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STOCHASTIC ORDERINGS BETWEEN SPACINGS OF GENERALIZED ORDER STATISTICS

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In this article, we establish stochastic comparisons between normalized spacings of generalized order statistics. These comparisons allow us to extend and unify some results obtained by other authors for ordinary order statistics and record values. Furthermore, we can compare spacings of different models (i.e., between ordinary order statistics and sequential order statistics, record values and Pfeifer's record values, and so forth).

1. INTRODUCTION

Generalized order statistics (gOSs) have been of interest during the last few years because of the known models they include (see Kamps [12]), which are important in Reliability Theory and other statistics fields.

Order statistics and record values are widely used in statistical modeling and inference, and they can be embedded in the concept of gOSs as a general framework for models of ordered random variables (r.v.'s). In Reliability, the $(n-r+1)$th-order statistic represents the life length of some r-out-of-n system.

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Record values are closely connected with the occurrence times of some corresponding nonhomogeneous Poisson process and are used in shock models. Other various models of ordered r.v.'s are introduced in Kamps [12], such as order statistics with nonintegral sample size, sequential order statistics, *k*th record values, Pfeifer's record model, k_n record from nonidentical distributions, and ordered r.v.'s which arise from truncation of distributions.

These models can be applied in Reliability Theory. For instance, sequential order statistics are an extension of ordinary order statistics and they describe certain dependencies or interactions among the components of the system caused by failures of themselves. Likewise, the different types of record values extend the model of ordinary record values and, consequently, they are more flexible in practical situations.

Some results on distribution properties, characterization of distributions, and inequalities for moments of gOSs have been given in Kamps [12]. Likewise, Ahsanullah [1,2] studied the gOSs from two-parameter exponential and uniform distributions, Keseling [13] characterized continuous distributions by conditional expectations of gOSs, and Gajek and Okolewski [8,9] gave bounds for expected values of gOSs.

On the other hand, other articles have been devoted to the study of orderings among order statistics or record values, in order to compare the reliability of different systems and certain aging processes; see, for instance, Barlow and Proschan [6], Gupta and Kirmani [10], Kochar [17,18], Shaked and Shanthikumar [22], Raqab and Amin [21], Khaledi and Kochar [15], and Belzunce, Franco, Ruiz, and Ruiz [7].

Particularly, many researchers have obtained stochastic comparison results among spacings of order statistics and record values. Barlow and Proschan [5] and Kochar and Kirmani [20] compared normalized spacings of order statistics from a random sample with decreasing failure rate (DFR) distribution. In the case of two samples, Kochar [19] and Khaledi and Kochar [14] established comparisons between spacings of order statistics when the populations were ordered and one of them belonged to a certain aging class. Hu and Wei [11] considered the problem of stochastic comparisons of generalized spacings according to various kinds of stochastic order. In addition, Kochar [17] obtained sufficient conditions to compare spacings of record values in the stochastic order.

The purpose of this article is to establish some stochastic comparison results among normalized spacings of gOSs formed from two distributions F and G. We show that the results obtained are general in the sense that they hold for many ordered random variables, including ordinary order statistics and record values. Furthermore, we can compare spacings of different models (i.e., between ordinary order statistics and sequential order statistics, between record values and Pfeifer's record values, and so forth).

In Section 2, we define the concept of generalized order statistics and recall the definition of some stochastic orders and aging classes. In Section 3, we study the preservation of some orderings between gOSs based on one and two distributions. Finally, in Section 4, normalized and simple spacings of gOSs are compared and different applications of these results are given.

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2. DEFINITIONS AND PREVIOUS RESULTS

First, we give the definition of the rth gOS based on a distribution function F using its marginal density function, which was introduced originally in Kamps [12] through the definition of uniform gOS.

Let the function $g_m(x)$, $m \in \mathbb{R}$, be defined as follows:

$$g_m(x) = \begin{cases} \dfrac{1}{m+1}\,(1-(1-x)^{m+1}) & \text{if } m \neq -1 \\ \\ \log\left(\dfrac{1}{1-x}\right) & \text{if } m = -1 \end{cases} \qquad \text{for all } x \in [0,1). \tag{2.1}$$

DEFINITION 2.1: *Let $n \in \mathbb{N}$, $1 \le r \le n$, $k \ge 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, and $M_s = \sum_{j=s}^{n-1} m_j$, $1 \le s \le n-1$, be parameters such that*

$$\gamma_s = k + n - s + M_s \ge 1 \quad \text{for all } s \in \{1, \ldots, n-1\},$$

$$m_1 = m_2 = \cdots = m_{r-1} = m$$

and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \ge 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary if $n = 1$). The marginal density function of the rth gOS $X_{(r,n,\tilde{m},k)}$ based on F is given by

$$f_{X_{(r,n,\tilde{m},k)}}(x) = \varphi_{r:n}(F(x))\, f(x), \tag{2.2}$$

where

$$\varphi_{r:n}(x) = \frac{c_{r-1}}{(r-1)!}\,(1-x)^{\gamma_r - 1} g_m^{r-1}(x), \quad x \in (0,1), \tag{2.3}$$

and $c_{r-1} = \prod_{i=1}^{r} \gamma_i$.

Note that the corresponding marginal distribution function of the rth gOS based on F is

$$F_{X_{(r,n,\tilde{m},k)}}(x) = \phi_{r:n}(F(x)), \tag{2.4}$$

where

$$\phi_{r:n}(x) = 1 - c_{r-1}(1-x)^{k+n-r+M_r} \sum_{j=0}^{r} \frac{1}{j!\, c_{r-j-1}}\, g_m^j(x), \quad x \in (0,1). \tag{2.5}$$

Let $m_1 = \cdots = m_{n-1} = 0$ and $k = 1$; then, Eq. (2.2) reduces to the density of the rth ordinary order statistic of a random sample with distribution function F. Let $m_1 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$; then, the model of kth record values is obtained.

Let us now recall the definitions of some stochastic orderings and aging classes that we will use in this article; see, for instance, Shaked and Shanthikumar [22].

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Let X and Y be the absolutely continuous r.v. with lifetime distributions F and G, respectively, and with f and g their probability density functions. It is said that X is smaller than Y in the stochastic ordering, $X \leq_{\text{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x, where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ are the survival functions of X and Y, respectively. This is equivalent to $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing function ϕ for which the expectations exists. It is said that X is smaller than Y in the failure rate ordering, $X \leq_{\text{hr}} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in x. It is said that X is smaller than Y in the reversed failure rate ordering, $X \leq_{\text{rh}} Y$, if $G(x)/F(x)$ is increasing in x. It is said that X is smaller than Y in the likelihood ratio ordering, $X \leq_{\text{lr}} Y$, if $g(x)/f(x)$ is increasing in x, and X is said to be smaller than Y in the dispersive ordering, $X \leq_{\text{disp}} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$, where F^{-1} and G^{-1} are the right-continuous inverses of F and G, respectively.

The relationships among these orderings are shown in the following diagram:

$$\begin{array}{ccc} X \leq_{\text{lr}} Y & \Rightarrow & X \leq_{\text{hr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\text{rh}} Y & \Rightarrow & X \leq_{\text{st}} Y. \end{array}$$

For convenience, we write $F \leq_{\text{Order}} G$ instead of $X \leq_{\text{Order}} Y$, where $\leq_{\text{Order}}$ is one of the above stochastic orders.

On the other hand, it is said that F is increasing (decreasing) failure rate, IFR (DFR), if $\bar{F}(x+a)/\bar{F}(x)$ is decreasing (increasing) in x for all $a \geq 0$; F is decreasing (increasing) reversed failure rate, DRF (IRF), if $F(x+a)/F(x)$ is decreasing (increasing) in x for all $a \geq 0$; and F is increasing (decreasing) likelihood ratio, ILR (DLR), if $f(x+a)/f(x)$ is decreasing (increasing) in x for all $a \geq 0$.

Remark 2.1: Bagai and Kochar [3] proved the following result that relates the failure rate and dispersive orderings and which will be used in this article: (i) if $F \leq_{\text{hr}} G$ and either F or G is DFR, then $F \leq_{\text{disp}} G$; (ii) if $F \leq_{\text{disp}} G$ and either F or G is IFR, then $F \leq_{\text{hr}} G$.

We conclude this section by describing some results of different authors on orderings among certain models of gOSs (ordinary order statistics, record values, etc.) and particularly among their spacings.

Let $X_{(1,n)} \leq X_{(2,n)} \leq \cdots \leq X_{(n,n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a distribution F, and let $Y_{(1,m)} \leq Y_{(2,m)} \leq \cdots \leq Y_{(m,m)}$ be the order statistics of a random sample $Y_1, Y_2, \ldots, Y_m$ from another distribution G. Then, $C_{i:n} = (n-i+1)(X_{(i,n)} - X_{(i-1,n)})$ and $D_{j:m} = (m-j+1)(Y_{(j,m)} - Y_{(j-1,m)})$ are called normalized spacings from F and G, respectively.

There is much literature on sample spacings, from the study of them in the context of empirical processes to their applications in goodness-of-fit tests. Note that $X_{(i,n)} - X_{(i-1,n)}$ represents the time between the ith and $(i+1)$th failures when n independent and identically distributed (i.i.d.) items are put to a life test. Further, it may be considered as the additional lifetime to be gained on using an $(n-i)$-out-of-n system rather than an $(n-i+1)$-out-of-n system (see Kirmani [16]).

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For one sample and under the assumption that G is DFR, Barlow and Proschan [5] proved that $D_{j:m} \leq_{\mathrm{st}} D_{j+1:m}$ and $D_{j:m+1} \leq_{\mathrm{st}} D_{j:m}$ for $j = 1,\ldots,m-1$, whereas Kochar and Kirmani [20] strengthened idem results to the failure rate and dispersive orderings. Kochar [19] and Khaledi and Kochar [14] considered the problem of stochastic comparisons of the spacings of two samples, the first one in the particular case of $i = j$ and $n = m$. In Khaledi and Kochar [14], it was shown that if $F \leq_{\mathrm{hr}} (\leq_{\mathrm{lr}}) G$ and if either F or G is DFR, then $C_{i:n} \leq_{\mathrm{st}} (\leq_{\mathrm{hr}}) D_{j:m}$ for all $i \leq j$ and $j - i \geq m - n$. If, instead, $F \leq_{\mathrm{lr}} G$ and if either F or G is DLR, then $C_{i:n} \leq_{\mathrm{lr}} D_{j:m}$ for all $i \leq j$ and $j - i \geq m - n$. Hu and Wei [11] have extended some of the above results comparing generalized spacings of ordinary order statistics of one sample and two samples.

Let $X_{L(1)},\ldots,X_{L(r)},\ldots$ and $Y_{L(1)},\ldots,Y_{L(r)},\ldots$ be the record values based on the sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of i.i.d. r.v.'s with distribution functions F and G, respectively. Kochar [17] proved that if $F \leq_{\mathrm{disp}} G$, then $X_{L(n+k)} - X_{L(k)} \geq_{\mathrm{st}} Y_{L(n+k)} - Y_{L(k)}$, and if $F \leq_{\mathrm{hr}} G$ and either F or G is DFR, then $X_{L(n+k)} - X_{L(k)} \geq_{\mathrm{st}} Y_{L(n+k)} - Y_{L(k)}$.

3. ORDERINGS BETWEEN GENERALIZED ORDER STATISTICS

In this section, we first study conditions on the parameters to compare gOSs based on the same distribution, and then based on two different distributions. As consequences, we obtain the preservation of the stochastic, failure rate, and likelihood ratio ordering under the formation of gOSs, which extend the preservation results of other authors.

For that, we will denote $\tilde{m} = (m_1,\ldots,m_{n_1-1})$, $m_1 = \cdots = m_{r_1-1} = m$, $\widetilde{m'} = (m'_1,\ldots,m'_{n_2-1})$, $m'_1 = \cdots = m'_{r_2-1} = m'$, $\gamma_{r_1} = k_1 + n_1 - r_1 + M_{r_1}$, and $\gamma'_{r_2} = k_2 + n_2 - r_2 + M'_{r_2}$.

The following theorem extends some results of Bapat and Kochar [4] and Raqab and Amin [21] on ordinary order statistics, Gupta and Kirmani [10] and Kochar [17,18] on record values, and Raqab and Amin [21] on k record values. In addition, Kamps [12] proved the result when $\tilde{m} = \widetilde{m'}$.

THEOREM 3.1: *Let $X_{(r_1,n_1,\tilde{m},k_1)}$ and $X_{(r_2,n_2,\widetilde{m'},k_2)}$ be gOSs based on an absolutely continuous distribution function F. Then, $X_{(r_1,n_1,\tilde{m},k_1)} \leq_{\mathrm{lr}} X_{(r_2,n_2,\widetilde{m'},k_2)}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, and $m \geq m' \geq -1$.*

PROOF: If $m \geq m' > -1$, we have to prove that

$$\frac{f_{X_{(r_2,n_2,\widetilde{m'},k_2)}}(x)}{f_{X_{(r_1,n_1,\tilde{m},k_1)}}(x)} = \frac{c_{r_2-1}(r_1-1)!}{c_{r_1-1}(r_2-1)!}\,\frac{(m+1)^{r_1-1}}{(m'+1)^{r_2-1}}\,(\bar{F}(x))^{\gamma'_{r_2}-\gamma_{r_1}}\,\frac{(1-\bar{F}^{m'+1}(x))^{r_2-1}}{(1-\bar{F}^{m+1}(x))^{r_1-1}}$$

is increasing in x for all $r_2 \geq r_1$ and $\gamma_{r_1} \geq \gamma'_{r_2}$, so it is sufficient to prove that

$$\frac{(1-\bar{F}^{m'+1}(x))^{r_2-1}}{(1-\bar{F}^{m+1}(x))^{r_1-1}} \text{ is increasing in } x. \tag{3.1}$$

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Since $1-\bar{F}^{m+1} \leq_{\mathrm{lr}} 1-\bar{F}^{m'+1}$ for all $m \geq m' > -1$, we have that

$$(1-\bar{F}^{m+1})^{r_1-1} \leq_{\mathrm{lr}} (1-\bar{F}^{m'+1})^{r_2-1} \quad \text{for all } r_2 \geq r_1; \tag{3.2}$$

therefore, Eq. (3.1) holds from Eq. (3.2) and the implication among orderings $\leq_{\mathrm{lr}} \Rightarrow \leq_{\mathrm{rh}}$.

Using the same arguments when $m = m' = -1$ and $m > m' = -1$, the required result follows. ■

Next, theorems are devoted to stochastic comparisons of gOSs from distribution functions F and G. The first theorem is immediate taking into account Eq. (2.4), the definition of the stochastic order, and that $\phi_{r:n}(x)$ is a distribution function (see Kamps [12]).

THEOREM 3.2: *If* $F \leq_{\mathrm{st}} G$, *then* $X_{(r,n,\tilde{m},k)} \leq_{\mathrm{st}} Y_{(r,n,\tilde{m},k)}$.

THEOREM 3.3: *If* $F \leq_{\mathrm{hr}} G$, *then* $X_{(r,n,\tilde{m},k)} \leq_{\mathrm{hr}} Y_{(r,n,\tilde{m},k)}$.

PROOF: The failure rate function $h^F_{r,n,\tilde{m},k}$ of the rth gOS $X_{(r,n,\tilde{m},k)}$ based on F is given by

$$h^F_{r,n,\tilde{m},k}(x) = \frac{1}{(r-1)!}\,\frac{f(x)}{1-F(x)}\,\frac{g_m^{r-1}(F(x))}{\displaystyle\sum_{j=0}^{r-1}\frac{1}{j!\,c_{r-j-1}}\,g_m^j(F(x))}$$

(cf. Kamps [12, Remark 1.1]). Since $F \leq_{\mathrm{hr}} G \Rightarrow F \leq_{\mathrm{st}} G$ and $f(x)/(1-F(x)) \geq g(x)/(1-G(x))$ for all x, it is sufficient to prove that the function

$$\frac{g_m^{r-1}(t)}{\displaystyle\sum_{j=0}^{r-1}\frac{1}{j!\,c_{r-j-1}}\,g_m^j(t)} \text{ is increasing in } t. \tag{3.3}$$

Finally, note that Eq. (3.3) is equivalent to showing that

$$\sum_{j=0}^{r-1}\frac{1}{j!\,c_{r-j-1}}\,\frac{1}{g_m^{r-j-1}(t)} \text{ is decreasing in } t,$$

which follows, since $g_m(t)$ is increasing in t. ■

THEOREM 3.4: *If* $F \leq_{\mathrm{lr}} G$, *then* $X_{(r,n,\tilde{m},k)} \leq_{\mathrm{lr}} Y_{(r,n,\tilde{m},k)}$ *for all* $m \geq 0$.

PROOF: Using the definition of the likelihood ordering, Eqs. (2.2) and (2.3), we need to prove that

$$\frac{g_{Y_{(r,n,\tilde{m},k)}}(x)}{f_{X_{(r,n,\tilde{m},k)}}(x)} = \frac{\varphi_{r,n}(G(x))g(x)}{\varphi_{r,n}(F(x))f(x)} = \left(\frac{\bar{G}(x)}{\bar{F}(x)}\right)^{\gamma_r-1}\left(\frac{g_m(G(x))}{g_m(F(x))}\right)^{r-1}\frac{g(x)}{f(x)}$$

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is increasing in x. Then, from the assumption $F \leq_{\mathrm{lr}} G$ and the implication among orderings $\leq_{\mathrm{lr}} \Rightarrow \leq_{\mathrm{hr}}$, it is sufficient to see that

$$\left(\frac{g_m(G(x))}{g_m(F(x))}\right)^{r-1} \text{ is increasing in } x$$

for all $m \geq 0$; that is,

$$\left(\frac{1-\bar{G}^{m+1}(x)}{1-\bar{F}^{m+1}(x)}\right)^{r-1} \text{ is increasing in } x. \tag{3.4}$$

Finally, Eq. (3.4) holds because of the implication $F \leq_{\mathrm{lr}} G \Rightarrow 1-\bar{F}^{m+1} \leq_{\mathrm{lr}} 1-\bar{G}^{m+1}$ for all $m \geq 0$ and the relationship among orderings $\leq_{\mathrm{lr}} \Rightarrow \leq_{\mathrm{rh}}$. ■

Taking into account the previous theorems, we easily obtain the following results on preservation of orderings under the formation of gOSs with different parameters and based on F and G, which extend the results of Singh and Vijayasree [23] and Raqab and Amin [21] on ordinary order statistics.

THEOREM 3.5: *If* $F \leq_{\mathrm{st}} G$, *then* $X_{(r_1,n_1,\tilde{m},k_1)} \leq_{\mathrm{st}} Y_{(r_2,n_2,\tilde{m}',k_2)}$ *for all* $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, *and* $m \geq m' \geq -1$.

THEOREM 3.6: *If* $F \leq_{\mathrm{hr}} G$, *then* $X_{(r_1,n_1,\tilde{m},k_1)} \leq_{\mathrm{hr}} Y_{(r_2,n_2,\tilde{m}',k_2)}$ *for all* $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, *and* $m \geq m' \geq -1$.

THEOREM 3.7: *If* $F \leq_{\mathrm{lr}} G$, *then* $X_{(r_1,n_1,\tilde{m},k_1)} \leq_{\mathrm{lr}} Y_{(r_2,n_2,\tilde{m}',k_2)}$ *for all* $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, *and* $m \geq m' \geq 0$.

4. ORDERINGS BETWEEN SPACINGS

In this section, we establish stochastic comparisons among spacings of gOSs from restricted families of distributions.

For that, we define normalized spacings of gOSs from two absolutely continuous distribution functions F and G by $U_{r_1:n_1} = \gamma_{r_1}(X_{(r_1,n_1,\tilde{m},k_1)} - X_{(r_1-1,n_1,\tilde{m},k_1)})$ and $V_{r_2:n_2} = \gamma'_{r_2}(Y_{(r_2,n_2,\tilde{m}',k_2)} - Y_{(r_2-1,n_2,\tilde{m}',k_2)})$ for $2 \leq r_1 \leq n_1$ and $2 \leq r_2 \leq n_2$.

First, we state Remark 2.8 of Kamps [12] about the Markov chain property of gOSs.

Remark 4.1: The gOSs $X_{(r,n,\tilde{m},k)}$, $r = 2,\ldots,n$ form a Markov chain with transition probabilities

$$P(X_{(r,n,\tilde{m},k)} > t \,|\, X_{(r-1,n,\tilde{m},k)} = s) = \left(\frac{\bar{F}(t)}{\bar{F}(s)}\right)^{\gamma_r}, \quad t \geq s.$$

From this remark, the survival function of the normalized spacing, $U_{r:n}$, of gOSs is given by

$$\bar{F}_{U_{r:n}}(x) = \int_{\mathbb{R}} \left(\frac{\bar{F}\left(\frac{x}{\gamma_r}+y\right)}{\bar{F}(y)}\right)^{\gamma_r} d\phi_{r-1:n}(F(y)). \tag{4.1}$$

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Now, we give a theorem that will be used for proving the various results of this section and in which we give sufficient conditions to compare conditional spacings of gOSs in the failure rate order.

THEOREM 4.1: *Let $F \leq_{\mathrm{hr}} G$ and either F or G be DFR. Then, for $\gamma_{r_1} \geq \gamma'_{r_2}$ and $y \geq 0$,*

$$U_{y,r_1:n_1} = (U_{r_1:n_1} | X_{(r_1-1,n_1,\tilde{m},k_1)} = y) \leq_{\mathrm{hr}} (V_{r_2:n_2} | Y_{(r_2-1,n_2,\tilde{m}',k_2)} = y) = V_{y,r_2:n_2}.$$

PROOF: The result can be obtained as Lemma 2.1(b) of Khaledi and Kochar [14] and taking into account that the failure rates of $U_{y,r_1:n_1}$ and $V_{y,r_2:n_2}$ are given by

$$r_{U_{y,r_1:n_1}}(x) = r_F\left(\frac{x}{\gamma_{r_1}} + y\right) \quad \text{and} \quad r_{V_{y,r_2:n_2}}(x) = r_G\left(\frac{x}{\gamma_{r_2}} + y\right),$$

where r_F and r_G are the failure rates of F and G, respectively. ■

The following theorem establishes the stochastic order among spacings of two samples of gOSs, when the distribution functions are ordered in the hazard rate order and one of them is supposed to have a decreasing failure rate function.

THEOREM 4.2: *Let $F \leq_{\mathrm{hr}} G$ and either F or G be DFR. Then, $U_{r_1:n_1} \leq_{\mathrm{st}} V_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, and $m \geq m' \geq -1$.*

PROOF: The proof is similar to that of Theorem 2.1 in [14] using Theorem 3.5 and that the survival functions of $U_{r_1:n_1}$ and $V_{r_2:n_2}$ can be written as

$$\bar{F}_{U_{r_1:n_1}}(x) = \int_{\mathbb{R}} \left(\frac{\bar{F}\left(\frac{x}{\gamma_{r_1}} + y\right)}{\bar{F}(y)} \right)^{\gamma_{r_1}} d\phi_{r_1-1:n_1}(F(y))$$

and

$$\bar{G}_{V_{r_2:n_2}}(x) = \int_{\mathbb{R}} \left(\frac{\bar{G}\left(\frac{x}{\gamma'_{r_2}} + y\right)}{\bar{G}(y)} \right)^{\gamma'_{r_2}} d\phi_{r_2-1:n_2}(G(y)).$$ ■

Note that if $\tilde{m} = (0,\ldots,0)$, $\tilde{m}' = (0,\ldots,0)$, and $k_1 = k_2 = 1$, we obtain Theorem 1.2 of [19] and Theorem 2.1 of [14] on the ordering of spacings of ordinary order statistics, and if $\tilde{m} = (-1,\ldots,-1)$, $\tilde{m}' = (-1,\ldots,-1)$, and $k_1 = k_2 = 1$, we obtain Corollary 3.2 of [17] on record values.

Analogously, in the next theorem, we establish comparisons in the stochastic order between spacings of gOSs, when the distribution functions are ordered in the reversed hazard rate order and one of them has increasing reversed failure rate function.

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The proof is immediate taking into account Theorem 4.2 and the equivalences

$$X \leq_{\mathrm{rh}} Y \Leftrightarrow -X \geq_{\mathrm{hr}} -Y \quad \text{and} \quad X \in \mathrm{IRF} \Leftrightarrow -X \in \mathrm{DFR}.$$

THEOREM 4.3: *Let $F \leq_{\mathrm{rh}} G$ and either F or G be IRF. Then, $W_{r_1:n_1} \leq_{\mathrm{st}} Z_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$ and $m \geq m' \geq -1$, where $P(W \leq t) = \bar{G}(-t)$ and $P(Z \leq t) = \bar{F}(-t)$.*

In particular, we obtain the next corollary for spacings of ordinary order statistics, which is a particular case of Corollary 3.1 of [11] when $j = i + 1$, $q = p + 1$, $p \geq i$, and $p - i \geq m - n$.

COROLLARY 4.4: *Let $F \leq_{\mathrm{rh}} G$ and either F or G be IRF. Then,*

$$\begin{aligned}(n_1 - r_1 + 1)&(Y_{(n_1-r_1+2,\,n_1)} - Y_{(n_1-r_1+1,\,n_1)}) \\ &\leq_{\mathrm{st}} (n_2 - r_2 + 1)(X_{(n_2-r_2+2,\,n_2)} - X_{(n_2-r_2+1,\,n_2)})\end{aligned}$$

for all $r_2 \geq r_1$ and $n_2 - r_2 \leq n_1 - r_1$.

PROOF: Taking $\tilde{m} = (0,\ldots,0)$, $\widetilde{m'} = (0,\ldots,0)$, and $k_1 = k_2 = 1$ in Theorem 4.3, we have that

$$(n_1 - r_1 + 1)(W_{(r_1,n_1)} - W_{(r_1-1,n_1)}) \leq_{\mathrm{st}} (n_2 - r_2 + 1)(Z_{(r_2,n_2)} - Z_{(r_2-1,n_2)})$$

for all $r_2 \geq r_1$ and $n_2 - r_2 \leq n_1 - r_1$. Finally, the required result follows from the fact

$$W_{(r_1,n_1)} =_{\mathrm{st}} -Y_{(n_1-r_1+1,n_1)} \quad \text{and} \quad Z_{(r_2,n_2)} =_{\mathrm{st}} -X_{(n_2-r_2+1,n_2)}. \qquad \blacksquare$$

Let us see now comparisons between spacings of gOSs when their populations are ordered in the likelihood ratio ordering. For that, we shall need the following lemma of Kochar and Kirmani [20].

LEMMA 4.5: *Let $\psi_1(x,y)$ and $\psi_2(x,y)$ be positive real-valued functions such that*

(i) for $y_1 \leq y_2$, $\psi_2(x,y_2)/\psi_2(x,y_1)$ is increasing in x,
(ii) for $y_1 \leq y_2$, $\psi_1(x,y_2)/\psi_2(x,y_1)$ is increasing in x,
(iii) for each fixed x, $\psi_1(x,y)/\psi_2(x,y)$ is increasing in y.

Then, for functions ψ_1 and ψ_2 satisfying the above conditions, $Z_1 \leq_{\mathrm{lr}} Z_2$ implies that $E[\psi_1(x,Z_2)]/E[\psi_2(x,Z_1)]$ is increasing in x, provided the expectations exist.

THEOREM 4.6: *Let $F \leq_{\mathrm{lr}} G$ and either F or G be DFR. Then, $U_{r_1:n_1} \leq_{\mathrm{hr}} V_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, and $m \geq m' \geq 0$.*

PROOF: The result follows as Theorem 2.2 of [14] using Lemma 4.5 and that the ratio of the survival functions of $V_{r_2:n_2}$ and $U_{r_1:n_1}$ may be rewritten as

$$\frac{E[\psi_1(x, W_{(r_2-1,\,r_2-1)})]}{E[\psi_2(x, Z_{(r_1-1,\,r_1-1)})]},$$

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where

$$\psi_1(x,y) = \frac{c_{r_2-2}}{(r_2-1)!}\,\frac{1}{(m'+1)^{r_2-1}}\,(\bar{G}(x/\gamma'_{r_2}+y))^{\gamma'_{r_2}},$$

$$\psi_2(x,y) = \frac{c_{r_1-2}}{(r_1-1)!}\,\frac{1}{(m+1)^{r_1-1}}\,(\bar{F}(x/\gamma_{r_1}+y))^{\gamma_{r_1}},$$

and W and Z are random variables with distribution functions $H_W(y) = 1 - \bar{G}^{m'+1}(y)$ and $H_Z(y) = 1 - \bar{F}^{m+1}(y)$, respectively. ■

Note that Theorem 2.1 of [19] and Theorem 2.2 of [14] can be obtained from Theorem 4.6 taking $\widetilde{m} = (0,\ldots,0)$, $\widetilde{m'} = (0,\ldots,0)$, and $k_1 = k_2 = 1$.

Analogously, if the distributions are likelihood ordered and one of them has an increasing reversed failure rate function, we get the next result. It follows from Theorem 4.6 and the equivalences

$$F \leq_{\mathrm{lr}} G \Leftrightarrow -F \geq_{\mathrm{lr}} -G \quad \text{and} \quad F \in \mathrm{IRF} \Leftrightarrow -F \in \mathrm{DFR}.$$

THEOREM 4.7: *Let $F \leq_{\mathrm{lr}} G$ and either F or G be IRF. Then, $W_{r_1:n_1} \leq_{\mathrm{hr}} Z_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, and $m \geq m' \geq 0$, where $P(W \leq t) = \bar{G}(-t)$ and $P(Z \leq t) = \bar{F}(-t)$.*

In particular, we obtain the next corollary for spacings of ordinary order statistics.

COROLLARY 4.8: *Let $F \leq_{\mathrm{lr}} G$ and either F or G be IRF. Then,*

$$(n_1 - r_1 + 1)(Y_{(n_1-r_1+2,\,n_1)} - Y_{(n_1-r_1+1,\,n_1)}) \leq_{\mathrm{hr}} (n_2 - r_2 + 1)(X_{(n_2-r_2+2,\,n_2)} - X_{(n_2-r_2+1,\,n_2)})$$

for all $r_2 \geq r_1$ and $n_2 - r_2 \leq n_1 - r_1$.

Normalized spacings from a DFR distribution are DFR, as it is shown in Theorem 1.15 of [12]. Thus, from Theorem 4.6 and Remark 2.1 we obtain the following corollary.

COROLLARY 4.9: *Let $F \leq_{\mathrm{lr}} G$ and either F or G be DFR. Then, $U_{r_1:n_1} \leq_{\mathrm{disp}} V_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} \geq \gamma'_{r_2}$, and $m \geq m' \geq 0$.*

Note that if $F \leq_{\mathrm{hr}} G$, F is DFR, and G is IFR, then $F \leq_{\mathrm{lr}} G$. Thus, we can apply Theorem 4.6 and Corollary 4.9, obtaining, in the first case, sufficient conditions to preserve the failure rate order under the formation of spacings of gOSs.

However, there are examples of distributions F and G such that $F \leq_{\mathrm{hr}} G$, F is IFR, and G is DFR, whereas $F \not\leq_{\mathrm{lr}} G$ (see Example 3.1 of [19]). Therefore, Theorem 4.6 is not applicable. In this setting, Hu and Wei [11] established some comparison results between normalized spacings of ordinary order statistics, which we extend to the model of gOSs in the following theorem and corollaries.

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THEOREM 4.10: *If there exists an exponential random variable Z with distribution function H_Z such that $F \leq_{\text{hr}} H_Z \leq_{\text{hr}} G$, then $U_{r_1:n_1} \leq_{\text{hr}} V_{r_2:n_2}$ for all parameters $(r_1, n_1, \tilde{m}, k_1)$ and $(r_2, n_2, \widetilde{m'}, k_2)$ such that $(m+1)^{r_1-1}(m'+1)^{r_2-1} \geq 0$.*

PROOF: The result can be obtained as Theorem 3.2 of [11] taking into account that the ratio of the survival functions of $V_{r_2:n_2}$ and $U_{r_1:n_1}$ can be written as (see proof of Theorem 4.6)

$$\frac{\tilde{C}(r_2:n_2)E[\hat{\psi}_1(x, W_{(r_2-1,r_2-1)})]}{\tilde{C}(r_1:n_1)E[\hat{\psi}_2(x, Z_{(r_1-1,r_1-1)})]},$$

where

$$\tilde{C}(r_2:n_2) = \frac{c_{r_2-2}}{(r_2-1)!}\,\frac{1}{(m'+1)^{r_2-1}},$$

$$\tilde{C}(r_1:n_1) = \frac{c_{r_1-2}}{(r_1-1)!}\,\frac{1}{(m+1)^{r_1-1}},$$

$$\hat{\psi}_1(x,y) = (\bar{G}(x/\gamma'_{r_2} + y))^{\gamma'_{r_2}}$$

and

$$\hat{\psi}_2(x,y) = (\bar{F}(x/\gamma_{r_1} + y))^{\gamma_{r_1}}. \qquad \blacksquare$$

As a consequence of Theorem 4.10, we easily obtain the next corollary when the distributions are ordered in the reversed failure rate ordering just taking into account the equivalences

$$F \leq_{\text{rh}} G \Leftrightarrow -F \geq_{\text{hr}} -G$$

and

$$F \in \text{IFR(DFR)} \Leftrightarrow -F \in \text{DRF(IRF)}.$$

COROLLARY 4.11: *Let $F \leq_{\text{rh}} G$. If one of F and G is IRF and the other is DRF, then $W_{r_1:n_1} \leq_{\text{hr}} Z_{r_2:n_2}$ for all parameters $(r_1, n_1, \tilde{m}, k_1)$ and $(r_2, n_2, \widetilde{m'}, k_2)$ such that $(m+1)^{r_1-1}(m'+1)^{r_2-1} \geq 0$, where $P(W \leq t) = \bar{G}(-t)$ and $P(Z \leq t) = \bar{F}(-t)$.*

Now, using Theorem 4.10, Remark 2.1, and that normalized spacings of gOSs from a DFR distribution have the DFR property, we get the next corollary on preservation of the dispersive ordering under the formation of spacings of gOSs.

COROLLARY 4.12: *Let $F \leq_{\text{disp}} G$. If one of F and G is IFR and the other is DFR, then $U_{r_1:n_1} \leq_{\text{disp}} V_{r_2:n_2}$ for all parameters $(r_1, n_1, \tilde{m}, k_1)$ and $(r_2, n_2, \widetilde{m'}, k_2)$ such that $(m+1)^{r_1-1}(m'+1)^{r_2-1} \geq 0$.*

Finally, we give sufficient conditions to preserve the likelihood ratio ordering under the formation of spacings of gOSs.

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THEOREM 4.13: *Let $F \leq_{\mathrm{lr}} G$ and either F or G be DLR. Then, $U_{r_1:n_1} \leq_{\mathrm{lr}} V_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} = \gamma'_{r_2}$, and $m \geq m' \geq 0$.*

PROOF: Note that the ratio of the density functions of $V_{r_2:n_2}$ and $U_{r_1:n_1}$ is given by

$$\frac{\tilde{C}(r_2:n_2)E[g(x/\gamma'_{r_2} + W_{(r_2-1,r_2-1)})(\bar{G}(x/\gamma'_{r_2} + W_{(r_2-1,r_2-1)}))^{\gamma'_{r_2}-1}]}{\tilde{C}(r_1:n_1)E[\,f(x/\gamma_{r_1} + Z_{(r_1-1,r_1-1)})(\bar{F}(x/\gamma_{r_1} + Z_{(r_1-1,r_1-1)}))^{\gamma_{r_1}-1}]}, \qquad \textbf{(4.2)}$$

which can be expressed as

$$\frac{E[\tilde{\psi}_1(x, W_{(r_2-1,r_2-1)})]}{E[\tilde{\psi}_2(x, Z_{(r_1-1,r_1-1)})]},$$

where $\tilde{\psi}_1(x,y) = \tilde{C}(r_2:n_2)g(x/\gamma'_{r_2} + y)(\bar{G}(x/\gamma'_{r_2} + y))^{\gamma'_{r_2}-1}$ and $\tilde{\psi}_2(x,y) = \tilde{C}(r_1:n_1)f(x/\gamma_{r_1} + y)(\bar{F}(x/\gamma_{r_1} + y))^{\gamma_{r_1}-1}$. From Lemma 4.5, the result follows as Theorem 4.6. ■

Once again, Theorem 2.3 of [14] can be obtained from Theorem 4.13 taking $\tilde{m} = (0,\ldots,0)$, $\widetilde{m'} = (0,\ldots,0)$, and $k_1 = k_2 = 1$.

Remark 4.14: The results given in this section also hold for simple spacings of gOSs $\tilde{U}_{r_1:n_1} = X_{(r_1,n_1,\tilde{m},k_1)} - X_{(r_1-1,n_1,\tilde{m},k_1)}$ and $\tilde{V}_{r_2:n_2} = Y_{(r_2,n_2,\widetilde{m'},k_2)} - Y_{(r_2-1,n_2,\widetilde{m'},k_2)}$.

To conclude this work, we show some practical applications of the results of this section, comparing spacings of the same ordered r.v. model and comparing spacings of different ordered r.v. models.

5. APPLICATIONS

1. Choosing $\tilde{m} = (-1,\ldots,-1)$, $\widetilde{m'} = (-1,\ldots,-1)$, and $k_1 = k_2 = 1$ in Theorem 4.2, we get comparisons in the stochastic order between spacings of record values. Let us consider two systems formed from two different types of i.i.d. components (i.e., with distribution functions F and G). This result allows us to compare the waiting times between two consecutive failures of the components of a system with the waiting times between another two consecutive failures of the components of the other system. Likewise, suppose that the systems are subject to shocks viewed as realizations of two different sequences of i.i.d. random variables with distribution functions F and G, respectively. Then, we can compare stochastically the waiting times between consecutive shocks of both systems or the intensity differences between consecutive shocks of the two systems.
2. Given positive real numbers $\alpha_1,\ldots,\alpha_{n_1}$ and $\beta_1,\ldots,\beta_{n_2}$, we take $m_i = (n_1 - i+1)\alpha_i - (n_1 - i)\alpha_{i+1} - 1$, $i = 1,\ldots,n_1-1$, $k_1 = \alpha_{n_1}$, $m'_i = (n_2 - i+1)\beta_i - (n_2 - i)\beta_{i+1} - 1$, $i = 1,\ldots,n_2-1$, and $k_2 = \alpha_{n_2}$ to obtain the sequential order statistics based on $F_s(x) = 1 - (1 - F(x))^{\alpha_s}$, $1 \leq s \leq n$. This model is more flexible and adequate for describing $(n - r + 1)$-out-of-n systems than ordinary order statistics, because it takes a specific dependence structure into

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consideration. If some component of the system fails, this may have an influence on the life-length distributions of the remaining components; that is, consider n components with life-length distribution F_1. At time x, we observe the first failure. Then, we suppose that the remaining components possess the life-length distribution F_2 truncated on the left at x, and so on. From results of Section 4, we can compare the additional lifetime to be gained on using a modified $(n_1 - r_1)$-out-of-n_1 system rather than a modified $(n_1 - r_1 + 1)$-out-of-n_1 system, both formed from $F_s(x) = 1 - (1 - F(x))^{\alpha_s}$, with the additional lifetime to be gained on using a modified $(n_2 - r_2)$-out-of-n_2 system rather than a modified $(n_2 - r_2 + 1)$-out-of-n_2 system, both formed from $G_s(x) = 1 - (1 - G(x))^{\beta_s}$.

3. Choosing suitable parameters (see Kamps [12]), we can compare spacings of Pfeifer's record values and we obtain similar comments to those made in Application 1, but each shock is now allowed to influence the magnitude of the subsequent one. Moreover, we can obtain this kind of application in a more general shock model using k_n records.
4. Finally, choosing suitable $\widetilde{m}$, $\widetilde{m'}$, k_1, and k_2, we can establish comparisons between the spacings of different ordered r.v. models. For instance, we can compare the additional lifetime to be gained on using an $(n_1 - r_1)$-out-of-n_1 system rather than an $(n_1 - r_1 + 1)$-out-of-n_1 system with the additional lifetime to be gained on using a modified $(n_2 - r_2)$-out-of-n_2 system rather than a modified $(n_2 - r_2 + 1)$-out-of-n_2 system. Also, we can compare spacings of records with spacings of Pfeifer's records or k_n records.

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