta x = 10^{-2}$



(4.3): $\Delta t = 2 \times 10^{-4}$, $\Delta x = 5 \times 10^{-3}$



(4.4): $\Delta t = 10^{-4}$, $\Delta x = 5 \times 10^{-3}$

FIGURE 4. Numerical solutions of (1.8): $\tau = 0.001$, $u(x, 0)$ is in dotted line, $u(x, 0.2)$ is in solid line.

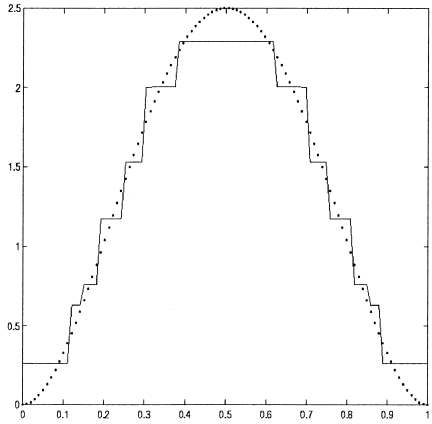
When $\tau = 0$, the above assumes the simpler form

$$u_n^{m+1} - u_n^m = \frac{\Delta t}{\Delta x}\left[\phi\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_n^m\right) - \phi\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_{n-1}^m\right)\right]. \tag{4.2}$$

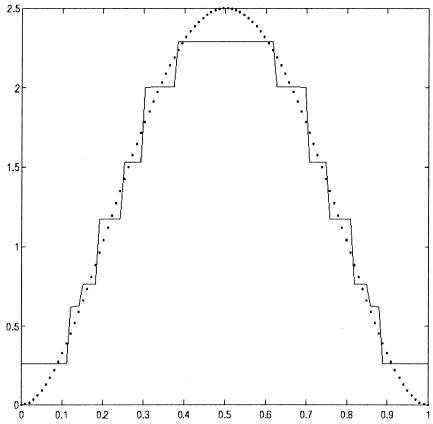
In the case of equation (1.3), the Crank–Nicolson method can be written as

$$\begin{aligned} u_n^{m+1} - u_n^m - \frac{\tau}{\Delta x}\big[&\psi(\Delta_x u_n^{m+1}/\Delta x) - \psi(\Delta_x u_{n-1}^{m+1}/\Delta x) \\ &-\psi(\Delta_x u_n^m/\Delta x) + \psi(\Delta_x u_{n-1}^m/\Delta x)\big] \\ = \frac{\Delta t}{\Delta x}&\left[\phi\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_n^m\right) - \phi\left(\frac{1}{\Delta x}\Upsilon_t \Delta_x u_{n-1}^m\right)\right] \end{aligned} \tag{4.3}$$

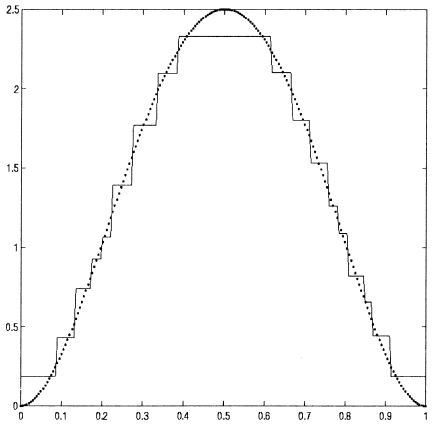
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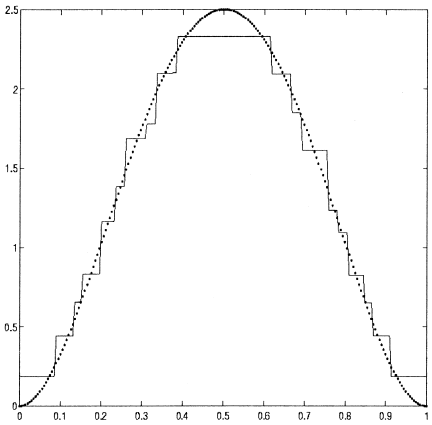
(5.1): $\Delta t = 2 \times 10^{-4}$, $\Delta x = 10^{-2}$



(5.2): $\Delta t = 10^{-4}$, $\Delta x = 10^{-2}$



(5.3): $\Delta t = 2 \times 10^{-4}$, $\Delta x = 5 \times 10^{-3}$



(5.4): $\Delta t = 10^{-4}$, $\Delta x = 5 \times 10^{-3}$

FIGURE 5. Numerical solutions of (1.8): $\tau = 0.001$ $u(x, 0)$ is in dotted line, $u(x, 0.2)$ is in solid line.

In each case, the Neumann boundary condition is discretized as

$$u_1^m = u_0^m, \quad u_{N-1}^m = u_N^m., \qquad m \geqslant 0.$$

According to Barenblatt *et al.* [1, 2], the function k_0 is usually assumed to be of the form

$$k_0(s) = \frac{\alpha}{\beta + s^2}. \tag{4.4}$$

Equation (1.2) becomes a backward heat equation at points where $|u_x| > \sqrt{\beta}$. In the following numerical examples, we let $\alpha = \beta = 1$. Numerical solutions of (3.1)–(3.3) and of (3.10)–(3.12) are displayed in Figures 1–4 for $t \in [0, 0.2]$, with initial functions of the form

$$f(x, t) = f(x) = Ax^2(3 - 2x), \qquad x \in [0, 1], \quad t \leqslant 0, \tag{4.5}$$

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where A is a constant. The following observations can be made:

(1) Figures 1–3 show how 'steps' are generated and then merged later on. This is not surprising. In fact, it has already been pointed out [9, 2] that step-like profiles are formulated when gradients of the initial function exceeds certain thresholds. In the case of (4.4), this threshold is $\sqrt{\beta} = 1$.

(2) Figures 1–3 demonstrate that the smaller τ is the faster steps are generated.

(3) The six plots in Figure 3, together with Figures 1.3 and 2.3, reveal that the number of steps is dependent on the gradients (with respect to x) of the initial functions. In general, the steeper the initial function is the more steps the solution formulates at some late times.

(4) Figure 4 displays numerical solutions of (3.1)–(3.3) at $t = 0.2$ with four different combinations of mesh sizes. It shows that the exact forms of step-like profiles can be very sensitive to changes in the temporal and spatial step sizes.

The fact that the initial function (4.5) is monotonic with respect to x is not essential to the generation of step-like profiles. For instance, when the initial function is of the form

$$f(x,t) = 40x^2(1-x)^2, \qquad x \in [0,1], \quad t \leqslant 0$$

step-like profiles formulate in finite times as demonstrated in Figure 5. Once again, we can see how sensitive to changes in the temporal and spatial step sizes those step-like profiles can be.

5 Conclusion

The heat equation (1.7) with a positive time delay τ may be suitable for modelling the heat and mass exchange in stably stratified turbulent shear flows. Its initial-boundary-value problem (1.8) has a unique global solution when the initial function is smooth. By contrast, the solution of (1.6) may become discontinuous in a finite time even with a smooth initial function, while it is still unknown whether the initial-Neumann boundary-value problems for (1.2) is well-posed or not. A semi-discretization of (1.8) is proved to have a unique asymptotically and globally stable equilibrium. However, the stability of the equilibrium of (1.8) is still unresolved. For initial conditions with large gradient, it is found by numerical computation that the solution of (1.8) develops step-like profiles in finite times.

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