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Generic chaos on dendrites

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Dedicated to the memory of Sylvie Ruette

Abstract. We characterize dendrites D such that a continuous selfmap of D is generically chaotic (in the sense of Lasota) if and only if it is generically ε-chaotic for some $\varepsilon > 0$. In other words, we characterize dendrites on which generic chaos of a continuous map can be described in terms of the behaviour of subdendrites with non-empty interiors under iterates of the map. A dendrite D belongs to this class if and only if it is completely regular, with all points of finite order (that is, if and only if D contains neither a copy of the Riemann dendrite nor a copy of the ω-star).

Key words: completely regular continuum, dendrite, generic chaos, scrambled pair
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1. *Introduction and main results*

During the last decades many interesting connections between dynamical systems and continuum theory have been studied. To illustrate this, we mention a few results.

Handel [24] has constructed a C^∞ area-preserving diffeomorphism of the plane with the pseudocircle as a minimal set.

Many authors have been investigating the problem of whether various classes of curves admit positive-entropy homeomorphisms. One of the first results was that Knaster continua [7] and the pseudoarc [34] have this property. On the other hand, every homeomorphism of a regular continuum has zero entropy [53], and even all group actions on regular continua are null [21]. By [43], also every homeomorphism of a chainable hereditarily decomposable continuum has zero entropy.

The question of whether a given class of curves admits an expansive homeomorphism has also attracted significant interest. The dyadic solenoid [61] and Plykin's attractors [48] are examples of continua that do admit expansive homeomorphisms, while tree-like continua [42] and hereditarily indecomposable continua [33] do not. For related results concerning continuum-wise expansive homeomorphisms, see [32] and references therein.

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The topology of curves appearing as inverse limit spaces of interval maps is a very active area of research; as a general reference see [31]. As an example of a deep result let us mention the proof of Ingram's conjecture stating that inverse limit spaces of tent maps with different slopes are non-homeomorphic [8].

One of the important classes of one-dimensional continua in dynamics are dendrites. Dendrites have long been studied in topology [37, 60] and it is of interest that they appear also in complex dynamics as Julia sets (see e.g. [38]). For us it is important that new dynamical phenomena, which are not possible on graphs, appear on dendrites. For instance, Ważewski's universal dendrite admits a weakly mixing, non-mixing system [26] which is proximal [1, Theorem 5.2] and has zero entropy [13]; such an example is not possible on dendrites having non-dense branch points [16]. The topology of dendrites admits ω-limit sets and minimal sets that are more complex than in simpler spaces; see [57, 6], respectively, for their topological characterizations.

Since there are many dynamical properties satisfied by all tree maps but not by all dendrite maps, it is natural to try to characterize the class of dendrites X such that every dynamical system on X possesses a given property. For example, in [30], it is proved that a dendrite has the PR-property (that is, the closure of the set of periodic points equals the closure of the set of recurrent points for every continuous selfmap) if and only if it has (at most) countably many endpoints. By [39], a dendrite has the ΩEP-property (that is, the set of non-wandering points is contained in the closure of the set of eventually periodic points for every continuous selfmap) if and only if it does not contain a null comb. (A *null comb* is any dendrite homeomorphic to the subgraph of the map $f\colon [0, 1] \to [0, 1]$ given by $f(x) = x$ for $x = 1/n$, $n \in \mathbb{N}$, and $f(x) = 0$ otherwise.)

The present paper provides another link between point-set topology (in particular continuum theory) and topological dynamics (in particular topological chaos). Namely, we characterize dendrites on which generic chaos is equivalent with generic ε-chaos; see Theorem A and the comment below it.

The notion of chaos in connection with a map was first introduced by Li and Yorke in [41], although they did not give a formal definition. As of today, Li–Yorke chaos is understood in the following way: for a dynamical system (X, f), X being a compact metric space with metric d and f being a continuous map $X \to X$, a pair (x, y) of points in X is called a *scrambled pair* if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. For every scrambled pair (x, y), there is an $\varepsilon > 0$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$ and then it is called an ε-scrambled pair. A set $S \subseteq X$ is *scrambled* or *ε-scrambled* if every pair (x, y) of distinct points in S is scrambled or ε-scrambled, respectively. The system (X, f) is *Li–Yorke chaotic* or *Li–Yorke ε-chaotic* if it has an uncountable scrambled or ε-scrambled set, respectively.

Li–Yorke chaos fits particularly well into interval dynamics, since maps of type 2^n in the Sharkovsky ordering are not Li–Yorke chaotic, maps of type greater than 2^∞ are Li–Yorke chaotic and maps of type 2^∞ may or may not be Li–Yorke chaotic. Moreover, the existence of just one scrambled pair on the interval implies the existence of a Cantor ε-scrambled set for some $\varepsilon > 0$ and hence Li–Yorke ε-chaos [36]. Also, as shown in [54], Li–Yorke chaos turns out to be the minimal requirement for a continuous selfmap of an interval to be 'chaotic'; indeed, an interval map f is either Li–Yorke ε-chaotic for

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some $\varepsilon > 0$, or all trajectories of f are approximable by periodic orbits. All of this shows that, on the interval, Li–Yorke chaos is quite a natural notion, albeit a very weak form of chaos.

Outside the interval, Li–Yorke chaos seems to be less natural. For instance, though the existence of a scrambled pair implies the existence of an uncountable (in this case even a Cantor) ε-scrambled set also on graphs [51], this is no longer true for dendrite maps [35] and for triangular maps in the square [19]. Though on graphs Li–Yorke chaos and Li–Yorke ε-chaos coincide [51], this is not true in general, as for instance Floyd's minimal system shows (see [18] and [5, pp. 24–27]); this is a minimal homeomorphism on a non-homogeneous space whose connected components are singletons and arcs. The maximal scrambled sets coincide with those arcs (cf. [15, Proposition 2]) but there is no infinite ε-scrambled set for any $\varepsilon > 0$ due to linearity of the map on the arcs. Using techniques from [15, §2.3], one can embed Floyd's system into the Cantor fan (that is, the cone over the Cantor set) in such a way that the new system still has no infinite ε-scrambled set. As regards dendrites, in the appendix we show that even every dendrite with uncountably many endpoints admits a Li–Yorke chaotic map with no infinite ε-scrambled set; see Proposition A.3.

One can ask why uncountability, and not some other 'size', is required in the definition of Li–Yorke chaos. A partial clarification for this can be found in [10], where it is shown that in many cases (not only for interval and graph maps) the uncountability of a scrambled set implies the existence of a Cantor scrambled set, which is perhaps a more natural choice of a 'large' scrambled set. Whether the existence of an uncountable scrambled set implies the existence of a Cantor scrambled set in general remains an open problem [10]. However, the existence of an uncountable ε-scrambled set in a Polish space implies the existence of a Cantor ε-scrambled set [10, Theorem 16].

It seems that Lasota was one of the first searching for a stronger type of formally defined chaos. The notion of generic chaos, suggested by him (as claimed by his student Piórek in [47]), is defined by the requirement for the set of all scrambled pairs to be generic, that is, residual, in the square $X \times X$ (recall that a subset of a metric space is residual if its complement is a set of the first category, that is, the union of countably many nowhere-dense sets).

It will be convenient to use the following notation. For $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{Prox}(f) &= \{(x, y) \in X^2 : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\},\\
\mathrm{nAs}(f) &= \{(x, y) \in X^2 : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0\},\\
\mathrm{nAs}(f, \varepsilon) &= \{(x, y) \in X^2 : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\},\\
\mathrm{LY}(f) &= \mathrm{Prox}(f) \cap \mathrm{nAs}(f),\\
\mathrm{LY}(f, \varepsilon) &= \mathrm{Prox}(f) \cap \mathrm{nAs}(f, \varepsilon).
\end{aligned}$$

Thus, $\mathrm{Prox}(f)$ is the *proximal relation*, $\mathrm{nAs}(f)$ is the complement of the *asymptotic relation* and $\mathrm{LY}(f)$ is sometimes called the *Li–Yorke relation* in the considered dynamical system. The elements of $\mathrm{LY}(f)$ are Li–Yorke or scrambled pairs, and the

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elements of $\mathrm{LY}(f, \varepsilon)$ are ε-Li–Yorke or ε-scrambled pairs. Thus, Lasota's definition is as follows.

- A map $f : X \to X$ is called *generically chaotic* if the set $\mathrm{LY}(f)$ is residual in X^2.

One of the closely related notions introduced and studied in [55, 56] is the following (recall also that Ruette simplified some ideas and results from [55, 56] and answered some open questions posed there; see e.g. [50]).

- A map $f : X \to X$ is called *generically ε-chaotic* if the set $\mathrm{LY}(f, \varepsilon)$ is residual in X^2.

The first examples of generically chaotic interval maps were found by Piórek [47]. Many examples follow from the fact, contained implicitly in [28, Theorem 3.5] and explicitly in [9, Proposition 2.11], that a weakly mixing map on a non-degenerate compact metric space X is generically ε-chaotic for every $0 < \varepsilon < \operatorname{diam} X$. Examples of generically ε-chaotic maps that are not weakly mixing can be found in [55].

The way generic chaos is defined can be viewed as a 'microscopic' definition in the following sense. In order to verify by the definition whether a system is generically chaotic, we should investigate trajectories of pairs of 'tiny' points in the whole space. However, it appears to be an almost impossible task to determine whether residually many of them are Li–Yorke pairs. Snoha [55] found a way to verify, on the interval, generic chaoticity in 'macroscopic' terms. More precisely, he showed that on the interval the notions of generic chaos and generic ε-chaos are equivalent, and that the latter (and, hence, also the former) can be checked by investigating the dynamics at a 'macroscopic' level, namely by investigating the trajectories of the (non-degenerate) subintervals rather than points. To study the behaviour of all subintervals and their pairs under iterates of the map is of course much easier than to do the same with residually many pairs of points. Therefore, it is not surprising that, as demonstrated in [55], this approach often enables successful verification of 'microscopically' defined generic chaoticity of an interval map purely 'macroscopically'. The situation seems to be similar to that of the point transitivity, which is also defined 'microscopically' (as the existence of a point with dense orbit) yet, in nice spaces, is equivalent with topological transitivity and so can be verified 'macroscopically' by examining the behaviour of non-empty open sets under the iterates of the map.

In [56], Snoha also remarked that it is possible to carry over some results concerning generic chaos from the interval to compact metric spaces. Then Murinová [40] proved these generalized statements for a class of metric spaces containing, in particular, all compact metric spaces; to state the main result of [40] (see Proposition 1.1 below), we first introduce some conditions.

If (X, f) is a dynamical system on a compact metric space X with metric d and if $\mathscr{S}$ is a family of non-degenerate subsets of X, we will consider the following three conditions which the family $\mathscr{S}$ may or may not satisfy:

(Prox) for all sets $S_1, S_2 \in \mathscr{S}$,

$$\liminf_{n\to\infty} d(f^n(S_1), f^n(S_2)) = 0;$$

(Sens_0) for every set $S \in \mathscr{S}$,

$$\limsup_{n\to\infty} \operatorname{diam} f^n(S) > 0;$$

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(Sens) there is $\eta > 0$ such that, for every set $S \in \mathscr{S}$,

$$\limsup_{n\to\infty} \operatorname{diam} f^n(S) > \eta. \tag{1.1}$$

Thus, (Prox) means that all sets $S_1, S_2 \in \mathscr{S}$ are proximal. The other two conditions are close to sensitivity. If $\mathscr{S}$ is the family of all open balls, then (Sens) is in fact equivalent with the sensitivity of the system. (A system (X, f) is *sensitive* if there is $\varepsilon > 0$ such that, for every $x \in X$ and every $\delta > 0$, there is $y \in X$ with $d(y, x) < \delta$ such that $d(f^n(y), f^n(x)) \geq \varepsilon$ for some $n \geq 0$ (then, by choosing y in a smaller neighbourhood of x if necessary, we may assume that n is as large as we require).) For this family $\mathscr{S}$, pointwise Lyapunov instability implies (Sens_0), but the converse is not true (for instance, the identity satisfies (Sens_0)). (A system (X, f) is *pointwise Lyapunov unstable* if for every $x \in X$ there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is $y \in X$ with $d(y, x) < \delta$ such that $d(f^n(y), f^n(x)) \geq \varepsilon$ for some $n \geq 0$.)

The above three conditions are closely related to generic chaos. First, realize that trivially

f is generically chaotic
$\Longrightarrow$ (Prox) and (Sens_0) hold for the family of all open balls,

because generic chaoticity of f implies that both $\text{Prox}(f)$ and $\text{nAs}(f)$ are dense in X^2. The converse implication does not hold even on the interval; see [55, Example 3.6]. However, we have the equivalence (of course, in the two boxed statements, the family of all open balls may be replaced by the family of all closed balls or by the family of all non-empty open sets)

f is generically ε-chaotic for some $\varepsilon > 0$
$\Longleftrightarrow$ (Prox) and (Sens) hold for the family of all open balls.

More precisely, Theorem A from [40] implies the following.

PROPOSITION 1.1. [40] *Let $X = (X, d)$ be a compact metric space and $f : X \to X$ be continuous. Then the following are equivalent:*

(1) *f is generically ε-chaotic for some $\varepsilon > 0$;*

(2) *(Prox) and (Sens) are satisfied by the family of all open (or closed) balls in X.*

Moreover, if f is generically ε-chaotic, then (Sens) holds for the family of all open (or closed) balls in X, with $\eta = \varepsilon$. Conversely, if (Prox) and (Sens) are satisfied by the family of all open (or closed) balls in X, then f is generically ε-chaotic for any $\varepsilon < \eta/2$.

The two boxed statements are crucial for understanding some proofs in the present paper. We will often use them without explicitly citing them. Also, we hope that no misunderstanding will arise if, for a generically chaotic map, we say that something is true by, for instance, (Sens_0). By this we of course mean that, due to the first boxed statement, for a generically chaotic map, any set with non-empty interior satisfies the inequality from (Sens_0).

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Thus, in spaces where generic chaos is equivalent to generic ε-chaos, we can check generic chaos 'macroscopically' using (Prox) and (Sens) for open or closed balls. We know from [**55**] that the interval is such a space. By [**59**], even graphs are such spaces. However, in general, this is not true for dendrites; indeed, by [**40**], an ω-star admits a generically chaotic selfmap that is not generically ε-chaotic for any $\varepsilon > 0$ (an *ω-star* is a (topologically unique) dendrite having exactly one branch point, and this branch point is of infinite order).

The two boxed statements show that the difference between generic chaos and generic ε-chaos lies in the sensitivity. Indeed, f is generically ε-chaotic for some $\varepsilon > 0$ if and only if it is generically chaotic and sensitive (that is, satisfies (Sens)). Thus, the above-mentioned map on an ω-star shows that a generically chaotic map need not be sensitive. Another such example was suggested by an anonymous referee; see Example 7.6.

Thus, already on dendrites, generic ε-chaos is stronger than generic chaos (this is not surprising; a similar fact is that for a continuous selfmap of a compact metric space, the whole space can be a scrambled set [**27**], while it is never an ε-scrambled set [**10**, Proposition 5] for any $\varepsilon > 0$ (compactness is essential here; see [**10**, Example 6])). The main aim of the present paper is to *characterize* dendrites on which generic chaos is equivalent to generic ε-chaos for some $\varepsilon > 0$, that is, to characterize dendrites on which generic chaos can be verified 'macroscopically', using the conditions (Prox) and (Sens).

We know that on an ω-star there exists a generically chaotic map which is not generically ε-chaotic for any $\varepsilon > 0$. By generalizing the construction from [**40**], we will show that such a map can be constructed on every dendrite having a branch point of infinite order; see Lemma 7.4. Thus, for all generically chaotic maps to be generically ε-chaotic, every branch point of a dendrite must be of finite order.

In Lemma 7.4, we will further show that another condition necessary for obtaining the equivalence between generic chaos and generic ε-chaos on a dendrite is complete regularity. Recall that a continuum is *completely regular* if every non-degenerate subcontinuum of it has non-empty interior (see [**45**]; see also [**29**]). A singleton is completely regular for trivial reasons. A subcontinuum of a completely regular continuum is completely regular. By [**37**, Theorem 4 on p. 301, Theorem 3 on p. 284], every dendrite as well as every completely regular continuum is regular (a continuum is *regular* if it has a basis consisting of open sets with finite boundary). Ważewski's universal dendrite is an example of a dendrite which is not completely regular. It is interesting that by [**46**], there is a universal completely regular dendrite, that is, a completely regular dendrite containing a copy of every completely regular dendrite.

Our main result is the following theorem showing that the two necessary conditions, when taken together, are also sufficient. To prove that generic chaos together with these two conditions imply generic ε-chaos, we first study invariant subcontinua for a generically chaotic map on a dendrite satisfying the two conditions. Similarly as on the interval [**55**] and graphs [**59**], we are able to prove that the invariant non-degenerate subdendrites have large diameters; see Lemma 4.3. Of course, the topology of dendrites makes the proof much more complicated, among other reasons due to the phenomenon described on p. 2127. Contrary to the interval and graph cases, there is still a long road to finish the proof. One of the new ideas is that now we need to use a non-trivial result from [**11**] about a dichotomy for the character of fixed points of continuous dendrite maps; see Lemma 3.4.

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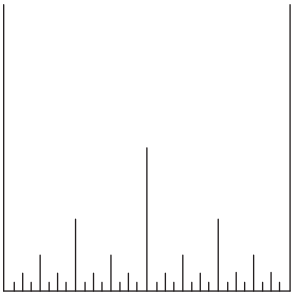


FIGURE 1. The Riemann dendrite has a nowhere-dense set of branch points but it is not completely regular.

This result is crucial in Lemmas 4.5 and 4.8, in which we study subdendrites having orbits containing no fixed and periodic points, respectively (the interval and graph cases did not require anything like this). Even using these two lemmas, the proof is still quite long.

THEOREM A. *Let D be a dendrite. Then the following are equivalent.*

(1) *For every continuous map $f\colon D \to D$, f is generically chaotic if and only if it is generically ε-chaotic for some $\varepsilon > 0$.*
(2) *The dendrite D is completely regular, with all points of finite order.*

Condition (2) can be reformulated by saying that the dendrite D contains neither a copy of the Riemann dendrite (see Proposition 2.13 and the definition of the Riemann dendrite just above it) nor a copy of the ω-star. Notice that if D is a singleton, then both (1) and (2) trivially hold.

Recall that if X is a compact metric space and $A \subseteq X$ is an arc (that is, a homeomorphic image of the interval $[0, 1]$), then A is called a *free arc* if the arc A without its endpoints is an open set in X; see e.g. [**16**]. Proposition 1.1 immediately gives the following corollary of Theorem A (see also Proposition 2.13).

COROLLARY B. *Let D be a non-degenerate completely regular dendrite with all points of finite order. Let $f\colon D \to D$ be a continuous map. Then the following conditions are equivalent:*

(1) *f is generically chaotic;*
(2) *f is generically ε-chaotic for some $\varepsilon > 0$;*
(3) *(Prox) and (Sens) are satisfied by the family of all open (or closed) balls in D;*
(4) *(Prox) and (Sens) are satisfied by the family of all free arcs in D;*
(5) *(Prox) and (Sens) are satisfied by the family of all non-degenerate subdendrites of D.*

Moreover, if f is generically ε-chaotic, then (Sens) in (3)–(5) hold with $\eta = \varepsilon$. Conversely, if (Prox) and (Sens) are satisfied in one of (3)–(5), then f is generically ε-chaotic for any $\varepsilon < \eta/2$.

Using Proposition 1.1, one can observe that conditions (2)–(4) in Corollary B are equivalent for a larger class of dendrites, namely for dendrites with dense free arcs (an example of such a dendrite is the Riemann dendrite; see Figure 1).

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In connection with Theorem A, a natural question is whether the equivalence between generic chaos and generic ε-chaos can be extended to a larger class of spaces. First of all, realize that some compact metric spaces X do not admit generically chaotic maps at all. For instance, this is true if X is *rigid* (that is, X admits no continuous selfmap other than the identity or a constant map). This is also true if X has an isolated point (an isolated point is a non-empty open set which does not satisfy the inequality from (Sens$_0$)).

We denote by $\mathcal{X}$ the system of all compact metric spaces X admitting at least one generically chaotic map and satisfying the condition

$$f\colon X \to X \text{ is generically chaotic} \iff f \text{ is generically } \varepsilon\text{-chaotic for some } \varepsilon > 0.$$

Thus, $\mathcal{X}$ is the system of those compacta on which generic chaos of continuous selfmaps can be checked macroscopically in the sense discussed above, that is, using (Prox) and (Sens).

The system $\mathcal{X}$ contains no zero-dimensional compact metric space (we already know this if it contains an isolated point; otherwise, see Proposition 7.5). We now consider one-dimensional spaces. By [**59**], $\mathcal{X}$ contains all finite graphs and, by Theorem A, it contains many, but not all, dendrites. We conjecture that Theorem A can be extended to local dendrites, that is, we conjecture that a local dendrite is in $\mathcal{X}$ if and only if it is completely regular, with all points of finite order. More generally, one can ask which locally connected curves belong to $\mathcal{X}$.

The square is an example of a two-dimensional space which admits a generically chaotic map but is not in $\mathcal{X}$; see the end of the proof of Theorem B in [**40**].

The paper is organized as follows. Sections 2 and 3 are preliminary. Section 4 contains technical lemmas which are then used in §5 to prove the implication (2) $\Rightarrow$ (1) in Theorem A. In §6, we construct special exact maps on dendrites which are not completely regular. Such maps are used in §7 to prove the implication (1) $\Rightarrow$ (2).

2. *Preliminaries*

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Let I be the unit interval $[0, 1]$. We sometimes use the symbol $\sqcup$ to denote a disjoint union. The cardinality of a set A is denoted by card A. By Bd A we denote the boundary of A.

If $X = (X, d)$ is a metric space and A, B are subsets of it, by $d(A, B)$ we mean the distance from A to B, that is, $d(A, B) = \inf\{d(a, b)\colon a \in A,\ b \in B\}$; if $A = \{a\}$ is a singleton, we write $d(a, B)$ instead of $d(\{a\}, B)$. The open ε-ball centred at x is denoted by $B_\varepsilon(x)$. A *regular closed set* is a set which is equal to the closure of its interior. A family of subsets of X is called a *null family* if for every $\varepsilon > 0$ only finitely many sets from this family have diameters larger than ε.

Let (X, f) be a *dynamical system*, that is, X is a compact metric space and $f\colon X \to X$ is a continuous map. The sets of all fixed points and all periodic points are denoted by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$, respectively. The *orbit* of a set $A \subseteq X$ is $\mathrm{Orb}_f(A) = \bigcup_{n=0}^{\infty} f^n(A)$. We say that f is:

- *transitive* if for every non-empty open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset$ (then there are infinitely many such positive integers n);
- *totally transitive* if f^n is transitive for every $n \in \mathbb{N}$;

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- *weakly mixing* if $f \times f\colon X \times X \to X \times X$ is transitive;
- *strongly mixing* if for every non-empty open sets $U, V \subseteq X$ there is $n_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq n_0$;
- *exact*, or *locally eventually onto*, if for every non-empty open set $U \subseteq X$ there is $n \in \mathbb{N}$ with $f^n(U) = X$.

If A, B are non-empty open subsets of X and $X = A \sqcup B$, we speak of a *disconnection of* X. Recall that X is connected if and only if it has no disconnection. A *continuum* is a non-empty, compact and connected metric space. We say that a continuum is *non-degenerate* if it has more than one point. A continuum X is *uniquely arcwise connected* provided that for every two distinct points of X there is exactly one arc in X joining these points. A continuum X is *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected (and hence a subcontinuum). X is *hereditarily unicoherent* if all its subcontinua are unicoherent. Equivalently, a continuum X is hereditarily unicoherent if and only if the intersection of any two subcontinua of X is connected.

A *simple closed curve* is a homeomorphic image of the unit circle. A *dendrite* is a locally connected continuum which contains no simple closed curve. Note that a singleton is also a dendrite but the empty set is not. We will use basic facts on dendrites from [14, 37, 44]. In particular, every subcontinuum of a dendrite is a dendrite and all dendrites are uniquely arcwise connected and hereditarily unicoherent. Since a dendrite is locally connected, the components of its open subsets are open. Recall also that dendrites have the fixed point property.

Let D be a dendrite and $x \in D$. The *order* of the point x, denoted by $\operatorname{ord}(x, D)$, is the cardinality of the set of (connected) components of $D \setminus \{x\}$ (see [44, Theorem 10.13]). The order can be either finite or infinite countable. An *endpoint* is a point of order 1. Any point with order greater than 1 or greater than 2 is called a *cutpoint* or a *branch point*, respectively. The sets of all endpoints, cutpoints and branch points of D are denoted by $\operatorname{End}(D)$, $\operatorname{Cut}(D)$ and $\operatorname{B}(D)$, respectively. Notice that any non-degenerate dendrite D has at least two endpoints, and every point of it is either an endpoint or a cutpoint; the set $B(D)$ is countable [37, Theorem 7, p. 302] and the set $\operatorname{End}(D)$ is totally disconnected [37, Theorem 2, p. 292]. The unique point in a degenerate dendrite has order 0. For distinct $x, y \in D$, let $[x, y]$ denote the unique arc with endpoints x and y, and let $[x, x]$ denote just the singleton $\{x\}$. For $x \neq y$, put $(x, y) = [x, y] \setminus \{x, y\}$, $(x, y] = [x, y] \setminus \{x\}$ and $[x, y) = [x, y] \setminus \{y\}$.

LEMMA 2.1. *Let S be a non-degenerate connected set in a dendrite D (and hence the closure $\overline{S}$ is a non-degenerate dendrite). Let $x \in \overline{S} \setminus S$. Then x is an endpoint of $\overline{S}$.*

Proof. If x is a cutpoint of $\overline{S}$, then there is a disconnection $\overline{S} \setminus \{x\} = A \sqcup B$. Hence, since $x \notin S$, $S = (A \cap S) \sqcup (B \cap S)$ is a disconnection of S. □

LEMMA 2.2. *Let X be a topological space and U be an open set. Let A be a connected subset of X such that $A \cap U \neq \emptyset$ and $A \cap \operatorname{Bd} U = \emptyset$. Then $A \subseteq U$.*

Proof. Otherwise $A = (A \setminus \overline{U}) \sqcup (A \cap U)$ is a disconnection of the connected space A. □

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The following simple lemma will be applied to dendrites.

LEMMA 2.3. *Let E be a hereditarily unicoherent continuum and $E_1, \dots, E_k$ be subcontinua of E. If the sets E_i intersect pairwise, then the intersection $\bigcap_{i=1}^{k} E_i$ is non-empty and hence it is a subcontinuum of E.*

Proof. We proceed by induction on k. For $k = 1$, the lemma is trivial; for $k = 2$, it follows from hereditary unicoherence of E. Now assume that $k \geq 3$ and that the lemma is valid for every family of fewer than k subcontinua of E. Take pairwise-intersecting subcontinua $E_1, \dots, E_k$. It is sufficient to prove that $\bigcap_{i=1}^{k} E_i \neq \emptyset$ (then, due to hereditary unicoherence of E, this intersection is a subcontinuum). By the induction hypothesis, the intersections $E_1^i = E_1 \cap \dots \cap E_i \neq \emptyset$ $(i < k)$ are subcontinua. Further, since $E_{k-1} \cap E_k \neq \emptyset$, also the set $E_{k-1} \cup E_k$ is a subcontinuum. Then

$$C = E_1^{k-2} \cap (E_{k-1} \cup E_k) \supseteq E_1^{k-1},$$

being the non-empty intersection of two subcontinua, is a subcontinuum due to hereditary unicoherence. Now suppose that $\bigcap_{i=1}^{k} E_i = \emptyset$. Then, since $E_1^{k-2} \cap E_k \neq \emptyset$ by the induction hypothesis,

$$C = E_1^{k-1} \sqcup (E_1^{k-2} \cap E_k)$$

is a disconnection of C, which is a contradiction. □

LEMMA 2.4. *Let D be a dendrite and $C_1, \dots, C_k$ $(k \geq 2)$ be connected subsets of D. If $C_i \cap C_j = \emptyset \neq \overline{C_i} \cap \overline{C_j}$ for every distinct i, j, then $\bigcap_{i=1}^{k} \overline{C_i}$ is a singleton.*

Proof. By Lemma 2.3, $C = \bigcap_{i=1}^{k} \overline{C_i}$ is a subdendrite of D. If C is non-degenerate, it contains an arc $[a, b]$. Hence, due to connectedness, every C_i contains (a, b), which is in contradiction with pairwise disjointness of them. Thus, C is a singleton. □

LEMMA 2.5. *Let D be a dendrite and U be an open connected subset with singleton boundary $\{u\}$. Let $a \in U$. Then, for every $x \in U$, $[x, u) \cap [a, u) \neq \emptyset$.*

Proof. Suppose, on the contrary, that $[x, u) \cap [a, u) = \emptyset$ for some $x \in U$. Thus, $[a, u] \cup [u, x]$ is an arc, containing u, with endpoints a and x and hence is equal to $[a, x]$ (because D is uniquely arcwise connected). However, connected sets in dendrites are arcwise connected; therefore, $[a, x] \subseteq U$ and so $u \notin [a, x]$. This is a contradiction. □

LEMMA 2.6. *Let D be a dendrite and E be a proper subdendrite of D. Then the components of $D \setminus E$ are open and form a (finite or infinite countable) null family. Moreover, if C is a component of $D \setminus E$ (and hence $\overline{C}$ is a subdendrite of D), then* $\mathrm{Bd}(C) = \{c\}$ *for some $c \in \mathrm{Bd}(E) \cap \mathrm{End}(\overline{C})$; if E is non-degenerate, then $c \in \mathrm{Cut}(D)$ and also* $\mathrm{Bd}(\overline{C}) = \{c\}$.

Proof. The subdendrite E is a so-called A-set by [**60**, (3.4), p. 69]. Thus, by [**60**, (3.31), p. 69], the components of $D \setminus E$ are open, form a (finite or infinite countable) null family and every component has a singleton boundary. Hence, $\mathrm{Bd}(C) = \{c\}$ for some $c \in D$; clearly, $c \in \mathrm{Bd}(E)$ and, by Lemma 2.1, $c \in \mathrm{End}(\overline{C})$.

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Now assume that E is non-degenerate. Then $D \setminus \{c\}$ has at least two components: one containing C and one intersecting the non-empty set $E \setminus \{c\}$. Thus, c is a cutpoint of D. Trivially, $\mathrm{Bd}(\overline{C}) \subseteq \mathrm{Bd}(C) = \{c\}$. On the other hand, $c \in \mathrm{Bd}(\overline{C})$ because every neighbourhood of C contains infinitely many points of the non-degenerate set E. □

The boundary of a subdendrite E of a dendrite D may be uncountable, even in the case when D is completely regular. For example, let $f\colon [0, 1] \to [0, 1]$ have values $1/(n + 1)$ at the endpoints of the contiguous intervals of range n of the Cantor ternary set, and be zero otherwise. Let D be the *subgraph* of f (that is, the set $\{[x, y] \in \mathbb{R}^2\colon x \in [0, 1],\ 0 \leq y \leq f(x)\}$) and $E = [0, 1] \times \{0\}$. Then the boundary of E in the space D is the Cantor ternary set (multiplied by $\{0\}$). However, we have at least the following.

LEMMA 2.7. *Let D be a dendrite and E be a proper subdendrite of D. Let C_j ($j \in J$) be the components of $D \setminus E$. Put*

$$\mathrm{Bd}^*(E) = \{c_j\colon j \in J\} \quad \textit{where } \{c_j\} = \mathrm{Bd}(C_j)$$

and, for every $c \in \mathrm{Bd}^*(E)$, put

$$J_c = \{j \in J\colon c_j = c\} \quad \textit{and} \quad B_c = \{c\} \sqcup \bigsqcup_{j \in J_c} C_j = \bigcup_{j \in J_c} \overline{C_j}.$$

Then $\mathrm{Bd}^*(E)$ is at most countable. It is a dense subset of $\mathrm{Bd}(E)$ and every B_c is a non-degenerate subdendrite of D with $\mathrm{Bd}(B_c) = \{c\}$, so B_c is regular closed. Moreover, the subdendrites B_c ($c \in \mathrm{Bd}^*(E)$) are the components of $\bigcup_{j \in J} \overline{C_j}$ and $d(B_c, B_{c'}) > 0$ for every $c \neq c'$.

Examples show that the obvious inclusion $\bigcup\{B_c\colon c \in \mathrm{Bd}^*(E)\} = \bigcup_{j \in J} \overline{C_j} \subseteq \overline{D \setminus E}$ is in general strict.

A point x of $\mathrm{Bd}(E)$ belongs to $\mathrm{Bd}^*(E)$ if and only if one can 'escape from E through x', since a component of $D \setminus E$ is 'attached' to it. Therefore, $\mathrm{Bd}^*(E)$ is said to be the *escape-boundary* of E in D.

Proof. Since the sets C_j are disjoint and open in D, the set J (and hence also $\mathrm{Bd}^*(E)$) is at most countable. The union of $\overline{C_j}$ meets $\mathrm{Bd}(E)$ in a dense set. Hence, $\mathrm{Bd}^*(E)$ is a dense subset of $\mathrm{Bd}(E)$.

Clearly, every B_c is connected, non-degenerate and $\mathrm{Bd}(B_c) = \{c\} \subseteq B_c$; thus, B_c is a non-degenerate subdendrite. Further, the subdendrites B_c are obviously pairwise disjoint and hence $d(B_c, B_{c'}) > 0$ for every distinct $c, c' \in \mathrm{Bd}^*(E)$.

Put $\tilde{C} = \bigcup_{j \in J} \overline{C_j}$ and choose any $c \in \mathrm{Bd}^*(E)$. Let L be the component of $\tilde{C}$ intersecting (and hence containing) B_c. Suppose that there is $x \in L \setminus B_c$. Let $j' \in J$ be such that $x \in \overline{C_{j'}}$; put $c' = c_{j'}$. Then $\overline{C_{j'}} \subseteq L$ and so $c' \in L$. Hereditary unicoherence of D implies that $[c, c'] \subseteq L \cap E$. Since $\tilde{C} \cap E = \mathrm{Bd}^*(E)$ is countable, we have $c' = c$ and so $x \in B_c$. This contradicts the choice of x and so we have that $L = B_c$, that is, B_c is a component of $\tilde{C}$. □

LEMMA 2.8. *Let D be a dendrite and $c_1, c_2 \in D$. Let C_1 and C_2 be components of $D \setminus \{c_1\}$ and $D \setminus \{c_2\}$, respectively. Then exactly one of the following conditions holds:*

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(1) $c_1 = c_2$ *and* $C_1 = C_2$*;*
(2) $c_1 = c_2$ *and* $C_1 \cap C_2 = \emptyset$*;*
(3) $c_1 \neq c_2$, $c_1 \in C_2$ *and* $c_2 \in C_1$*; in this case,* $C_1 \cap C_2 \supseteq (c_1, c_2)$ *is a non-empty proper subset of both* C_1 *and* C_2*;*
(4) $c_1 \neq c_2$, $c_1 \notin C_2$ *and* $c_2 \notin C_1$*; in this case,* $C_1 \cap C_2 = \emptyset$*;*
(5) $c_1 \neq c_2$, $c_1 \notin C_2$ *and* $c_2 \in C_1$*; in this case,* $C_2 \subsetneq C_1$*;*
(6) $c_1 \neq c_2$, $c_1 \in C_2$ *and* $c_2 \notin C_1$*; in this case,* $C_1 \subsetneq C_2$.

Proof. Recall that C_i is an open connected set with boundary $\{c_i\}$, $i = 1, 2$. It is trivial that if $c_1 = c_2$, then either (1) or (2) is true.

Let the assumptions in (3) be fulfilled. Then, for $i = 1, 2$, $\overline{C_i}$ contains both c_1 and c_2 and hence contains the arc $[c_1, c_2]$. Thus, $C_1 \cap C_2 \supseteq (c_1, c_2)$. Since $c_1 \in C_2 \setminus C_1$ and $c_2 \in C_1 \setminus C_2$, the intersection $C_1 \cap C_2$ is a proper subset of both C_1 and C_2.

If we are in (4), suppose, on the contrary, that there is $d \in C_1 \cap C_2$. Then $[d, c_i) \subseteq C_i$ for $i = 1, 2$. The union of the arcs $[c_1, d]$ and $[d, c_2]$ is a path from c_1 to c_2. Replacing d by another point from the intersection of these arcs, if necessary, we may assume that the mentioned path is an arc. Since dendrites are uniquely arcwise connected, this arc containing d coincides with the arc $[c_1, c_2]$. It follows that $d \in [c_1, c_2]$. Then $(c_1, c_2]$ is a connected set intersecting C_1 and not intersecting the boundary of C_1. Now Lemma 2.2 yields that $(c_1, c_2] \subseteq C_1$, which is in contradiction with $c_2 \notin C_1$.

Since (6) is analogous to (5), it suffices to prove (5). So, assume that $c_1 \neq c_2$, $c_1 \notin C_2$ and $c_2 \in C_1$. Since C_1 is a neighbourhood of c_2 and $c_2 \in \overline{C_2}$, C_2 intersects C_1 but does not intersect the boundary of C_1. By Lemma 2.2, $C_2 \subseteq C_1$. Clearly, $C_2 \neq C_1$. □

LEMMA 2.9. *Let* D *be a dendrite and* A_i, B_i $(i = 1, 2)$ *be non-degenerate subdendrites of* D *such that* $A_i \cap B_i = \emptyset$ *for* $i \in \{1, 2\}$ *and the intersections* $A_1 \cap A_2$, $B_1 \cap B_2$ *and* $A_1 \cap B_2$ *are non-empty. Denote by* C_i *the (unique) component of* $D \setminus B_i$ *containing* A_i*. Then*

$$\mathrm{Bd}(C_i) = \mathrm{Bd}(\overline{C_i}) = \{c_i\} \quad \text{for some } c_i \in B_i \cap \mathrm{Cut}(D) \quad (i \in \{1, 2\}). \tag{2.1}$$

Moreover, $c_2 \in A_1 \cap B_2$, $c_1 \notin \overline{C_2}$ *and* $C_2 \subsetneq C_1$.

Proof. The claim (2.1) follows from Lemma 2.6. Further, by the assumptions, $C_1 \cap C_2 \supseteq A_1 \cap A_2 \neq \emptyset$ and $C_1 \setminus C_2 \supseteq A_1 \cap B_2 \neq \emptyset$. Thus, we have either (3) or (5) from Lemma 2.8. In particular, $c_1 \neq c_2$.

We show that (3) is also impossible. Suppose, on the contrary, that $c_1 \in C_2$ and $c_2 \in C_1$. Then, by Lemma 2.8(3), $(c_1, c_2) \subseteq C_1 \cap C_2$ and so $(c_1, c_2) \cap (B_1 \cup B_2) = \emptyset$. Choose any $b \in B_1 \cap B_2$; clearly, $b \notin C_1 \cup C_2$. By (2.1), $c_i \in B_i$ and hence $[b, c_i] \subseteq B_i$ for $i \in \{1, 2\}$. Clearly, $[b, c_1] \cup [c_1, c_2] \cup [c_2, b]$ contains a simple closed curve (because the first and the third arcs are in $B_1 \cup B_2$, while the second arc (minus the endpoints) is disjoint from $B_1 \cup B_2$), which is a contradiction.

We have showed that (5) from Lemma 2.8 is true. Thus, $c_2 \in C_1$, $c_1 \notin C_2$ and $C_2 \subsetneq C_1$. Since also $c_2 \neq c_1$ and $\mathrm{Bd}(C_2) = \{c_2\}$, we have that $c_1 \notin \overline{C_2}$. Since $c_2 \in B_2$, it remains to

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show that $c_2 \in A_1$. Since A_1 is connected and intersects both B_2 and the superset C_2 of A_2, Lemma 2.2 gives that A_1 intersects $\mathrm{Bd}(C_2) = \{c_2\}$ and thus $c_2 \in A_1$. □

For a metric space (X, d), the Hausdorff one-dimensional measure is denoted by $\mathcal{H}^1_d$; if (X, d) is a closed real interval with the Euclidean metric, we write simply $|\cdot|$ instead of $\mathcal{H}^1_d(\cdot)$.

By [25] (see also [12]), every dendrite D admits a convex metric d such that D has *finite length* with respect to this metric, that is, $\mathcal{H}^1_d(D)$ is finite. Here *convex* means that for any $x, y \in D$, there exists $z \in D$ such that $d(x, z) = d(z, y) = d(x, y)/2$.

LEMMA 2.10. *Let D be a non-degenerate dendrite with a convex metric d. Then:*

(1) *for every distinct $x, y \in D$, the arc $[x, y]$ is a* geodesic *in the sense that $\mathcal{H}^1_d([x, y]) = d(x, y)$;*

(2) *$d(x, y) = d(x, z) + d(z, y)$ for every distinct $x, y \in D$ and every $z \in [x, y]$.*

Proof. For (1), see [17]. Then (2) follows from (1) (alternatively, one can start by a repeated use of the definition of a convex metric). □

Let E be a subdendrite of a dendrite D and let $r\colon D \to E$ be the first point map; see [44, Theorem 10.26]. If $x \in D$, then $r(x)$ will be called the *projection* of x into E and denoted by $\mathrm{proj}(x, E)$.

LEMMA 2.11. *Let D be a non-degenerate dendrite with a convex metric d. Let E be a subdendrite of D and $x \in D$.*

(1) *If $x \notin E$, then, for every $e \in E$,*

$$\mathrm{proj}(x, E) \in [x, e] \quad \text{and} \quad [x, e] \cap E = [\mathrm{proj}(x, E), e].$$

(2) *The projection* $\mathrm{proj}(x, E)$ *is the unique point $e_0 \in E$ such that*

$$d(x, E) = d(x, e_0).$$

(3) *If $x \notin E$ and E is non-degenerate, then* $\mathrm{proj}(x, E)$ *is a cutpoint of D.*

(4) *If $x \notin E$, then* $\mathrm{proj}(x, E) \in \mathrm{Bd}(E)$.

Proof. (1) For the first part, see [44, Lemma 10.24]. The intersection $[x, e] \cap E$ is a subcontinuum of D containing both $\mathrm{proj}(x, E)$ and e. If it is a singleton, then $\mathrm{proj}(x, E) = e$ and we are done. Otherwise it is a subarc $[e_1, e]$ of $[x, e]$ with $e_1 \in E \cap [x, e]$. By the first part, $\mathrm{proj}(x, E) \in [x, e_1]$. Since $[x, e_1] \cap E = ([x, e_1] \cap [x, e]) \cap E = [x, e_1] \cap [e_1, e] = \{e_1\}$, we get $e_1 = \mathrm{proj}(x, E)$ and we are done.

(2) By (1) and convexity, $d(x, \mathrm{proj}(x, E)) = d(x, E)$. Take any $e \in E$, $e \neq \mathrm{proj}(x, E)$. Then, by Lemma 2.10(2), $d(x, e) = d(x, \mathrm{proj}(x, E)) + d(\mathrm{proj}(x, E), e) > d(x, E)$.

(3) Choose a point $e \in E$, $e \neq \mathrm{proj}(x, E)$. By (1), $\mathrm{proj}(x, E) \in (x, e)$. Hence, $\mathrm{proj}(x, E)$ is a cutpoint of D.

(4) This follows from (1) and Lemma 2.2. □

LEMMA 2.12. *Let D be a dendrite of finite length and $f\colon D \to D$ be a continuous map. Let $E \subseteq D$ be a connected set with* $\limsup_{n\to\infty} \mathrm{diam}\, f^n(E) > 0$.

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(1) *There exists a least non-negative integer n_0 such that, for some positive integer k,*

$$f^{n_0}(E) \cap f^{n_0+k}(E) \neq \emptyset.$$

(2) *Let k be any positive integer with that property. Then the sets*

$$K_i = \bigcup_{j=0}^{\infty} f^{n_0+i+jk}(E) \quad \text{for } i \in \{0, 1, \ldots, k-1\}$$

are connected subsets of D such that

$$f(K_i) = K_{i+1} \quad \text{for } i \in \{0, 1, \ldots, k-2\} \text{ and } f(K_{k-1}) \subseteq K_0. \tag{2.2}$$

(3) *The orbit $\mathrm{Orb}_f(f^{n_0}(E))$ has components L_0 ($\supseteq K_0 \supseteq f^{n_0}(E)$), $L_1, \ldots, L_{r-1}$, where r is a divisor of k,*

$$f(L_j) = L_{j+1} \quad \text{for } j \in \{0, 1, \ldots, r-2\} \text{ and } f(L_{r-1}) \subseteq L_0. \tag{2.3}$$

Moreover, for every $j \in \{0, 1, \ldots, r-1\}$, $L_j = \bigcup_{\ell=0}^{k/r-1} K_{j+\ell r}$.

(4) $\mathrm{Orb}_f(E) = E \sqcup f(E) \sqcup \cdots \sqcup f^{n_0-1}(E) \sqcup L_0 \sqcup L_1 \sqcup \cdots \sqcup L_{r-1}$.

Proof. (1) By the assumption, there are $\varepsilon > 0$ and positive integers $n_1 < n_2 < \ldots$ such that $\operatorname{diam} f^{n_i}(E) > \varepsilon$ for every i. For every i, choose $a_i, b_i \in f^{n_i}(E)$ with $d(a_i, b_i) > \varepsilon$. By connectedness, the arc $A_i = [a_i, b_i]$ is a subset of $f^{n_i}(E)$. Thus, $\mathcal{H}^1_d(A_i) > \varepsilon$ for every i. Since the length of D is finite, the arcs A_i cannot be disjoint and hence we obtain (1).

(2) The properties of the sets K_i are obvious.

(3) It follows that $\mathrm{Orb}_f(f^{n_0}(E)) = \bigcup_{i=0}^{k-1} K_i$ has at most k components; let $L_0, L_1, \ldots, L_{r-1}$ be the list of them. Clearly, every L_j is the union of some of the connected sets K_i. Each of the components L_j is mapped to a component. Fewer than r components cannot form a cycle since every point from every K_i visits every K_j repeatedly. Therefore, with appropriate notation, we have (2.3). Choose a point $x_0 \in L_0$. It comes back to L_0 only in times which are multiples of r. However, it belongs to some $K_i \subseteq L_0$ and so it comes back to L_0 also in the time k by (2.2). Hence, k is a multiple of r.

We may assume that $L_0 \supseteq K_0$ and $f(L_j) \subseteq L_{j+1 \bmod r}$. Then clearly $L_j \supseteq \bigcup_{\ell=0}^{k/r-1} K_{j+\ell r}$ for all j. Since we have used all the sets K_i here, these inclusions are in fact equalities.

(4) This follows from (3) and definition of n_0. □

The above three lemmas indicate that it will be useful to adopt the following convention (recall that it is possible due to [**12**, **25**]).

Convention. From now on, until the end of §5, we will always assume that the metric d on a dendrite D is convex and such that D has finite length with respect to this metric.

It is important to realize that, when proving Theorem A, we are allowed to adopt this convention, that is, to replace the original metric on D by an equivalent metric from the

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convention. Indeed, the notions of 'generic chaos' and 'generic ε-chaos for some $\varepsilon > 0$' are invariants of topological conjugacy on compact metric spaces (this is a simple consequence of uniform continuity of the conjugating homeomorphism, as was observed already in [55]).

We will also use the simple fact that if f is a continuous selfmap of a compact metric space and $g = f^k$ for some positive integer k, then f is generically chaotic (generically ε-chaotic) if and only if so is g.

Since completely regular dendrites appear in Theorem A, the following simple observation will be useful (and repeatedly used). To state it, first recall the definition of the Riemann dendrite. Let $r : [0, 1] \to \mathbb{R}$ be the Riemann function (called also the Thomae function), that is, the function defined by $r(x) = 0$ if x is irrational and $r(x) = 1/q$ if $x = p/q$, where q is a positive integer, p is a non-negative integer and p and q are relatively prime. Then the subgraph of r is a dendrite; we call it the *Riemann dendrite*; see Figure 1.

PROPOSITION 2.13. *Let D be a dendrite. Then the following six conditions are equivalent.*

(1) *D is completely regular (that is, every non-degenerate subdendrite of D has non-empty interior).*
(2) *Every non-degenerate subdendrite of D is a regular closed set.*
(3) *Every arc in D has non-empty interior in D.*
(4) *Every arc in D contains a subarc which is a free arc in D.*
(5) *There is no arc A in D such that the set $A \cap B(D)$ is dense in A.*
(6) *D does not contain a copy of the Riemann dendrite.*

Further, we have the implications

$$\mathrm{B}(D) \text{ is discrete} \implies D \text{ is completely regular} \implies \mathrm{B}(D) \text{ is nowhere dense},$$

while the converse implications do not hold.

Proof. The equivalence of the six conditions is obvious.

Suppose that $\mathrm{B}(D)$ is discrete and A is an arc in D. Then there is a subarc A' of A such that it does not contain any branch point of D. However, then A' is a free arc in D and we get (4). Now suppose that D is completely regular and that $\mathrm{B}(D)$ is dense in an open set $\emptyset \neq U \subseteq D$. Fix an arc $A \subseteq U$. Since $\overline{\mathrm{B}(D)} \supseteq A$, A does not contain any free arc, in contradiction with (4).

The Riemann dendrite has a nowhere-dense set of branch points but is not completely regular.

Finally, let $f : [-1, 1] \to \mathbb{R}$ be the function defined by $f(1/n) = 1/n$ for every $n \in \mathbb{N}$, $f(0) = 1$ and $f(x) = 0$ for all other points $x \in [-1, 1]$. Then the subgraph of f is a completely regular dendrite whose set of branch points is not discrete; see Figure 2. □

In general, generic chaos is not carried over to invariant subsets (for instance, a generically chaotic map on a dendrite may be equal to the identity on a subarc; see Proposition 6.1 below). However, the following is true.

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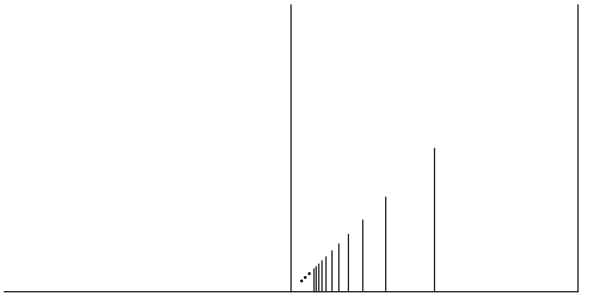


FIGURE 2. A completely regular dendrite whose set of branch points is not discrete.

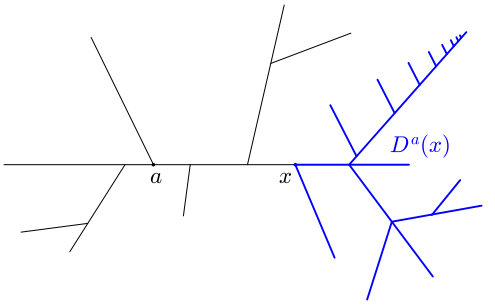


FIGURE 3. The set $D^a(x)$.

LEMMA 2.14. [**59**, Proposition 13] *Let X be a compact metric space, $f\colon X \to X$ be continuous and Y be a regular closed f-invariant subset of X. If f is generically chaotic or generically ε-chaotic, then so is $f|_Y\colon Y \to Y$.*

Clearly, a singleton does not admit a generically chaotic selfmap. Therefore, in auxiliary results in the next sections, we will always assume that the phase space is a *non-degenerate* dendrite.

3. *Fixed points of dendrite maps*

Let D be a non-degenerate dendrite and $a, b, x \in D$. We say that x *separates* a and b if a and b lie in different components of $D \setminus \{x\}$. For $a \in D$, let $\leq_a$ be the partial order on D defined by $x \leq_a y$ whenever $x \in [a, y]$; if $x \leq_a y$ and $x \neq y$, we will write $x <_a y$. If E is a subdendrite of D and $a \in D \setminus E$, then, by Lemma 2.11(1) (recall the Convention),

$$\operatorname{proj}(a, E) = \inf_{\leq_a} E.$$

For any $x \in D$, the set

$$D^a(x) = \{y \in D\colon x \leq_a y\} \tag{3.1}$$

is a subdendrite of D; see Figure 3. Notice that if $x \neq a$, then $D^a(x)$ is the set containing x and all those points of D which are separated from a by x. Since the metric d is convex by the Convention, if $y \in D^a(x)$, then $d(y, a) \geq d(x, a)$ and so $x = \operatorname{proj}(a, D^a(x))$.

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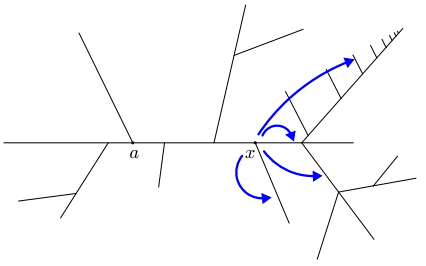


FIGURE 4. x evades a.

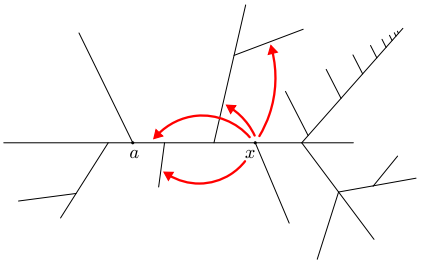


FIGURE 5. x admires a.

For any distinct $a, b \in D$, let us define a subdendrite $D_{[a,b]} \subseteq D$ by

$$D_{[a,b]} = \overline{D \setminus (D^b(a) \cup D^a(b))}. \tag{3.2}$$

One can imagine it as the subdendrite 'enclosed by a and b'. Put $D_{(a,b]} = D_{[a,b]} \setminus \{a\}$, $D_{[a,b)} = D_{[a,b]} \setminus \{b\}$ and $D_{(a,b)} = D_{[a,b]} \setminus \{a, b\}$. Let $D_{[a,a]}$ denote the singleton $\{a\}$.

Now consider a continuous map $f : D \to D$ on a non-degenerate dendrite D. Note that, for $a \neq x$ in D, there are four mutually exclusive possibilities:

- $f(x) = x$, that is, x is a fixed point of f;
- $f(x) \in D^a(x) \setminus \{x\}$, in this case we say that x *evades* a; see Figure 4;
- $f(x) \in D_{[a,x)}$, in this case we say that x *admires* a; see Figure 5;
- a separates x from $f(x)$, in this case we say that x *jumps over* a.

If a is an endpoint of D, then the fourth possibility cannot occur and so we have a trichotomy for the points $a \neq x$: either x is fixed or x evades a or x admires a. In particular, if also x is an endpoint of D, then we have only a dichotomy: either x is fixed or x admires a.

Definition 3.1. See [11, Definition 5.3.2] Let D be a non-degenerate dendrite and $f : D \to D$ be continuous. Let $a \in D$ be a fixed point of f and B be a component of $D \setminus \{a\}$. Then a is called a *weakly repelling fixed point of f for B* if at least one of the following two conditions holds:

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- in B, arbitrarily close to a there is a cutpoint of D which is fixed by f; or
- in B, arbitrarily close to a there is a cutpoint x separating a from $f(x)$ (that is, x evades a).

We omit the obvious proof of the following lemma.

LEMMA 3.2. *Let D be a non-degenerate dendrite and $f\colon D \to D$ be continuous. Let $a \in D$ be a fixed point of f and B be a component of $D \setminus \{a\}$. Then a is* not *weakly repelling for B if and only if there exists a neighbourhood U of a in D such that, for every cutpoint $x \in U \cap B$:*

- *either x admires a;*
- *or x jumps over a.*

The following fact is well known; see [**52**, Lemma 3.4] or [**11**, Theorem 7.2.2(1)].

LEMMA 3.3. *Let D be a non-degenerate dendrite and $f\colon D \to D$ be continuous. If $x, y \in D$ are such that x evades y and y evades x, then f has a fixed point in (x, y).*

LEMMA 3.4. See [**11**, Lemma 7.2.5] *Let D be a non-degenerate dendrite and $f\colon D \to D$ be continuous. Then:*

- *either there is a fixed point of f which is a cutpoint of D;*
- *or there is a fixed point $a \in D$ which is an endpoint of D and is not weakly repelling of f for $D \setminus \{a\}$.*

LEMMA 3.5. *Let D be a non-degenerate dendrite and $f\colon D \to D$ be continuous such that* $\mathrm{Fix}(f) \subseteq \mathrm{End}(D)$. *Then there exists an endpoint a fixed by f and such that every $y \in D$ which is not fixed by f admires a. Such an endpoint a is unique.*

Proof. Lemma 3.4 guarantees the existence of an endpoint a fixed by f which is not a weakly repelling point of f for $D \setminus \{a\}$. Then, by Lemma 3.2, there exists a neighbourhood U of a such that every cutpoint in U admires a. We need to prove that, in fact, all points in $D \setminus \{a\}$ which are not fixed by f admire a. Suppose, on the contrary, that some point $w \in D \setminus \{a\}$ evades a. So, w is a cutpoint because endpoints never evade a and then w is in $D \setminus U$. Consider any cutpoint $z \in U \cap (a, w)$. We know that z admires a and hence z evades w. Clearly, w evades z. Then, by Lemma 3.3, f has a fixed point in (z, w), which is obviously a cutpoint, which is a contradiction.

Suppose that there are two distinct endpoints a_1, a_2 with the property from the lemma. Choose a point $x \in (a_1, a_2)$. Since $x \notin \mathrm{End}(D)$, it is admired both by a_1 and a_2, that is, $f(x) \in D_{[a_1,x)} \cap D_{[a_2,x)}$. However, this intersection is obviously empty, which is a contradiction. □

LEMMA 3.6. *Let D be a non-degenerate dendrite and $f\colon D \to D$ be continuous. Suppose that p and q are two distinct endpoints fixed by f, which are not weakly repelling of f for $D \setminus \{p\}$ and $D \setminus \{q\}$, respectively. Then there exists a cutpoint $r \in (p, q)$ fixed by f.*

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Proof. By Lemma 3.2, there exist disjoint neighbourhoods U_p and U_q of p and q, respectively, such that every cutpoint in U_p admires p and every cutpoint in U_q admires q. We can suppose that diam $U_p < d(p,q)/2$ and diam $U_q < d(p,q)/2$. Fix $p^* \in U_p \cap (p,q)$ and $q^* \in U_q \cap (p,q)$. Clearly, p^* and q^* are cutpoints and so they admire p and q, respectively. As the diameters of U_p and U_q are less than $d(p,q)/2$, the points p, p^*, q^* and q are positioned on $[p,q]$ in this order. The fact that p^* admires p gives us $f(p^*) \in D_{[p,p^*)}$. Similarly, $f(q^*) \in D_{(q^*,q]}$. Then, by Lemma 3.3, there is a fixed point $r \in (p^*,q^*) \subseteq (p,q)$. Obviously, r is a cutpoint. □

4. *Generic chaos on completely regular dendrites*

The first lemma in this section does not need the assumption of complete regularity.

LEMMA 4.1. *Let D be a non-degenerate dendrite and $f: D \to D$ be continuous such that the condition (Prox) is satisfied by the family of non-degenerate subdendrites. Then the intersection of any non-empty family of invariant non-degenerate subdendrites is an invariant (possibly degenerate) subdendrite and hence it contains a fixed point of f.*

Proof. The intersection of a family of invariant non-degenerate subdendrites is trivially invariant and closed. Using unicoherence of D, it is a subdendrite if it is non-empty. Hence, it is sufficient to prove that the intersection is non-empty. Since D is compact, it is in fact sufficient to prove that any family of invariant non-degenerate subdendrites of D has the finite intersection property. So, we need to show that any finite family of invariant non-degenerate subdendrites has non-empty intersection. In view of Lemma 2.3, it is sufficient to prove that if D_1 and D_2 are two invariant non-degenerate subdendrites, then $D_1 \cap D_2 \neq \emptyset$. However, this is obvious because, by (Prox), $d(D_1, D_2) = 0$ and since D_1 and D_2 are compact sets, we have $D_1 \cap D_2 \neq \emptyset$. □

A generically chaotic map on an ω-star may have arbitrarily small invariant non-degenerate subdendrites [40]. The next two lemmas show that on some dendrites this cannot happen.

LEMMA 4.2. *Let D be a completely regular non-degenerate dendrite with all points of finite order. Let $f: D \to D$ be generically chaotic and $p \in D$ be a fixed point of f. Then there exists $\delta > 0$ such that every invariant non-degenerate subdendrite $S \subseteq D$ containing p has diameter at least δ.*

Proof. Suppose, on the contrary, that for every $\delta > 0$ there exists an invariant non-degenerate subdendrite $S(\delta) \subseteq D$ such that $p \in S(\delta)$ and diam $S(\delta) < \delta$. To obtain a contradiction, we proceed in three steps.

Step 1. We construct a nested sequence of invariant non-degenerate subdendrites $\{T_n\}_{n=0}^{\infty}$ containing p and converging to p.

Let $\mathcal{C}$ be the set of components of $D \setminus \{p\}$. Since $\mathcal{C}$ is finite, there are a subsequence $\{S_n\}_{n=0}^{\infty}$ of $\{S(1/k)\}_{k=1}^{\infty}$ and a non-empty set $\mathcal{B} \subseteq \mathcal{C}$ with the following property: for every n, the dendrite S_n intersects a component of $D \setminus \{p\}$ if and only if this component belongs to $\mathcal{B}$. Clearly, diam $S_n \to 0$.

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For every n, $T_n = \bigcap_{i=0}^{n} S_i$ is an invariant non-degenerate subdendrite of D and $T_{n+1} \subseteq T_n$. Every T_n intersects each component from $\mathcal{B}$ and no component from $\mathcal{C} \setminus \mathcal{B}$. Since $\operatorname{diam} T_n \to 0$, by passing to a subsequence if necessary, we may assume that $\operatorname{diam} T_{n+1} < \operatorname{diam} T_n$ for every n.

Step 2. We show that, for every $N \in \mathbb{N}$, the set

$$A_N = \{x \in T_0 \colon \ \operatorname{Orb}_f(x) \cap T_N = \emptyset\}$$

is nowhere dense in T_0. (Thus, generic chaos on D implies that most of the points from $T_0 \setminus T_N$ enter T_N. This is not trivial. For instance, let D be the dendrite from Figure 2 where the horizontal arc has endpoints $(-1, 0)$ and $(1, 0)$, with $p = (0, 0)$ being the leftmost branch point. Imagine that T_0 consists of p and the points $(a, b) \in D$ with $a > 0$, and T_n is the arc joining p and $(1/(n + 1), 0)$, $n = 1, 2, \ldots$. If for instance (x, y) is a Li–Yorke pair with $x \in T_0 \setminus T_1$ and $y \in T_2$, the point x is proximal to the invariant set T_2 but this does not mean that x necessarily enters T_2 or at least T_1. Indeed, a priori it is not excluded that all points $f(x)$, $f^2(x)$, $\ldots$ are in vertical arcs of D and none of them is in the horizontal one.)

Fix N. To simplify the notation, denote T_N and A_N just by T and A, respectively. From now on, we will consider T_0 as the underlying space to work with. Since f is generically chaotic on D and T_0 is an invariant regular closed subset by Proposition 2.13, f is generically chaotic on T_0 by Lemma 2.14. Now suppose, on the contrary, that A is dense in a non-empty open subset U of T_0. Since dendrites are locally connected, we may additionally assume that U is connected. By continuity and (Sens$_0$), $f^n(U)$ is connected and non-degenerate for every n.

By Lemma 2.7, the escape-boundary $\operatorname{Bd}^*(T) = \{c_j \colon j \in J\}$ of (the proper subdendrite) T in T_0 is at most countable. Put

$$T^* = \bigcup \{\overline{C} \colon C \text{ is a component of } T_0 \setminus T\} \subseteq \overline{T_0 \setminus T}.$$

For every $j \in J$, denote by B_j the component of T^* containing c_j (that is, B_j is the union of the closures of those components of $T_0 \setminus T$ whose boundaries are $\{c_j\}$). By Lemma 2.7, every B_j is a non-degenerate subdendrite of T_0 with $\operatorname{Bd}(B_j) = \{c_j\}$ and $d(B_i, B_j) > 0$ for any distinct $i, j \in J$. By the choice of T_n's, $p \notin B_j$ for any $j \in J$.

Since A is dense in U, we have that $f^n(A)$ is dense in $f^n(U)$ for every $n \in \mathbb{N}_0$. By definition of A,

$$f^n(A) \cap T = \emptyset \quad \text{for every } n \in \mathbb{N}_0. \tag{4.1}$$

Now fix $n \in \mathbb{N}_0$. We will show that

$$f^n(U) \cap T \text{ contains at most one point}$$
$$\text{and this point, if it exists, belongs to } \operatorname{Bd}^*(T). \tag{4.2}$$

First suppose that there are two distinct points $x, y \in f^n(U) \cap T$. Since both $f^n(U)$ and T are connected, the whole arc $[x, y] \subseteq f^n(U) \cap T$. As $[x, y]$ is a regular closed set by Proposition 2.13, the set $f^n(A)$ is dense in $[x, y] \subseteq T$ and so $f^n(A) \cap T \neq \emptyset$, in contradiction with (4.1). We have thus shown that $f^n(U) \cap T$ has at most one element. Now suppose that $f^n(U) \cap T = \{y\}$. The set $f^n(U)$ is non-degenerate, connected,

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contains the point $y \in T$ and $f^n(U) \setminus \{y\} \subseteq T_0 \setminus T$. Thus, there is a component B_{j_0} of T^* such that $f^n(U) \subseteq B_{j_0}$; clearly, $\{y\}$ coincides with $\mathrm{Bd}(B_{j_0}) = \{c_{j_0}\}$. So, y belongs to the escape-boundary $\mathrm{Bd}^*(T)$. We have proved (4.2).

Fix any $n \in \mathbb{N}_0$. The set $f^n(U)$ is connected and non-degenerate. If $f^n(U) \cap T \neq \emptyset$, then, by (4.2), $f^n(U)$ contains some c_j and so $f^n(U) \subseteq B_j$. If $f^n(U) \cap T = \emptyset$, then $f^n(U)$ is a subset of some component of $T_0 \setminus T$, so again there is $j \in J$ with $f^n(U) \subseteq B_j$. That is,

$$\text{for every } n \in \mathbb{N}_0 \text{ there is a unique } j \in J \text{ such that } f^n(U) \subseteq B_j. \tag{4.3}$$

Since f is generically chaotic on T_0 and U is non-empty open in T_0, $\limsup_{n\to\infty} \operatorname{diam} f^n(U) > 0$ by (Sens_0). Since U is connected, Lemma 2.12 yields the existence of a minimal $n_0 \in \mathbb{N}_0$ and a corresponding minimal $k \in \mathbb{N}$ such that

$$f^{n_0}(U) \cap f^{n_0+k}(U) \neq \emptyset.$$

Then, in view of (4.3), the sets $f^{n_0}(U)$ and $f^{n_0+k}(U)$ lie in the same component of T^*. It follows that for any $i \in \{0, 1, \ldots, k-1\}$ and for any $\ell \in \mathbb{N}$, the sets $f^{n_0+i}(U)$ and $f^{n_0+i+\ell k}(U)$ lie in the same component of T^*. This implies that the orbit of U intersects only finitely many components of T^*, that is,

$$\mathrm{Orb}_f(U) \subseteq \bigcup_{j\in J'} B_j$$

for some finite subset J' of J. Hence, $\mathrm{Orb}_f(U)$ is disjoint from some ε-neighbourhood $B_\varepsilon(p)$ of p.

Since $\operatorname{diam}(T_m) \to 0$, we can choose m with $T_m \subseteq B_{\varepsilon/2}(p)$. Then $\mathrm{Orb}_f(T_m) \subseteq T_m$ has positive distance from $\mathrm{Orb}_f(U)$. Since both T_m and U have non-empty interiors in T_0, this contradicts the fact that $f\colon T_0 \to T_0$ is generically chaotic. This finishes Step 2 of the proof.

Step 3. We finish the proof by finding a contradiction with generic chaoticity of f on T_0.

By Step 2, each of the sets $A_n \subseteq T_0$ ($n \in \mathbb{N}$) of points whose orbits do not intersect T_n is nowhere dense in T_0. Clearly, $\{A_n\}_{n=1}^\infty$ is an increasing sequence in the sense that $A_n \subseteq A_{n+1}$. Consider the set

$$M = \bigcup_{n=1}^{\infty} A_n.$$

Then $K = T_0 \setminus M$ is the set of points from T_0 whose orbits intersect T_n for every $n \in \mathbb{N}$. The trajectory of every such point actually converges to p, because all the sets T_n are invariant, contain p and their diameters tend to zero. Consequently, for every $x, y \in K$, we have $d(f^n(x), f^n(y)) \to 0$. Since each A_n is nowhere dense in T_0, the set K is residual in T_0 and K^2 is residual in T_0^2. This contradicts the fact that f is generically chaotic on T_0. □

LEMMA 4.3. *Let D be a completely regular non-degenerate dendrite with all points of finite order. Let $f\colon D \to D$ be generically chaotic. Then there exists $\delta > 0$ such that the diameter of each invariant non-degenerate subdendrite is at least δ.*

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Proof. Suppose, on the contrary, that there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of invariant subdendrites with diam $D_n \to 0$. By Lemma 4.1 (note that our assumptions imply that the condition (Prox) is satisfied by the family of non-degenerate subdendrites), there exists a fixed point $p \in \bigcap_{n=1}^{\infty} D_n$. This contradicts Lemma 4.2. □

The following lemmas show some connections between generic chaos and the orbits of subdendrites.

LEMMA 4.4. *Let D be a completely regular non-degenerate dendrite with all points of finite order. Let $f: D \to D$ be generically chaotic. Then there exists $\delta > 0$ such that every non-degenerate subdendrite $E \subseteq D$ with* $\mathrm{Orb}_f(E)$ *containing a fixed point satisfies* $\limsup_{n\to\infty} \operatorname{diam} f^n(E) > \delta$.

Proof. Suppose, on the contrary, that there exists a sequence of non-degenerate subdendrites $\{D_i\}_{i=1}^{\infty}$ such that for every i there is some fixed point $p_i \in \mathrm{Orb}_f(D_i)$ and

$$\varepsilon_i \to 0 \quad \text{where } \varepsilon_i = \limsup_{n\to\infty} \operatorname{diam} f^n(D_i).$$

Here $\varepsilon_i > 0$ because D_i has non-empty interior and f is generically chaotic; see (Sens$_0$). For every $i \in \mathbb{N}$, we can pick $n_i \in \mathbb{N}$ such that $f^n(D_i)$ contains p_i and diam $f^n(D_i) < 2\varepsilon_i$ whenever $n \geq n_i$. Consider the subdendrites

$$E_i = \overline{\bigcup_{n=n_i}^{\infty} f^n(D_i)}.$$

It can be easily concluded that diam $E_i \leq 4\varepsilon_i$. So, the sets E_i are invariant non-degenerate subdendrites with diam $E_i \to 0$, which contradicts Lemma 4.3. □

LEMMA 4.5. *Let D be a completely regular non-degenerate dendrite with all points of finite order. Let $f: D \to D$ be generically chaotic. Then there exists $\delta > 0$ such that*

$$\limsup_{n\to\infty} \operatorname{diam} f^n(E) > \delta$$

whenever $E \subseteq D$ is a (non-degenerate) subdendrite satisfying $E \cap f(E) \neq \emptyset$ and having $\mathrm{Orb}_f(E)$ *fixed point free.*

Proof. Suppose, on the contrary, that there exists a sequence of non-degenerate subdendrites $\{D_i\}_{i=1}^{\infty}$ such that $D_i \cap f(D_i) \neq \emptyset$, $\mathrm{Orb}_f(D_i)$ is fixed point free and

$$\varepsilon_i \to 0 \quad \text{where } \varepsilon_i = \limsup_{n\to\infty} \operatorname{diam} f^n(D_i).$$

By (Sens$_0$), $\varepsilon_i > 0$. Now fix $i \in \mathbb{N}$.

Step 1. We define a subdendrite F_i.

The set $F_i = \overline{\mathrm{Orb}_f(D_i)}$ is an invariant non-degenerate subdendrite of D since $\mathrm{Orb}_f(D_i)$ is connected. From now on we will work with F_i (with relative topology). Since $\mathrm{Orb}_f(D_i)$ is fixed point free, by Lemma 2.1,

$$\emptyset \neq \mathrm{Fix}(f) \cap F_i \subseteq \mathrm{End}(F_i).$$

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Step 2. We find an appropriate endpoint $a_i \in F_i \setminus \mathrm{Orb}_f(D_i)$ fixed by f, and points $x_j \in f^j(D_i)$ $(j \in \mathbb{N}_0)$.

By Lemma 3.5, there exists a fixed endpoint $a_i \in F_i$ such that every non-fixed $y \in F_i$ admires a_i. By the assumption, $a_i \notin \mathrm{Orb}_f(D_i)$ and every point from $\mathrm{Orb}_f(D_i)$ admires a_i. For every $j \in \mathbb{N}_0$, put

$$x_j = \mathrm{proj}(a_i, f^j(D_i)) \in f^j(D_i).$$

Since $x_j \in f^j(D_i) \subseteq \mathrm{Orb}_f(D_i)$, it is not fixed by f.

Step 3. We prove that $x_{j-1} \in f^j(D_i)$ for every $j \in \mathbb{N}$.

Suppose, on the contrary, that $x_{j-1} \notin f^j(D_i)$ for some $j \in \mathbb{N}$. As $x_{j-1} \in f^{j-1}(D_i)$, we have $f(x_{j-1}) \in f^j(D_i)$, and x_{j-1} as a non-fixed point of F_i admires a_i. In other words, $f(x_{j-1}) \in (F_i)_{[a_i, x_{j-1})}$ (see (3.2)). Since $D_i \cap f(D_i) \neq \emptyset$, there is $y \in f^{j-1}(D_i) \cap f^j(D_i)$. Obviously $y \neq x_{j-1}$ as $x_{j-1} \notin f^j(D_i)$. Then, by definition of x_{j-1}, we get $y \in (F_i)^{a_i}(x_{j-1}) \setminus \{x_{j-1}\}$ (see (3.1)). It follows that $x_{j-1} \in (f(x_{j-1}), y) \subseteq f^j(D_i)$, which is a contradiction. So, indeed,

$$x_{j-1} \in f^j(D_i).$$

Step 4. We prove that $x_j \in (a_i, x_{j-1})$ for every $j \in \mathbb{N}$.

By Step 3, $x_{j-1} \in f^j(D_i)$. However, $x_j = \mathrm{proj}(a_i, f^j(D_i))$ and so $x_j \in [a_i, x_{j-1}]$ by Lemma 2.11(1). Since $x_j \in \mathrm{Orb}_f(D_i)$ and $a_i \notin \mathrm{Orb}_f(D_i)$, we get $x_j \neq a_i$. Further, x_{j-1} admires a_i, that is, $f(x_{j-1}) \in (F_i)_{[a_i, x_{j-1})}$. Now $f(x_{j-1}) \in f^j(D_i)$ and the definition of x_j give that $x_j \in (F_i)_{[a_i, x_{j-1})}$. Thus, $x_j \neq x_{j-1}$.

Step 5. We prove that $\lim_{j\to\infty} x_j = a_i$.

By Step 4, all points x_j are in $(a_i, x_0]$ with $x_{j+1} <_{a_i} x_j$. Therefore, the sequence $\{x_j\}_{j=1}^{\infty}$ monotonically converges to some $x \in [a_i, x_0)$. We want to show that $x = a_i$. For a contradiction, let $x \in (a_i, x_0)$. Then x is a cutpoint of $F_i = \overline{\mathrm{Orb}_f(D_i)}$ and so $x \in \mathrm{Orb}_f(D_i)$ by Lemma 2.1. Hence, x admires a_i, that is, $f(x) \in (F_i)_{[a_i, x)}$. The continuity of f implies that for some index K, the point x_K, which is close enough to x, has its image $f(x_K)$ in $(F_i)_{[a_i, x)}$. As $f(x_K) \in f^{K+1}(D_i)$, we get $f^{K+1}(D_i) \cap (F_i)_{[a_i, x)} \neq \emptyset$ and hence $x_{K+1} \in (F_i)_{[a_i, x)}$. On the other hand, $x_{K+1} \in [a_i, x_0)$ by Step 4. This gives that $x_{K+1} \in [a_i, x)$, which contradicts the definition of x.

Step 6. We construct a non-degenerate invariant subdendrite E_i of D with $\mathrm{diam}\, E_i \leq 6\varepsilon_i$.

Let U_i be the ε_i-neighbourhood of a_i (in F_i). By Step 4, choose $N_1 \in \mathbb{N}$ such that $x_n \in U_i$ whenever $n \geq N_1$. Hence, $f^n(D_i) \cap U_i \neq \emptyset$ for every $n \geq N_1$. By the definition of ε_i, there is $N_2 \in \mathbb{N}$ such that $\mathrm{diam}\, f^n(D_i) < 2\varepsilon_i$ for every $n \geq N_2$. Put $N = \max\{N_1, N_2\}$; then

$$E_i = \overline{\bigcup_{n=N}^{\infty} f^n(D_i)}$$

is a non-degenerate invariant subdendrite of F_i (and hence also of D) with diameter at most $6\varepsilon_i$.

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Step 7. We finish the proof.

Since fixed $i \in \mathbb{N}$ was arbitrary, we have a sequence $\{E_i\}_{i=1}^{\infty}$ of non-degenerate invariant subdendrites of D with diameters converging to zero. By Lemma 4.3, this contradicts generic chaoticity of f. □

LEMMA 4.6. *Let D be a non-degenerate dendrite. Let $f\colon D \to D$ be continuous and $E \subseteq D$ be a subdendrite with* $\limsup_{n\to\infty} \operatorname{diam} f^n(E) > 0$. *Let $L_0, L_1, \ldots, L_{r-1}$ be as in Lemma 2.12. If f is generically chaotic, $r > 1$ and $L_0, L_1, \ldots, L_{r-1}$ have non-empty interiors, then $\bigcap_{i=0}^{r-1} \overline{L_i}$ is a singleton.*

Proof. The map $g = f^r$ is generically chaotic. Take any distinct $i, j \in \{0, 1, \ldots, r-1\}$. Since L_i and L_j have non-empty interiors, there exists a Li–Yorke pair in $L_i \times L_j$. This implies that $d(L_i, L_j) = 0$ and hence $\overline{L_i} \cap \overline{L_j} \neq \emptyset$. So, $L_i \cap L_j = \emptyset \neq \overline{L_i} \cap \overline{L_j}$ for any distinct i, j. By Lemma 2.4, $\bigcap_{i=0}^{r-1} \overline{L_i}$ is a singleton. □

LEMMA 4.7. *Let D be a completely regular non-degenerate dendrite. Let $f\colon D \to D$ be generically chaotic and $E \subseteq D$ be a non-degenerate subdendrite such that the following conditions are satisfied:*

(1) $\operatorname{Orb}_f(E)$ *is connected;*

(2) *there exists $k \in \mathbb{N}$ such that $E \cap f^k(E) \neq \emptyset$.*

Denote $K_j = \bigcup_{i=0}^{\infty} f^{j+ik}(E)$, $j \in \{0, 1, \ldots, k-1\}$. Then $\bigcap_{j=0}^{k-1} K_j \neq \emptyset$.

Proof. We assume that $k \geq 2$, otherwise the lemma is trivial. The sets K_j are non-degenerate and connected. The map $g = f^k$ is generically chaotic and the sets K_j are g-invariant. The subdendrites $\overline{K_j}$ are regular closed sets by Proposition 2.13. So, for any distinct $l, m \in \{0, 1, \ldots, k-1\}$, we can find a Li–Yorke pair in $\overline{K_l} \times \overline{K_m}$ and hence $\overline{K_l} \cap \overline{K_m} \neq \emptyset$. By Lemma 2.3, $\bigcap_{j=0}^{k-1} \overline{K_j} \neq \emptyset$.

For a contradiction, suppose that $\bigcap_{j=0}^{k-1} K_j = \emptyset$. By Lemma 2.4, $\bigcap_{j=0}^{k-1} \overline{K_j}$ is a singleton $\{p\}$. By Lemma 2.12 (with $n_0 = 0$), the sets K_j are cyclically permuted by f in the sense of (2.2). Then also their closures are cyclically permuted by f and, since $p \in \bigcap_{j=0}^{k-1} \overline{K_j}$, then also $f(p) \in \bigcap_{j=0}^{k-1} \overline{K_j} = \{p\}$. Hence, p is fixed by f.

The point p belongs to some K_j, otherwise $\operatorname{Orb}_f(E) = \bigcup_{j=0}^{k-1} K_j$ would not be connected. But then the point p, being fixed for f, belongs to every K_j by (2.2). This contradicts the assumption that $\bigcap_{j=0}^{k-1} K_j = \emptyset$. □

Let D be a non-degenerate dendrite and $f\colon D \to D$ be generically chaotic. Given a non-degenerate subdendrite $E \subseteq D$, in the following lemma we will work with the set

$$M_f(E) = \{l \in \mathbb{N}\colon \text{there exists } n \geq 0 \text{ such that } f^n(E) \cap f^{n+l}(E) \neq \emptyset\}. \tag{4.4}$$

If D is completely regular, E has non-empty interior and so $\limsup_{n\to\infty} \operatorname{diam} f^n(E) > 0$ by (Sens$_0$). Then $M_f(E) \neq \emptyset$ by Lemma 2.12.

LEMMA 4.8. *Let D be a completely regular non-degenerate dendrite. Let $f\colon D \to D$ be generically chaotic. Let $E \subseteq D$ be a non-degenerate subdendrite and $k = \min M_f(E)$. Assume that the following conditions are satisfied:*

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(1) $\mathrm{Orb}_f(E)$ *is connected and periodic point free;*
(2) $E \cap f^k(E) \neq \emptyset$.
Then $k = 1$.

Proof. Due to (1), the set $\overline{\mathrm{Orb}_f(E)}$ is a subdendrite of D. Since it is invariant and, by Proposition 2.13, also regular closed, the restriction of f to this set is generically chaotic by Lemma 2.14. Therefore, without loss of generality, we may assume that

$$D = \overline{\mathrm{Orb}_f(E)},$$

otherwise we just restrict our dynamical system to the subdendrite $\overline{\mathrm{Orb}_f(E)}$. Notice that $\mathrm{Orb}_f(E)$, being a connected dense set, contains all cutpoints of D.

Suppose that $k \geq 2$. To get a contradiction, we proceed in several steps.

Step 1. We find a subdendrite $\widetilde{E}$ that is an appropriate iterate of E, and define subdendrites A_n, B_n ($n \in \mathbb{N}_0$).

Since f satisfies (Sens$_0$), the condition (2) and Lemma 2.12 (with $n_0 = 0$) imply that

$$\mathrm{Orb}_f(E) = \bigcup_{j=0}^{k-1} K_j \quad \text{where } K_j = \bigcup_{i=0}^{\infty} f^{j+ik}(E) \quad (j \in \{0, 1, \dots, k-1\}).$$

Fix two distinct $j, j' \in \{0, 1, \dots, k-1\}$. By Lemma 4.7, we know that $K_j \cap K_{j'} \neq \emptyset$. So, there are non-negative integers i, i' such that

$$f^{j+ik}(E) \cap f^{j'+i'k}(E) \neq \emptyset. \tag{4.5}$$

Choose i and i' satisfying (4.5) which minimize the quantity $\Delta = |(j' + i'k) - (j + ik)|$. Clearly, $\Delta > 0$ and so $\Delta \geq k$ due to the definition of k. We have in fact $\Delta \geq k + 1$ because $0 < |j' - j| < k$. Without loss of generality, we may assume that $j' + i'k > j + ik$, so

$$\Delta = (j' + i'k) - (j + ik) \geq k + 1.$$

Consider the (non-degenerate) dendrite

$$\widetilde{E} = f^{j+ik}(E).$$

Put $m = \Delta - k$. Then $m \geq 1$ and m is not a multiple of k. In the following, we will denote $g = f^k$ and

$$A_n = f^{nk}(\widetilde{E}) = g^n(\widetilde{E}) \quad \text{and} \quad B_n = f^{nk+m}(\widetilde{E}) = g^n(f^m(\widetilde{E})) \quad \text{for } n \in \mathbb{N}_0.$$

The sets A_n, B_n are non-degenerate subdendrites.

Step 2. We study g-iterates of the sets A_n and B_n from the point of view of their intersections.

We claim that

$$A_0 \cap B_1 = \widetilde{E} \cap f^{k+m}(\widetilde{E}) \neq \emptyset \quad \text{and} \quad A_1 \cap B_1 = f^k(\widetilde{E}) \cap f^{k+m}(\widetilde{E}) = \emptyset. \tag{4.6}$$

The first part is (4.5). Further, $f^k(\widetilde{E}) \cap f^{k+m}(\widetilde{E}) = f^{j+(i+1)k}(E) \cap f^{j'+i'k}(E)$ and $(j' + i'k) - (j + (i+1)k) = m < k + m = \Delta$ and so $A_1 \cap B_1 = \emptyset$ by minimality of Δ.

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By applying $g = f^k$ to the non-empty intersection in (4.6) and, respectively, by minimality of Δ, we get

$$A_1 \cap B_2 = f^k(\widetilde{E}) \cap f^{2k+m}(\widetilde{E}) \neq \emptyset \quad \text{and} \quad A_2 \cap B_2 = f^{2k}(\widetilde{E}) \cap f^{2k+m}(\widetilde{E}) = \emptyset.$$

By induction, for all $n \in \mathbb{N}$,

$$A_{n-1} \cap B_n \neq \emptyset \quad \text{and} \quad A_n \cap B_n = \emptyset. \tag{4.7}$$

Further, by (2),

$$A_n \cap A_{n+1} = f^{nk}(\widetilde{E}) \cap f^{(n+1)k}(\widetilde{E}) \neq \emptyset$$

and

$$B_n \cap B_{n+1} = f^{nk+m}(\widetilde{E}) \cap f^{(n+1)k+m}(\widetilde{E}) \neq \emptyset.$$

Step 3. We partially describe the 'position' of the sets A_n, B_n in D.
Using (4.7), we get that for all $n \in N$,

$$A_n \subseteq C_n \quad \text{for some (unique) component } C_n \text{ of } D \setminus B_n.$$

Choose $n \in \mathbb{N}$ and apply Lemma 2.9 to the sets A_n, A_{n+1} and B_n, B_{n+1}. Denoting the boundary of C_i by $\{x_i\}$ $(i = n, n+1)$, we have

$$A_n \subseteq C_n, \quad C_n \supsetneq C_{n+1},$$

$$x_{n+1} \in A_n \cap B_{n+1}, \quad x_n \notin \overline{C_{n+1}}, \quad \{x_{n+1}\} = \mathrm{Bd}(\overline{C_{n+1}}). \tag{4.8}$$

It also follows that $x_{n+1} \notin C_{n+1}$ since C_{n+1} is open. Further,

$$x_n \neq x_{n+1} \quad \text{for every } n \in \mathbb{N}. \tag{4.9}$$

Step 4. We study the intersection $C = \bigcap_{n=1}^{\infty} C_n$.
Since $\widetilde{E}$ is a subdendrite with non-empty interior and $g = f^k$ is generically chaotic,

$$\delta = \limsup_{n\to\infty} \operatorname{diam} A_n = \limsup_{n\to\infty} \operatorname{diam} g^n(\widetilde{E}) > 0.$$

Since $C_1 \supsetneq C_2 \supsetneq \cdots$ is a nested sequence and $C_n \supseteq A_n$, it follows that for all n we have $\operatorname{diam} C_n \geq \delta$. Notice that

$$C = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \overline{C_n}.$$

Indeed, this follows from the fact that, for every n, the boundary of C_n is $\{x_n\}$ and $x_n \notin C_{n+1}$. Thus, C is a non-degenerate dendrite.

Step 5. By considering the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, we get a contradiction.
By (4.8), for $n \geq 2$ we have $\{x_n\} = \mathrm{Bd}(\overline{C_n})$ and $x_1 \notin \overline{C_n}$. So, by Lemma 2.11(4),

$$x_n = \operatorname{proj}(x_1, \overline{C_n}).$$

Put

$$x = \operatorname{proj}(x_1, C).$$

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Let $n \geq 2$. Then $x \in C \subseteq \overline{C_n}$. On the other hand, $x_1 \notin \overline{C_n}$ by (4.8). Hence, by Lemma 2.11(1),

$$x_n \in [x_1, x]. \tag{4.10}$$

By (4.10), (4.9) and the fact that $\{\overline{C_n}\}_{n=1}^{\infty}$ is a nested sequence, we get that

$$\{x_n\}_{n=1}^{\infty} \text{ is a linearly ordered set with } x_{n+1} <_x x_n. \tag{4.11}$$

We claim that (see (3.1))

$$C = \{y \in D\colon x \leq_{x_1} y\} = D^{x_1}(x). \tag{4.12}$$

Indeed, by the choice of x, we have $C \subseteq D^{x_1}(x)$. Now let $y \in D^{x_1}(x)$. To show that $y \in C$, we fix n and show that $y \in C_n$. Put $A = (x_n, y]$. This is a connected set that intersects C_n but does not contain the point x_n, which is the unique boundary point of C_n. Thus, $A \subseteq C_n$ by Lemma 2.2 and so $y \in C_n$.

By Lemma 2.11(1),

$$[x_n, x] \subseteq \overline{C_n} \subseteq \{y \in D\colon x_n \leq_{x_1} y\} = D^{x_1}(x_n)$$

for every $n \geq 2$. This, together with (4.10), (4.11) and the definitions of C and x, yield

$$\lim_{n\to\infty} x_n = x. \tag{4.13}$$

Since C is non-degenerate, x is a cutpoint of D by Lemma 2.11(3) and hence $x \in \mathrm{Orb}_f(E)$. As $\mathrm{Orb}_f(E)$ is periodic point free, x is not a fixed point for g. Further, by (4.8), $x_n \in A_{n-1}$; hence, $g(x_n) \in g(A_{n-1}) = A_n \subseteq C_n \subseteq \overline{C_n}$. Then, since $\{\overline{C_n}\}_{n=1}^{\infty}$ is a nested sequence, (4.13) and (4.12) give

$$g(x) = \lim_{n\to\infty} g(x_n) \in \bigcap_{n=1}^{\infty} \overline{C_n} = C = D^{x_1}(x).$$

As $g(x) \neq x$, we have $g(x) \in D^{x_1}(x) \setminus \{x\}$. By continuity, $g(x_n) \in D^{x_1}(x) \setminus \{x\} \subseteq C$ for sufficiently large n. On the other hand, by (4.8), for every $n \geq 2$ we have $x_n \in B_n$ and so $g(x_n) \in B_{n+1}$. It follows that $g(x_n) \notin C_{n+1}$, whence $g(x_n) \notin C$, which is a contradiction. □

5. *Proof of Theorem A: (2) implies (1)*

We assume that D is a completely regular (non-degenerate) dendrite with all points of finite order and $f\colon D \to D$ is a generically chaotic map. We are going to prove that f is generically ε-chaotic for some $\varepsilon > 0$. By Proposition 1.1 and the assumptions on D, it is sufficient to show that (Prox) and (Sens) are satisfied by the family of all non-degenerate subdendrites of D. Since f is generically chaotic, the condition (Prox) is satisfied trivially (also (Sens$_0$) is satisfied trivially).

To prove (Sens), suppose, on the contrary, that there exists a sequence $\{\tilde{D}_i\}_{i=1}^{\infty}$ of non-degenerate subdendrites of D such that

$$\lim_{i\to\infty} \limsup_{n\to\infty} \operatorname{diam} f^n(\tilde{D}_i) = 0, \tag{5.1}$$

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where, by (Sens$_0$), for every i we have

$$\limsup_{n\to\infty} \operatorname{diam} f^n(\tilde{D}_i) > 0. \tag{5.2}$$

To get a contradiction, we proceed in several steps.

Step 1. For $i \in \mathbb{N}$, we define positive integers m_i and replace $\tilde{D}_i$ by D_i.

The set $M_f(\tilde{D}_i)$ from (4.4) is non-empty due to (Sens$_0$); put $m_i = \min M_f(\tilde{D}_i)$. Fix $n_i \geq 0$ such that $f^{n_i}(\tilde{D}_i) \cap f^{n_i+m_i}(\tilde{D}_i) \neq \emptyset$ and put

$$D_i = f^{n_i}(\tilde{D}_i) \quad (i \in \mathbb{N}).$$

Then (5.1) and (5.2) still hold for D_i instead of $\tilde{D}_i$. Moreover,

$$D_i \cap f^{m_i}(D_i) \neq \emptyset \quad (i \in \mathbb{N}). \tag{5.3}$$

Clearly, $m_i = \min M_f(D_i)$.

Step 2. For $i \in \mathbb{N}$, we define positive integers r_i, connected sets K_i^k and L_i^j and points p_i.

By (5.3), for $i \in \mathbb{N}$ the sets

$$K_i^k = \bigcup_{n=0}^{\infty} f^{k+nm_i}(D_i) \quad (k \in \{0, 1, \ldots, m_i - 1\})$$

are connected and, by (5.2), they are non-degenerate. By Lemma 2.12, for any $i \in \mathbb{N}$ there exist $r_i \in \mathbb{N}$ (a divisor of m_i) and connected sets $L_i^0, L_i^1, \ldots, L_i^{r_i-1}$ such that

$$\operatorname{Orb}_f(D_i) = L_i^0 \sqcup L_i^1 \sqcup \cdots \sqcup L_i^{r_i-1},$$

$f(L_i^0) = L_i^1$, $f(L_i^1) = L_i^2, \ldots, f(L_i^{r_i-2}) = L_i^{r_i-1}$ and $f(L_i^{r_i-1}) \subseteq L_i^0$. Here,

$$L_i^j = \bigcup_{k=0}^{m_i/r_i-1} K_i^{j+kr_i} \quad (j \in \{0, 1, \ldots, r_i - 1\}).$$

The closures $\overline{L_i^j}$ are f^{r_i}-invariant non-degenerate subdendrites. By Lemma 4.6, if $r_i > 1$, then there exists a unique p_i such that

$$\bigcap_{j=0}^{r_i-1} \overline{L_i^j} = \{p_i\}.$$

Since the sets L_i^j $(j \in \{0, \ldots, r_i - 1\})$ are pairwise disjoint, they belong to different components of $D \setminus \{p_i\}$. Thus,

$$\operatorname{ord}(p_i, D) \geq r_i. \tag{5.4}$$

Step 3. To finish the proof, that is, to get a contradiction, we distinguish two cases depending on whether the sequence $\{r_i\}_{i=1}^{\infty}$ is bounded or unbounded.

Case I. Assume that the sequence $\{r_i\}_{i=1}^{\infty}$ is bounded.

By passing to a subsequence if necessary, we may assume that the sequence is constant: $r_i = r$ for every i. Then r divides every m_i. The map $g = f^r$ is generically chaotic. Put $L_i = L_i^0$ for every i. Recall that each L_i is non-degenerate, g-invariant and connected,

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with

$$L_i = \mathrm{Orb}_g(D_i) = \bigcup_{k=0}^{m_i/r-1} K_i^{rk}. \tag{5.5}$$

From (5.1) and generic chaoticity of g, we get

$$\lim_{i\to\infty} \varepsilon_i = 0 \quad \text{where } \varepsilon_i = \limsup_{n\to\infty} \operatorname{diam} g^n(D_i) > 0. \tag{5.6}$$

By (5.6) and Lemma 4.4, $\mathrm{Orb}_g(D_i)$ contains a fixed point of g only for finitely many indices i. We may assume that

$$\mathrm{Orb}_g(D_i) \cap \mathrm{Fix}(g) = \emptyset \quad \text{for every } i \in \mathbb{N}. \tag{5.7}$$

If $m_i = r$ for infinitely many i, then $D_i \cap g(D_i) \neq \emptyset$ for these indices, which together with (5.6) contradicts Lemma 4.5. Therefore, we may assume that

$$m_i > r \quad \text{for all } i \in \mathbb{N}. \tag{5.8}$$

By definition of m_i,

$$g^n(D_i) \cap g^{n+1}(D_i) = \emptyset \quad \text{for every } n \in \mathbb{N}_0. \tag{5.9}$$

Fix any $i \in \mathbb{N}$. Each $\overline{L_i}$ is a non-degenerate g-invariant dendrite. It is regular closed by Proposition 2.13. Moreover, $\overline{L_i}$ shares the properties of D; it is completely regular, with all points of finite order. From now on till the end of Case I we will work with this dendrite $\overline{L_i}$ and we will write just g rather than $g|_{\overline{L_i}}$. Note that by Lemma 2.14, this restriction is generically chaotic.

By (5.8), in (5.5) we have the union of at least two sets. Recall that $\mathrm{Orb}_g(D_i)$ in (5.5) is connected and by (5.3) we have $D_i \cap g^{m_i/r}(D_i) = D_i \cap f^{m_i}(D_i) \neq \emptyset$. Notice also that $K_i^{rk} = \bigcup_{n=0}^{\infty} f^{rk+nm_i}(D_i) = \bigcup_{n=0}^{\infty} g^{k+nm_i/r}(D_i)$, $k = 0, \ldots, m_i/r - 1$. Thus, we may apply Lemma 4.7 to get that

$$\bigcap_{k=0}^{m_i/r-1} K_i^{rk} \neq \emptyset.$$

By (5.7), fixed points of g exist only in $\mathrm{End}(\overline{L_i}) \setminus \mathrm{Orb}_g(D_i)$. Therefore, Lemma 3.5 ensures the existence of a point

$$a_i \in \mathrm{End}(\overline{L_i}) \setminus \mathrm{Orb}_g(D_i) \tag{5.10}$$

which is fixed for g and such that

$$\text{every point from } \mathrm{Orb}_g(D_i) \text{ admires } a_i. \tag{5.11}$$

By (5.5), $a_i \in \overline{K_i^{rk}}$ for some $k \in \{0, 1, \ldots, m_i/r - 1\}$. As $g(a_i) = a_i$ and the sets K_i^{rk} in (5.5) are cyclically permuted by g, we have

$$a_i \in \bigcap_{k=0}^{m_i/r-1} \overline{K_i^{rk}}.$$

Denote $a = a_i$. We distinguish two subcases.

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Subcase I(1). Assume that there exists an open (in $\overline{L_i}$) neighbourhood U of a with a singleton boundary $\{u\}$ such that

$$\text{for any } n \in \mathbb{N}_0, \text{ if } g^n(D_i) \cap U \neq \emptyset \text{ then } g^n(D_i) \cap (a, u) \neq \emptyset. \tag{5.12}$$

As $a \in \overline{\mathrm{Orb}_g(D_i)}$, there exists n_0 with $g^{n_0}(D_i) \cap U \neq \emptyset$. By (5.12), $g^{n_0}(D_i) \cap (a, u) \neq \emptyset$. Let

$$x_0 = \mathrm{proj}(a, g^{n_0}(D_i)).$$

By (5.10), $x_0 \neq a$. Choose $y \in g^{n_0}(D_i) \cap (a, u)$ and use Lemma 2.11(1) to get $x_0 \in (a, y] \subseteq (a, u)$. By (5.11) and (5.10), $g(x_0) \in (\overline{L_i})_{(a,x_0)}$. Thus,

$$x_0 \in (a, u) \quad \text{and} \quad g(x_0) \in (\overline{L_i})_{(a,x_0)}.$$

Then, since $g(x_0) \in g^{n_0+1}(D_i)$, we have that $x_1 = \mathrm{proj}(a, g^{n_0+1}(D_i)) \in (\overline{L_i})_{(a,x_0)}$. Similarly as above we can show that also $x_1 \in (a, u)$. It follows that $x_1 \in (a, x_0)$. Therefore, by (5.9), $g^{n_0+1}(D_i) \subseteq (\overline{L_i})_{(a,x_0)}$.

By induction, denoting $x_n = \mathrm{proj}(a, g^{n_0+n}(D_i))$, we get that the sequence $\{x_n\}_{n=0}^{\infty}$ lies in (a, u) and is such that, for every $n \in \mathbb{N}_0$,

$$a \ <_a \ x_{n+1} \ <_a \ x_n \quad (n \in \mathbb{N})$$

and

$$g^{n_0+n+1}(D_i) \subseteq (\overline{L_i})_{(a,x_n)}. \tag{5.13}$$

We claim that $x_n \to a$. Indeed, by (5.5) and (5.10), $\liminf_{n\to\infty} d(a_i, g^{n_0+n}(D_i)) = 0$; hence, $\liminf_{n\to\infty} d(a_i, x_n) = 0$; now monotonicity yields that $x_n \to a$. Since $x_n \to a$, $\mathrm{diam}(\overline{L_i})_{(a,x_n)} \to 0$ and (5.13) yields that $\mathrm{diam}\, g^n(D_i) \to 0$, which contradicts generic chaoticity of g.

Subcase I(2). Assume that for every open (in $\overline{L_i}$) neighbourhood U of a with a singleton boundary $\{u\}$

$$\text{there is } n_U \in \mathbb{N}_0 \text{ such that } g^{n_U}(D_i) \cap U \neq \emptyset \quad \text{and} \quad g^{n_U}(D_i) \cap (a, u) = \emptyset.$$

If such a neighbourhood U is connected, $g^{n_U}(D_i) \subseteq U$. (Otherwise, by Lemma 2.2, $g^{n_U}(D_i)$ contains u. Choose $x \in g^{n_U}(D_i) \cap U$. By Lemma 2.5, $g^{n_U}(D_i) \cap [a, u) \supseteq [x, u) \cap [a, u) \neq \emptyset$, which is a contradiction.) Since a is an endpoint of $\overline{L_i}$, it has a basis of connected neighbourhoods with singleton boundaries. Consider a sequence of such neighbourhoods U_k ($k \in \mathbb{N}$) with corresponding singleton boundaries $\{u_k\}$ such that $u_k \to a$ and $u_{k+1} <_a u_k$ for any $k \in \mathbb{N}$. To simplify the notation, let $n_k = n_{U_k}$. Thus,

$$\text{for every } k \in \mathbb{N} \text{ there is } n_k \in \mathbb{N}_0 \text{ such that } g^{n_k}(D_i) \subseteq U_k \quad \text{and} \quad g^{n_k}(D_i) \cap (a, u_k) = \emptyset.$$

Since $a \in \overline{\mathrm{Orb}_g(D_i)} \setminus \mathrm{Orb}_g(D_i)$, we may assume that $n_{k+1} > n_k \geq k$.

We claim that $\mathrm{Orb}_g(D_i)$ does not contain any periodic point of g. Assume that the opposite is true, that is, $p \in \mathrm{Orb}_g(D_i)$ is a periodic point. Thus, $p \in g^N(D_i)$ for some N. Then $g^n(D_i) \cap \mathrm{Orb}_g(p) \neq \emptyset$ for all $n \geq N$. Also for some $m > N$ the neighbourhood U_m is small enough to be disjoint with the orbit of p, that is, $U_m \cap \mathrm{Orb}_g(p) = \emptyset$. But also $g^{n_m}(D_i) \subseteq U_m$ and, since $n_m \geq m > N$, $g^{n_m}(D_i) \cap \mathrm{Orb}_g(p) \neq \emptyset$, which is

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a contradiction. Hence, $L_i = \mathrm{Orb}_g(D_i)$ is connected and periodic point free. Further, $D_i \cap g^{m_i/r}(D_i) \neq \emptyset$ by (5.3). Since $m_i = \min M_f(D_i)$, we have $m_i/r = \min M_g(D_i)$ (see (4.4)). Then Lemma 4.8 implies $m_i/r = 1$, in contradiction with (5.8).

Case II. Assume that the sequence $\{r_i\}_{i=1}^{\infty}$ is unbounded.

By passing to a subsequence, we may assume that $1 < r_i < r_{i+1}$ for $i \in \mathbb{N}$. Fix i and consider the map $g = f^{r_i r_1}$. Recall that the sets L_i^j ($j \in \{0, 1, \ldots, r_i - 1\}$) and $L_1^{j'}$ ($j' \in \{0, 1, \ldots, r_1 - 1\}$) are g-invariant and

$$\bigcap_{j=0}^{r_i-1} \overline{L_i^j} = \{p_i\} \quad \text{and} \quad \bigcap_{j'=0}^{r_1-1} \overline{L_1^{j'}} = \{p_1\}.$$

Since g is generically chaotic on D, for any j, j' the sets L_i^j and $L_1^{j'}$ have zero distance, as they have non-empty interiors.

We claim that $p_i = p_1$. Suppose not. Since $r_i > 1$, there exists $J \in \{0, 1, \ldots, r_i - 1\}$ such that L_i^J lies in a component of $D \setminus \{p_i\}$ which does not contain p_1. Similarly, there exists $J' \in \{0, 1, \ldots, r_1 - 1\}$ such that $L_1^{J'}$ lies in a component of $D \setminus \{p_1\}$ which does not contain p_i. This gives $d(L_i^J, L_1^{J'}) \geq d(p_i, p_1) > 0$, which is a contradiction.

We have proved that $p_i = p_1$ for any i. Since $r_i \to \infty$, (5.4) gives that $\mathrm{ord}(p_1, D) = \infty$, which contradicts the assumption that all points of D are of finite order.

6. *Exact maps on dendrites*

The purpose of this section is to prove the following result, which will be used in §7.

PROPOSITION 6.1. *Let D be a dendrite and $A \subseteq D$ be either a singleton or a nowhere-dense arc. Then there is an exact map $f : D \to D$ such that*

$$f(x) = x \quad \textit{for every } x \in A.$$

We start by recalling some results from [58] which will be used in the proof of this proposition.

Let D be a non-degenerate dendrite. We say that a family $\mathcal{C}$ of non-degenerate subdendrites of D is *dense* if $D \in \mathcal{C}$ and every non-empty open set in D contains some $C \in \mathcal{C}$. The system of all non-degenerate closed subintervals of $I = [0, 1]$ is denoted by $\mathcal{C}_I$; we assume that I is equipped with the Euclidean metric d_I.

Definition 6.2. Let (D, d), (D', d') be non-degenerate dendrites and $\mathcal{C}, \mathcal{C}'$ be dense families of non-degenerate subdendrites of D, D', respectively. Let $\varrho > 1$. We say that a continuous map $f : D \to D'$ is *ϱ-length expanding (with respect to $\mathcal{C}, \mathcal{C}'$)* if for every $C \in \mathcal{C}$ we have $f(C) \in \mathcal{C}'$ and

$$f(C) = D' \quad \text{or} \quad \mathcal{H}^1_{d'}(f(C)) \geq \varrho \cdot \mathcal{H}^1_d(C).$$

The following is a part of [58, Theorem C].

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PROPOSITION 6.3. *Let $\varrho > 1$, D be a non-degenerate dendrite and $a \in D$. Then there is a convex metric $d_{D,a}$ on D compatible with the topology of D, and continuous surjections $\varphi_{D,a}\colon I \to D$, $\psi_{D,a}\colon D \to I$ such that the following are true:*

(1) $\mathcal{H}^1_{d_{D,a}}(D) = 1$;
(2) $\varphi_{D,a}(0) = \varphi_{D,a}(1) = a$ *and* $\psi_{D,a}(a) = 0$;
(3) *the family $\mathcal{C}_{D,a} = \varphi_{D,a}(\mathcal{C}_I)$ is a dense family of non-degenerate subdendrites of D;*
(4) *$\varphi_{D,a}$ and $\psi_{D,a}$ are ϱ-length expanding (with respect to $\mathcal{C}_I, \mathcal{C}_{D,a}$ and $\mathcal{C}_{D,a}, \mathcal{C}_I$, respectively).*

Proof of Proposition 6.1. We may assume that D is non-degenerate. If A is a singleton, the assertion follows from [58, Corollary E]. (Though not stated explicitly, an exact map constructed in the mentioned corollary can have one or two prescribed fixed points, since the proof of it is based on [58, Theorem D].) Assume now that A is an arc and fix $0 < q < 1 < \varrho$. The proof is divided into eight steps.

Step 1. We define subdendrites D_k ($k \geq 0$).

Put $\tilde{D} = \bigcup\{\overline{C}\colon C \text{ is a component of } D \setminus A\}$. Since A is non-degenerate and nowhere dense, the set $\tilde{D}$ has $\aleph_0$ components; denote them by D_k ($k \in \mathbb{N}$). By Lemma 2.7, every D_k is a non-degenerate dendrite such that $\mathrm{Bd}(D_k) = D_k \cap A$ is a singleton $\{x_k\}$; moreover, $x_k \neq x_h$ for every $k \neq h$. In the following, we sometimes write D_0 instead of A. We believe that it will be convenient to call the subdendrite D_k ($k \geq 1$) a *bush rooted at* x_k. Then D is the union of $D_0 = A$ (which is not a bush) and all the bushes.

Step 2. We define a convex metric d on D compatible with the topology of D, and surjective maps $\varphi_k\colon I \to D_k$ and $\psi_k\colon D_k \to I$ ($k \geq 0$) such that, for $k \geq 1$, φ_k and ψ_k are ϱ-length expanding (with respect to $\mathcal{C}_I, \mathcal{C}_k$ and $\mathcal{C}_k, \mathcal{C}_I$, respectively).

For every $k \in \mathbb{N}$, consider the metric $d_k = d_{D_k,x_k}$ and maps $\varphi_k = \varphi_{D_k,x_k}$, $\psi_k = \psi_{D_k,x_k}$ obtained from Proposition 6.3 applied to the dendrite D_k and the point x_k. Then

$$\varphi_k(0) = \varphi_k(1) = x_k \quad \text{and} \quad \psi_k(x_k) = 0, \tag{6.1}$$

and

$$\mathcal{C}_k = \varphi_k(\mathcal{C}_I) \quad \text{is a dense family of non-degenerate subdendrites of } D_k. \tag{6.2}$$

Fix a homeomorphism $\varphi_0\colon I \to A = D_0$ and denote $\psi_0 = \varphi_0^{-1}$. Put $\mathcal{C}_0 = \varphi_0(\mathcal{C}_I)$. Let d_0 be a convex metric on A given by $d_0(x, x') = d_I(\psi_0(x), \psi_0(x'))$; thus, $\mathcal{H}^1_{d_0}(A) = 1$.

For every $k \geq 0$, put $\lambda_k = (1 - q)q^k$. Define a metric d on D by

$$d(x, x') = \begin{cases} \lambda_k d_k(x, x') & \text{if } x, x' \in D_k \text{ for } k \in \mathbb{N}_0; \\ \lambda_0 d_0(x, x_k) + \lambda_k d_k(x_k, x') & \text{if } x \in A, x' \in D_k \text{ for } k \in \mathbb{N}; \\ \lambda_k d_k(x, x_k) + \lambda_0 d_0(x_k, x') & \text{if } x \in D_k \text{ for } k \in \mathbb{N}, x' \in A; \\ \lambda_k d_k(x, x_k) + \lambda_0 d_0(x_k, x_h) + \lambda_h d_h(x_h, x') & \text{if } x \in D_k, x' \in D_h \text{ for } k \neq h \text{ in } \mathbb{N}. \end{cases}$$

Since dendrites are uniquely arcwise connected, it is an easy exercise to show that d is a convex metric compatible with the topology of D and that

$$\operatorname{diam}_d D_k \leq \mathcal{H}^1_d(D_k) = \lambda_k \quad \text{for every } k \geq 0.$$

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Hence, $\operatorname{diam}_d D \le \mathcal{H}^1_d(D) = \sum_{k=0}^{\infty} \lambda_k = 1$. Notice that for $k, h \in \mathbb{N}_0$ we have

$$k < h \iff \mathcal{H}^1_d(D_k) > \mathcal{H}^1_d(D_h)$$

and $\mathcal{H}^1_d(D_k) \searrow 0$ as $k \to \infty$. In particular, if D_k and D_h are two bushes with $k < h$, it is reasonable to say that D_h is *smaller* than D_k and D_k is *larger* than D_h.

Step 3. For $k \ge 1$, we define subdendrites $E_k \supsetneq D_k$ of D such that $E_1 = D$ and each E_k ($k \ge 2$) will contain one bush D_{ℓ_k} larger than D_k and infinitely many bushes smaller than D_k.

For every $k \ge 2$, find an integer $1 \le \ell_k < k$ such that $d(x_k, x_{\ell_k}) \to 0$ and $\ell_k \to \infty$ as $k \to \infty$; this is possible since the set $\{x_k : k \in \mathbb{N}\}$ is dense in A. Put

$$N_k = \{k, \ell_k\} \sqcup \{h > k : x_h \in (x_k, x_{\ell_k})\}.$$

Define subdendrites E_k of D by

$$E_1 = D, \quad E_k = [x_k, x_{\ell_k}] \cup \bigsqcup_{h \in N_k} D_h \quad (k \ge 2).$$

Thus, E_k is the union of the arc $[x_k, x_{\ell_k}]$, bushes D_k and D_{ℓ_k}, and all those bushes rooted at the points strictly between x_k and x_{ℓ_k} which are smaller than the bush D_k.

For $k \ge 2$, we have $\mathcal{H}^1_d(E_k \cap A) = d(x_k, x_{\ell_k})$ and $\mathcal{H}^1_d(E_k \setminus A) = \sum_{h \in N_k} \lambda_h \le \lambda_{\ell_k} + \sum_{i \ge k} \lambda_i = \lambda_{\ell_k} + q^k$. Moreover, $\lambda_{\ell_k} \to 0$ because $\ell_k \to \infty$. Therefore, $\mathcal{H}^1_d(E_k) \to 0$, hence

$$\operatorname{diam}_d(E_k) \to 0. \tag{6.3}$$

Step 4. We define surjective maps $\tilde{\varphi}_k : I \to E_k$ ($k \ge 1$).

Fix $k \ge 2$. Denote

$$t_h = \psi_0(x_h) \quad \text{for every } h \in N_k, \quad \text{and} \quad J_k = [t_k, t_{\ell_k}] \subseteq I;$$

we will assume that $t_k < t_{\ell_k}$ (the case $t_k > t_{\ell_k}$ is analogous). Note that the set $\{t_h : h \in N_k\}$ is dense in J_k. We are going to define an auxiliary surjection g_k from a compact real interval J_k^+ (containing J_k) to E_k. The construction starts, roughly speaking, by taking the interval J_k and 'blowing up' every point t_h ($h \in N_k$) into a closed interval T_h of length λ_h. To be more precise, for every $h \in N_k$, put

$$\Lambda_h = \sum_{i \in N_k,\, t_i < t_h} \lambda_i$$

(here $\Lambda_k = 0$); note that every Λ_h is finite. Put $J_k^+ = [t_k, t_{\ell_k} + \Lambda_{\ell_k} + \lambda_{\ell_k}]$ and let $\mu_k : J_k^+ \to J_k$ be the unique non-decreasing (continuous) surjection such that $\mu_k^{-1}(t_h) = T_h = [t_h + \Lambda_h, t_h + \Lambda_h + \lambda_h]$ for every $h \in N_k$.

We are ready to define a continuous surjective map $g_k : J_k^+ \to E_k$ as follows.

- If $s \in T_h$ for some $h \in N_k$, then put $g_k(s) = \varphi_h(s')$, where the position of s' in I is the same as that of s in T_h; thus, $s' = (s - t_h - \Lambda_h)/\lambda_h \in I$. Note that $g_k(s) \in D_h \subseteq E_k$.
- If $s \in J_k^+ \setminus \bigcup_{h \in N_k} T_h$, then put $g_k(s) = \varphi_0(\mu_k(s))$; here we have that $g_k(s) \in [x_k, x_{\ell_k}] \setminus \{x_h : h \in N_k\} \subseteq [x_k, x_{\ell_k}] = E_k \cap A$.

The map $g_k : J_k^+ \to E_k$ has the following properties:

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(1) $g_k(T_h) = D_h$ for every $h \in N_k$, the endpoints of T_h being mapped to x_h by (6.1); in particular, $g_k(t_k) = x_k$;
(2) for $s \in J_k^+$ and $h \in N_k$, $g_k(s) \in D_h$ if and only if $s \in T_h$;
(3) if $h \in N_k$ and L is a non-degenerate closed interval in T_h, then $g_k(L) \in \mathcal{C}_h$ (indeed, $g_k|_{T_h} : T_h \to D_h$ is the composition of a non-constant linear map $T_h \to I$ followed by the map $\varphi_h : I \to D_h$ and so $g_k(L) = \varphi_h(L')$, where L' is a non-degenerate closed interval in I, that is, $L' \in \mathcal{C}_I$, whence $g_k(L) \in \varphi_h(\mathcal{C}_I) = \mathcal{C}_h$);
(4) $g_k(J_k^+ \setminus \bigcup_{h\in N_k} T_h) = [x_k, x_{\ell_k}] \setminus \{x_h : h \in N_k\}$;
(5) g_k is surjective by (1) and (4);
(6) g_k is continuous (to see this, use (1), (4) and the facts that the maps μ_k, φ_0 and φ_h ($h \in N_k$) are continuous, and the families $\{T_h : h \in N_k\}$ and $\{D_h : h \in N_k\}$ are null);
(7) for every closed non-degenerate subinterval K of J_k^+:
- if $g_k(K) \cap A$ is non-degenerate, then $g_k(K) \supseteq D_h$ for some (even for infinitely many) $h \in N_k$;
- if $g_k(K) \cap A$ is degenerate or empty, that is, K is a subinterval of some T_h, then $g_k(K) = D_h$ or $\mathcal{H}^1_d(g_k(K)) \geq \varrho|K|$. To see this, let $K' \subseteq I$ be the interval whose position in I is the same as the position of K in T_h; that is, $K' = (K - t_h - \Lambda_h)/\lambda_h$. Then $|K'| = |K|/\lambda_h$ and $g_k(K) = \varphi_h(K')$. Now, by ϱ-length expansiveness of the map $\varphi_h : I \to D_h$ and the definition of d, $g_k(K) = \varphi_h(K') = D_h$ or

$$\frac{1}{\lambda_h}\mathcal{H}^1_d(g_k(K)) = \mathcal{H}^1_{d_k}(\varphi_h(K')) \geq \varrho|K'| = \frac{\varrho}{\lambda_h}|K|.$$

Still for $k \geq 2$ we define a map $\tilde{\varphi}_k : I \to E_k$ via the composition $\tilde{\varphi}_k = g_k \circ \nu_k$, where $\nu_k : I \to J_k^+$ is a continuous map such that $\nu_k(0) = t_k$ and, for every closed non-degenerate subinterval J of I, $|\nu_k(J)| \geq |J|$. (For instance, ν_k can be a piecewise linear map with constant slope and sufficiently many laps, each of which is mapped onto J_k^+.) So, by (7),

$$\tilde{\varphi}_k(J) \text{ contains some } D_h, \quad \text{or} \quad \mathcal{H}^1_d(\tilde{\varphi}_k(J)) \geq \varrho|J| \quad \text{with } \operatorname{card}(\tilde{\varphi}_k(J) \cap A) \leq 1. \tag{6.4}$$

By (5) and (6), $\tilde{\varphi}_k$ is a continuous surjection and, since $g_k(t_k) = x_k$,

$$\tilde{\varphi}_k(0) = x_k. \tag{6.5}$$

So, we have defined $\tilde{\varphi}_k$ for every $k \geq 2$. For $k = 1$, the map $\tilde{\varphi}_1 : I \to E_1$ is defined analogously, as $\tilde{\varphi}_1 = g_1 \circ \nu_1$. The only differences are:
- in the definition of $g_1 : J_1^+ \to E_1$, we put $J_1 = I$ and $J_1^+ = [0, 1 + \Lambda]$, where $\Lambda = \sum_{h\in\mathbb{N}} \lambda_h = q$; now the analogue of the property $g_k(t_k) = x_k$ is $g_1(0) = x_1$;
- to get $\tilde{\varphi}_1(0) = x_1$, we choose $\nu_1 : I \to J_1^+$ with $\nu_1(0) = 0$.

Then (6.4) and (6.5) work also for $k = 1$.

Step 5. We define a continuous map $f : D \to D$ such that $f(D_1) = D$ and $f(D_k) \supseteq D_{\ell_k}$ for every $k \geq 2$.

Define f by

$$f(x) = \begin{cases} x & \text{if } x \in A; \\ \tilde{\varphi}_k \circ \psi_k(x) & \text{if } x \in D_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

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Since $\tilde{\varphi}_k \circ \psi_k(x_k) = \tilde{\varphi}_k(0) = x_k$ for every $k \geq 1$ by (6.1) and (6.5), the map f is well defined. Further, $f(D_k) = E_k$ for every $k \in \mathbb{N}$; thus,

$$f(D_1) = D \quad \text{and} \quad f(D_k) \supseteq D_{\ell_k} \quad \text{for every } k \geq 2. \tag{6.6}$$

The continuity of f is obvious since it is continuous on A and on each D_k, and the f-images of the bushes D_k form a null family by (6.3).

Step 6. We claim that, for every $k \in \mathbb{N}$ and every $C \in \mathcal{C}_k$ (see (6.2)), at least one of the following holds:

(a) $f(C)$ contains some D_h;
(b) $\mathcal{H}^1_d(f(C)) \geq \varrho^2 \cdot \mathcal{H}^1_d(C)$ and $f(C) \in \mathcal{C}_h$ for some $h \in \mathbb{N}$.

To see this, we use (6.4) and ϱ-length expansiveness of ψ_k. Indeed, fix any $k \in \mathbb{N}$ and $C \in \mathcal{C}_k$, and denote the (non-degenerate) closed interval $\psi_k(C)$ by J. Since $C \subseteq D_k$, we have

$$f(C) = \tilde{\varphi}_k(\psi_k(C)) = \tilde{\varphi}_k(J) = g_k(\nu_k(J)).$$

Assume that $f(C)$ does not satisfy (a), that is, it contains no D_h. Then $J \neq I$ (since otherwise $f(C) = \tilde{\varphi}_k(I) = E_k \supseteq D_k$) and so, by ϱ-length expansiveness of ψ_k, $J = \psi_k(C) \in C_I$ is non-degenerate and $|J| \geq \varrho \cdot \mathcal{H}^1_d(C)$. Further, since we assume that $\tilde{\varphi}_k(J)$ contains no D_h, (6.4) gives that $\mathcal{H}^1_d(\tilde{\varphi}_k(J)) \geq \varrho|J|$ and hence $\mathcal{H}^1_d(f(C)) \geq \varrho^2 \cdot \mathcal{H}^1_d(C)$ and $\operatorname{card}(\tilde{\varphi}_k(J) \cap A) \leq 1$. This cardinality condition gives that $\tilde{\varphi}_k(J) \subseteq D_h$ for some $h \in N_k$. Since $\tilde{\varphi}_k(J) = g_k(\nu_k(J))$, the property (2) of g_k shows that the (non-degenerate) closed interval $\nu_k(J)$ is a subset of T_h. Then, by the property (3) of g_k, we get that $f(C) = g_k(\nu_k(J)) \in \mathcal{C}_h$. So, we have (b).

Step 7. We prove that for every non-empty open set U in D there are $h \geq 1$ and $n \in \mathbb{N}_0$ such that

$$f^n(U) \supseteq D_h.$$

Since A is nowhere dense, we may assume that U is a subset of D_k for some $k \geq 1$. By (6.2), we may fix some $C \in \mathcal{C}_k$ lying in U. Since $\varrho > 1$ and D has finite length, the iterative use of Step 6 yields that $f^n(C) \supseteq D_h$ for some n and h.

Step 8. We finish the proof by showing that f is exact.

In view of Step 7, it is sufficient to prove that for every $h_0 \geq 1$ there is m such that

$$f^m(D_{h_0}) = D.$$

If $h_0 = 1$, then $f(D_1) = D$ by (6.6) and we are done with $m = 1$. Now let $h_0 \geq 2$ and put $h_1 = \ell_{h_0}$ (see the definition of integers ℓ_k in the beginning of Step 3). By (6.6), $f(D_{h_0}) \supseteq D_{h_1}$ and $1 \leq h_1 < h_0$. If $h_1 = 1$, then we are done with $m = 2$. If $h_1 > 1$, we continue by taking $h_2 = \ell_{h_1}$ and so on. Since $h_0 > h_1 > h_2 > \cdots \geq 1$, in a finite number n of steps we obtain that $h_n = 1$. Then $f^{n+1}(D_{h_0}) \supseteq f(D_1) = D$ and we are done with $m = n + 1$. This finishes the proof. □

7. *Proof of Theorem A: (1) implies (2)*

The following two lemmas generalize ideas from [**40**, p. 49].

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LEMMA 7.1. *Let (X, d) be a compact metric space and $f: X \to X$ be a continuous map. Assume that, for some $k \in \mathbb{N}$, there are f-invariant closed subsets $Z_1, \ldots, Z_k$ of X such that:*

(1) $X = Z_1 \cup \cdots \cup Z_k$;
(2) $Z_i \cap Z_j \neq \emptyset$ *for every* i, j;
(3) $f|_{Z_i}: Z_i \to Z_i$ *is strongly mixing for every* i.

Then f is generically ε-chaotic for any $0 < \varepsilon < (1/2) \min_i \operatorname{diam} Z_i$.

Proof. We may assume that X is non-degenerate, otherwise the claim is trivial. If $k = 1$, the lemma follows from the fact, mentioned already in §1, that a weakly mixing map on a (non-degenerate) compact metric space X is generically ε-chaotic for every $0 < \varepsilon < \operatorname{diam} X$. From now on assume that $k \geq 2$.

If B is an open ball in X, we can write $B = (B \cap Z_1) \cup \cdots \cup (B \cap Z_k)$. Since the sets $B \cap Z_i$ are closed in B, there is i such that $B \cap Z_i$ has non-empty interior in B. In other words,

$$\text{every ball contains a ball lying in one of the sets } Z_i. \tag{7.1}$$

To prove the lemma, it is clearly sufficient to show that f is generically ε-chaotic for any $\varepsilon < \eta/2$ whenever $0 < \eta < \min_i \operatorname{diam} Z_i$. Therefore, fix such an η. By Proposition 1.1, it is sufficient to prove that (Prox) and (1.1) in (Sens) are satisfied by the family of all open balls in X.

To prove (Prox), let $B_1, B_2 \subseteq X$ be open balls. To show that $\liminf_{n\to\infty} d(f^n(B_1), f^n(B_2)) = 0$, we may assume, in view of (7.1), that $B_1 \subseteq Z_i$ and $B_2 \subseteq Z_j$ for some i and j. Let $\delta > 0$. By (2), there is $x_0 \in Z_i \cap Z_j$. Since the restrictions of f to Z_i and Z_j are strongly mixing, there is n_0 such that, for every $n \geq n_0$, both $d(x_0, f^n(B_1))$ and $d(x_0, f^n(B_2))$ are smaller than $\delta/2$. Hence, $d(f^n(B_1), f^n(B_2)) < \delta$ for every $n \geq n_0$. We have proved that $\lim_{n\to\infty} d(f^n(B_1), f^n(B_2)) = 0$.

To prove (1.1) in (Sens), choose any open ball $B \subseteq X$. To show that $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > \eta$, again by (7.1) we may assume that $B \subseteq Z_i$ for some i. Put $g = f|_{Z_i}$. Since $\eta < \operatorname{diam} Z_i$, there are non-empty open sets $U, V \subseteq Z_i$ such that $d(U, V) > \eta$. By transitivity of $g \times g$, there is an increasing sequence $\{n_k\}_k$ of positive integers such that every $(g \times g)^{n_k}(B \times B)$ intersects $U \times V$. Hence, $\operatorname{diam} g^{n_k}(B) \geq d(U, V) > \eta$ for every k, which proves that $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > \eta$. □

LEMMA 7.2. *Let X be a compact metric space and $f: X \to X$ be a continuous map. Assume that there is an increasing sequence of f-invariant closed sets X_i ($i \in \mathbb{N}$) such that $X = \bigcup_{i=1}^{\infty} X_i$ and $f|_{X_i}: X_i \to X_i$ is generically chaotic for every i. Then f is generically chaotic.*

Proof. Denote by $\mathrm{nLY}(f)$ the set of pairs $(x, y) \in X^2$ which are not Li–Yorke for f. Since every $f|_{X_i}$ is generically chaotic, the sets $X_i^2 \cap \mathrm{nLY}(f)$ are of the first category in X_i^2 and hence of the first category in X^2. Since $X^2 = \bigcup_{i=1}^{\infty} X_i^2$ due to the fact that $X_i \subseteq X_{i+1}$ for every i, the set $\mathrm{nLY}(f)$ is of the first category in X^2. This shows that f is generically chaotic. □

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One can see that a slightly stronger lemma is true. It is sufficient to assume that $X = \bigcup_{i=1}^{\infty} X_i \cup Y$, where Y is of the first category in X.

By combining the previous two lemmas, we get the following proposition.

PROPOSITION 7.3. *Let X be a compact metric space and $f: X \to X$ be a continuous map. Assume that there are f-invariant closed sets X_i ($i \in \mathbb{N}$) such that:*

(1) $X = \bigcup_{i=1}^{\infty} X_i$;
(2) *every X_i has non-empty interior;*
(3) $\operatorname{diam} X_i \to 0$;
(4) $X_i \cap X_j \neq \emptyset$ *for every* $i \neq j$;
(5) $f|_{X_i}: X_i \to X_i$ *is strongly mixing for every* i.

Then f is generically chaotic but not generically ε-chaotic for any $\varepsilon > 0$.

Proof. Denote $Y_k = \bigcup_{i=1}^{k} X_i$, $k \in \mathbb{N}$. By Lemma 7.1, $f|_{Y_k}: Y_k \to Y_k$ is generically chaotic for every k. Since $X = \bigcup_{k=1}^{\infty} Y_k$, f is generically chaotic by Lemma 7.2. By (2) and (3), there are arbitrarily small invariant sets with non-empty interiors. This implies that (Sens) is not satisfied by the family of open balls in X. Therefore, by Proposition 1.1, f is not generically ε-chaotic for any $\varepsilon > 0$. □

LEMMA 7.4. *Let D be a non-degenerate dendrite and $A \subseteq D$ be either a nowhere-dense non-degenerate subdendrite, or a singleton $\{a\}$ such that a is of infinite order in D. Then there is a generically chaotic map $f: D \to D$ which is not generically ε-chaotic for any $\varepsilon > 0$.*

Proof. We may assume that A is either a nowhere-dense arc or $A = \{a\}$. By the assumptions, in either case the set $D \setminus A$ has infinitely many components C_i ($i \in \mathbb{N}$). By Lemma 2.6, $\operatorname{Bd}(C_i) = \{c_i\} \subseteq A$ and, since A is nowhere dense, the set $\{c_i : i \in \mathbb{N}\}$ is dense in A. Further, the open connected sets C_i form a null family.

Assume first that $A = \{a\}$. By Proposition 6.1, every dendrite $\overline{C_i}$ admits an exact map f_i fixing the point a. The map $f: D \to D$ such that $f(x) = f_i(x)$ if $x \in \overline{C_i}$ is well defined and continuous. By Proposition 7.3, f is generically chaotic but not generically ε-chaotic for any $\varepsilon > 0$.

Now let A be a nowhere-dense arc. Fix $a \in A$; for simplicity, we can choose $a \notin \{c_i : i \in \mathbb{N}\}$. Choose a nested sequence of subarcs A_j of A such that $A_1 = A$ and $\bigcap_{j=1}^{\infty} A_j = \{a\}$. By induction, one can obviously construct a partition $\mathbb{N} = \bigsqcup_{j=1}^{\infty} N_j$ with infinite sets N_j such that the (non-degenerate) dendrites E_j ($j \in \mathbb{N}$) defined by

$$E_j = A_j \sqcup \bigsqcup_{i \in N_j} C_i$$

are such that:

(a) $A_j = E_j \cap A$ is nowhere dense in E_j for every j.

We clearly have the following:

(b) $D = \bigcup_{j=1}^{\infty} E_j$;
(c) $a \in E_j$ for every j;
(d) $E_j \cap E_k = A_k \subseteq A$ for every $j < k$;

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(e) diam $E_j \to 0$ as $j \to \infty$ (because $a \notin \{c_i : i \in \mathbb{N}\}$);
(f) E_j has non-empty interior in D for every j (in fact, every E_j contains some C_i and C_i is non-empty and open in D);
(g) there is a sequence of positive reals $\delta_j \to 0$ such that $E_j \subseteq B_{\delta_j}(a)$ for every j, that is, in the Hausdorff metric the sequence E_j converges to the singleton $\{a\}$ (this follows from (c) and (e)).

Using (a) and Proposition 6.1, for every j there is an exact map

$$f_j : E_j \to E_j \quad \text{such that} \quad f_j(x) = x \quad \text{for every } x \in A_j. \tag{7.2}$$

Define $f : D \to D$ by $f(x) = f_j(x)$ provided $x \in E_j$ ($j \in \mathbb{N}$). This map is well defined by (b), (d) and (7.2). We prove that f is continuous. If $x \neq a$, then, by (g), there is an open neighbourhood of x covered by finitely many sets E_j; thus, f is continuous at x by the pasting lemma. To prove continuity at the point a, fix a neighbourhood U of a. By (g), there exists N such that $\bigcup_{j>N} E_j \subseteq U$. Since $f_1, \ldots, f_N$ are continuous at a and a is their common fixed point, there is a neighbourhood $V \subseteq U$ of a such that $f_j(V) \subseteq U$ for every $j \leq N$. Then $f_j(V) \subseteq U$ for all j and so $f(V) \subseteq U$.

By Proposition 7.3, f is generically chaotic but not generically ε-chaotic for any $\varepsilon > 0$. □

Proof of Theorem A: (1) ⇒ (2). Assume that (2) is not true, that is, D contains a nowhere-dense non-degenerate subdendrite or D contains a point of infinite order. In either case, Lemma 7.4 shows that D admits a generically chaotic selfmap which is not generically ε-chaotic for any $\varepsilon > 0$. Thus, (1) from Theorem A is not satisfied. □

We add the following simple observation used at the end of §1.

PROPOSITION 7.5. *A Cantor set admits a generically chaotic map which is not generically ε-chaotic for any $\varepsilon > 0$.*

Proof. Let D be an ω-star with branch point z. Let A_i ($i \in \mathbb{N}$) be the closures of the components of $D \setminus \{z\}$. In each A_i, choose a Cantor set Z_i containing z. Then $X = \bigcup_{i=1}^{\infty} Z_i$ is a Cantor set. Every Z_i admits a strongly mixing map $f_i : Z_i \to Z_i$ with $f_i(z) = z$. Then $f : X \to X$ defined by $f(x) = f_i(x)$ for every $i \in \mathbb{N}$ and $x \in Z_i$ is continuous. By Proposition 7.3, f is generically chaotic but not generically ε-chaotic for any $\varepsilon > 0$. □

Finally, here we present the example, suggested by an anonymous referee and mentioned in §1, of a generically chaotic map which is not sensitive.

Example 7.6. Cf. [**28**, Corollary 4.2] Take any topologically transitive, non-minimal homeomorphism h on a compact metric space X which is almost equicontinuous (for a construction of such a homeomorphism, see e.g. [**2**, Theorem 4.2]). Then the transitive points are equicontinuity points [**2**, Theorem 2.4], h is uniformly rigid [**2**, Corollary 3.7] and the union M of all minimal sets is not dense [**2**, Theorem 2.5]. The closure $\overline{M}$ is an invariant nowhere-dense set (if $\overline{M}$ has non-empty interior, then it contains a transitive

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point and so $\overline{M} = X$, contradicting the fact that M is not dense). By collapsing $\overline{M}$ to a point, we obtain a system $(\tilde{X}, \tilde{h})$, where $\tilde{h}$ is a homeomorphism of a compact metric space $\tilde{X}$. This new system has the following properties. First, since all the equicontinuity points (that is, the transitive points) of h are outside the invariant nowhere-dense closed set $\overline{M}$, it is straightforward to show that also $\tilde{h}$ is almost equicontinuous (alternatively, one can use [22, Lemma 1.6]). Further, $\tilde{h}$ has a fixed point which is the unique minimal set and so the new system is proximal [3, Proposition 2.2]. Finally, $\tilde{h}$ is uniformly rigid [2, Corollary 3.7] and so every pair of points is recurrent. That is, the whole space $\tilde{X}$ is strongly scrambled (meaning that every pair of points is proximal and recurrent) and so the system $(\tilde{X}, \tilde{h})$ is obviously generically chaotic. Being almost equicontinuous, it is not sensitive.

A. *Appendix: Li–Yorke chaotic dendrite map which is not Li–Yorke ε-chaotic*

The purpose of this appendix is to show that an analogue of Theorem A does not hold with generic chaos replaced by Li–Yorke chaos. We start with a construction of a system which resembles that of Floyd and Auslander [5, 18]; see also [23]. However, our pattern for producing a family of subrectangles from a given rectangle is very different. Therefore, the obtained homeomorphism will not be minimal (even not transitive) and in fact will have an appropriate invariant Cantor set.

Consider the alphabet $\mathcal{A} = \{0, 1, 2\}$, the set $\mathcal{A}^* = \bigcup_{n\in\mathbb{N}_0} \mathcal{A}^n$ of finite words (here $\mathcal{A}^0$ contains just the empty word $\emptyset$) and the set $\Sigma = \mathcal{A}^{\mathbb{N}_0}$ of infinite words over the alphabet $\mathcal{A}$. The elements $\alpha \in \Sigma$ are infinite sequences, $\alpha = \alpha_0\alpha_1 \ldots$ (α_i being the ith coordinate of α). Analogously, the finite words are written in the form of finite sequences; in the usual way we can concatenate them. The set Σ together with addition mod3 with carry from the left to the right is the 3-adic group. Below we abbreviate $\alpha + 10^\infty$ to $\alpha + 1$; then $\alpha + n$ has the usual meaning for every $n \in \mathbb{Z}$.

For a rectangle $K = [a, b] \times [c, d]$, let K_i ($i \in \mathcal{A}$) be the subrectangles of K given by

$$K_i = \left[a + \frac{2i(b-a)}{5}, a + \frac{(2i+1)(b-a)}{5}\right] \times [c, c + \theta_i(d-c)],$$

where $\theta_i = 1$ if $i = 1$ and $\theta_i = 1/3$ otherwise. Starting with $K_\emptyset = [0, 1]^2$ and applying this pattern inductively, we obtain rectangles K_a for every $a \in \mathcal{A}^*$; for every $a \in \mathcal{A}^*$ and $i \in \mathcal{A}$, we put $K_{ai} = (K_a)_i$ (recall that $\emptyset i = i$). For every $n \in \mathbb{N}_0$, define $X_n = \bigcup_{a\in\mathcal{A}^n} K_a$, see Figures 6 and 7, and $X = \bigcap_{n\in\mathbb{N}_0} X_n$. One can see that

$$X = \bigcup_{\alpha\in\Sigma} K_\alpha, \quad \text{where } K_\alpha = \{x_\alpha\} \times J_\alpha,$$

with $x_\alpha = \sum_{i=0}^{\infty}(2\alpha_i)/5^{i+1}$ and

$$J_\alpha = [0, 3^{-\ell_\alpha}], \quad \ell_\alpha = \operatorname{card}\{i \in \mathbb{N}_0 \colon \alpha_i \in \{0, 2\}\}$$

(we adopt the convention $3^{-\aleph_0} = 0$). The sets K_α will be called *(vertical) fibres* of X; K_α is the fibre above x_α.

Define a map $H \colon X \to X$ by

$$H(x_\alpha, y) = (x_{\alpha+1}, h_\alpha(y)) \quad (\alpha \in \Sigma,\ y \in J_\alpha),$$

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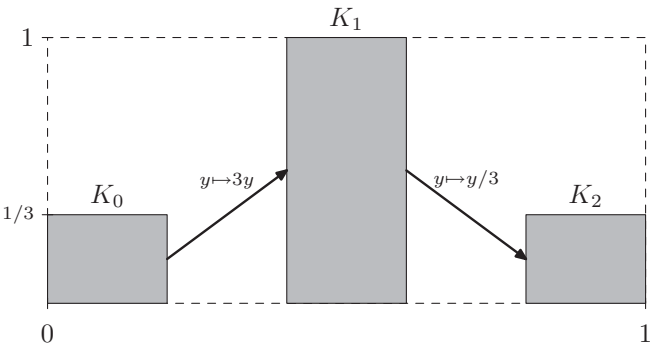


FIGURE 6. The set $X_1 = K_0 \cup K_1 \cup K_2$.

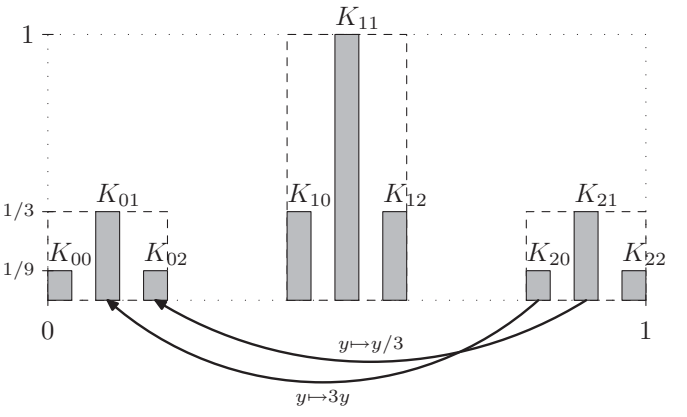


FIGURE 7. The set $X_2 = K_{00} \cup K_{01} \cup \cdots \cup K_{22}$.

where

$$h_\alpha \colon J_\alpha \to J_{\alpha+1}, \quad h_\alpha(y) = 3^{\ell_\alpha - \ell_{\alpha+1}} y, \tag{A.1}$$

is an increasing linear surjection. Figures 6 and 7 show the form of the fibre maps h_α except on the fibres lying in the rightmost rectangle.

Basic properties of X and H are summarized in the following lemma. We omit a proof, since it is straightforward and analogous to the proof of the corresponding properties of the Floyd–Auslander system (the property (5) can be proved similarly as [**15**, Proposition 2(b)]).

LEMMA A.1. *Let X and H be defined as above. Then the following hold.*

(1) *X is a compact subset of the unit square. The projection of X onto the first coordinate is a Cantor set $C_1 = \{x_\alpha : \alpha \in \Sigma\}$ and the connected components of X are the fibres K_α ($\alpha \in \Sigma$).*

(2) *Every degenerate fibre is a singleton with second coordinate zero. The fibre K_α is non-degenerate if and only if $\alpha = 1^\infty + m$ for some $m \in \mathbb{Z}$ (that is, if only finitely many coordinates of α are different from 1). Thus, the non-degenerate fibres are the*

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fibres above the full orbit (under the homeomorphism $x_\alpha \mapsto x_{\alpha+1}$) of the point x_α, $\alpha = 1^\infty$.

(3) *The map H is a homeomorphism and $H^n(K_\alpha) = K_{\alpha+n}$ for every $\alpha \in \Sigma$ and $n \in \mathbb{Z}$.*

(4) *For every non-degenerate fibre K_α, we have*

$$\liminf_{n\to\infty} \operatorname{diam} H^n(K_\alpha) = 0, \quad \limsup_{n\to\infty} \operatorname{diam} H^n(K_\alpha) = \tfrac{1}{3}.$$

(5) *Maximal scrambled sets of H are the non-degenerate vertical fibres K_α.*

(6) *For every $\varepsilon > 0$, there is $n_\varepsilon > 0$ such that every ε-scrambled set of H has cardinality at most n_ε.*

Now let C_2 denote the Cantor ternary set. Then, by (A.1), the (closed) set $Y = X \cap (C_1 \times C_2)$ is strongly H-invariant, that is, $H(Y) = Y$. Obviously, Y is a Cantor set and $K_\alpha \cap Y$ is a Cantor set for every non-degenerate fibre K_α. Then Lemma A.1 immediately yields the following corollary.

COROLLARY A.2. *There is a Cantor homeomorphism which is Li–Yorke chaotic but not Li–Yorke ε-chaotic for any $\varepsilon > 0$.*

Let (Y, G) be the system from Corollary A.2 and let D be the Gehman dendrite [**20**, p. 42]. (Recall that the *Gehman dendrite* is a topologically unique dendrite D all of whose branch points are of order 3 and the set of endpoints $\operatorname{End}(D)$ is a Cantor set.) We may assume that $\operatorname{End}(D) = Y$. By [**49**, Theorem B], there is a continuous map $F\colon D \to D$ such that the restriction of F onto $\operatorname{End}(D) = Y$ is G, and every point of $D \setminus \operatorname{End}(D)$ is eventually mapped to one fixed point which also lies in $D \setminus \operatorname{End}(D)$. This clearly implies that the scrambled sets of F coincide with the scrambled sets of G. Hence, (D, F) is Li–Yorke chaotic but not Li–Yorke ε-chaotic for any $\varepsilon > 0$. Thus, the Gehman dendrite is a (completely regular) dendrite (with all points of finite order) on which Li–Yorke chaos and Li–Yorke ε-chaos are not equivalent. However, more can be said.

Since every dendrite with uncountably many endpoints contains a copy of the Gehman dendrite [**4**, Proposition 6.8], and dendrites are absolute retracts, we immediately get the following fact.

PROPOSITION A.3. *Every dendrite with uncountably many endpoints admits a Li–Yorke chaotic map which is not Li–Yorke ε-chaotic for any $\varepsilon > 0$.*

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