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ON THE DESCENT CONJECTURE FOR RATIONAL POINTS AND ZERO-CYCLES

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Abstract The descent method is one of the approaches to study the Brauer–Manin obstruction to the local–global principle and to weak approximation on varieties over number fields, by reducing the problem to 'descent varieties'. In recent lecture notes by Wittenberg, he formulated a 'descent conjecture' for torsors under linear algebraic groups. The present article gives a proof of this conjecture in the case of connected groups, generalizing the toric case from the previous work of Harpaz–Wittenberg. As an application, we deduce directly from Sansuc's work the theorem of Borovoi for homogeneous spaces of connected linear algebraic groups with connected stabilizers. We are also able to reduce the general case to the case of finite (étale) torsors. When the set of rational points is replaced by the Chow group of zero-cycles, an analogue of the above conjecture for arbitrary linear algebraic groups is proved.

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1. Introduction

1.1. Brauer–Manin obstruction and the descent conjecture

Let X be a smooth variety defined over a number field k. We say that X satisfies the *Hasse principle* (or the *local–global principle*) if either $X(k_v) = \varnothing$ for some completion k_v of k or $X(k) \neq \varnothing$. It was Manin who introduced a general method to study the failure of the Hasse principle on X, as follows. Assuming X geometrically integral, we associate to X the cohomological Brauer group $\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ (which generalizes the Brauer group $\mathrm{Br}(k) = \mathrm{H}^2(k, \bar{k}^\times)$ of the field k), and consider the unramified subgroup $\mathrm{Br}_{\mathrm{nr}}(X) \subseteq \mathrm{Br}(X)$ (see Subsection 1.3 below). One then defines the *Brauer–Manin pairing* [Man71]

$$\langle -, - \rangle_{\mathrm{BM}} : X(k_\Omega) \times \mathrm{Br}_{\mathrm{nr}}(X) \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_v, \alpha) \mapsto \sum_v \alpha(x_v), \tag{$\mathrm{BM}_{\mathrm{nr}}$}$$

where $X(k_\Omega)$ is the product of all the $X(k_v)$'s, and where $\mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ *via* the invariant map from local class field theory. By the global reciprocity law (Albert–Brauer–Hasse–Noether), the pairing ($\mathrm{BM}_{\mathrm{nr}}$) vanishes on the image of $\mathrm{Br}(k)$ in $\mathrm{Br}_{\mathrm{nr}}(X)$ and on the diagonal image of $X(k)$ in $X(k_\Omega)$. If we denote by $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ the subset of families of local points of X orthogonal to $\mathrm{Br}_{\mathrm{nr}}(X)$, then its vacuity is an obstruction to the existence of k-points on X, the so-called *Manin obstruction*. If either $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \varnothing$ or $X(k) \neq \varnothing$ (that is, the Manin obstruction to the Hasse principle for X is the only one), we shall say that X *satisfies the property* (**BMHP**).

We are also interested in the question of weak approximation, namely, the density of $X(k)$ in $X(k_\Omega)$ (equipped with the product of v-adic topologies). By continuity of ($\mathrm{BM}_{\mathrm{nr}}$), the closure $\overline{X(k)}$ is contained in the (closed) set $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$. Thus, a family of local points on X not orthogonal to $\mathrm{Br}_{\mathrm{nr}}(X)$ cannot be approximated by k-rational points. This is known as the *Brauer–Manin obstruction* to weak approximation. If $\overline{X(k)} = X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ (that is, the Brauer–Manin obstruction to weak approximation for X is the only one), then we say that X *satisfies the property* (**BMWA**). Note that we included here the case where both sides are empty (hence, the property (**BMWA**) implies the property (**BMHP**)). The properties (**BMHP**) and (**BMWA**) are stable birational invariants, thanks to the stable birational invariance of the unramified Brauer group, the theorem of Nishimura–Lang [Nis55] and the Artin–Whaples approximation lemma.

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Definition 1.1 (Campana–Kollár–Miyaoka–Mori). A smooth variety X over a field k is said to be *rationally connected* if, for every algebraically closed field K containing k, two general points $x_0, x_1 \in X(K)$ can be connected by a K-rational map $\mathbb{A}^1_K \dashrightarrow X \times_k K$.

Rational connectedness is a stable birational invariant. Examples of rationally connected varieties include (geometrically) unirational varieties. For example, in characteristic 0, this is the case for homogeneous spaces of connected linear algebraic groups (indeed, the celebrated theorem of Chevalley asserts that such groups are geometrically rational, and even unirational over their fields of definition [Che54]). An important conjecture concerning arithmetics of rationally connected varieties over number fields is the following.

Conjecture A (Colliot-Thélène). *Every smooth rationally connected variety over a number field satisfies the properties* (**BMHP**) *and* (**BMWA**).

One of the reasons why Conjecture A is so strong is that its truth would imply that any finite group is a Galois group over k. This is the inverse Galois problem over number fields, long known to be open. The reader is invited to take a look at the recent notes [Wit24] by Wittenberg. One of the techniques to attack Conjecture A is the *descent theory*. Namely, let $f : Y \to X$ be a *torsor* under a linear algebraic group G defined over a number field k, that is, f is an fppf morphism between smooth k-varieties, where Y equipped with a right action $Y \times_k G \to Y$ of G such that f is G-equivariant and $G(\bar{k})$ acts simply transitively on the fibre $f^{-1}(x)$ over each geometric point $x \in X(\bar{k})$. For each Galois cocycle $\sigma : \mathrm{Gal}(\bar{k}/k) \to G(\bar{k})$, we have a *twisted torsor* $f^\sigma : Y^\sigma \to X$ under an inner k-form G^σ of G. Then, we have the following partition of $X(k)$ (see *e.g.* [Sko01, p. 22]):

$$X(k) = \bigsqcup_{[\sigma] \in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k)).$$

The idea of the descent theory is to establish an equality of the form

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{[\sigma] \in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)})}$$

in the topological space $X(k_\Omega)$. In particular, if the 'descent varieties' Y^σ satisfy (**BMHP**) (resp. (**BMWA**)), then it is also the case for X. In the very same notes, Wittenberg formulated the following conjecture, which we shall call the 'descent conjecture' [Wit24, Conjecture 3.7.4].

Conjecture B. *With the above notations and assumptions, suppose in addition that Y (hence also X) is rationally connected. If the varieties Y^σ satisfy the property* (**BMHP**) *(resp.* (**BMWA**)*), then X itself satisfies the property* (**BMHP**) *(resp.* (**BMWA**)*).*

Descent theory was originally developed in the foundational work [CTS87] of Colliot-Thélène and Sansuc for torsors under a torus T, over a variety X with $\bar{k}[X]^\times = \bar{k}^\times$. From its birth, this approach has given many nice results on the Brauer–Manin obstruction for (geometrically) rational varieties, notably for Châtelet surfaces as in the works [CTSSD87a, CTSSD87b] by Colliot-Thélène, Sansuc and Swinnerton-Dyer. As noted by Skorobogatov (see [Sko01, §6.1]), the torus T can be replaced by a group of multiplicative

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type S. The torsors under consideration (*i.e.*, the twists of a given torsor) are the torsors of the same 'type'. The notion of extended type, introduced later by Harari and Skorobogatov [HS13], enabled descent theory (under a group of multiplicative type) for varieties with nonconstant invertible functions. Other generalizations in this direction are found in the various works of Stoll [Sto06], Demarche [Dem09], Skorobogatov [Sko09], Poonen [Poo10], Wei [Wei16] and Cao–Demarche–Xu [CDX19]…, where there is notably the notion of *étale Brauer–Manin obstruction* for torsors under a finite (étale) group scheme.

Conjecture B was established by Harpaz and Wittenberg for torsors under a torus [HW20] or a finite supersolvable group (scheme) [HW24]. In the present article, we prove this conjecture for torsors under an arbitrary connected linear algebraic group. More precisely, we shall prove

Theorem A (Theorem 3.11). *Let k be a number field, X a smooth rationally connected k-variety, and G a connected linear algebraic k-group. Let $Y \to X$ be a torsor under G, then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)})}$$

in the topological space $X(k_\Omega)$. In particular, if the varieties Y^σ satisfy (**BMHP**) *(resp.* (**BMWA**)*) for all $[\sigma] \in \mathrm{H}^1(k,G)$, then it is also the case for X.*

The proof of Theorem A relies on the technique of Harpaz–Wittenberg in [HW20] and Borovoi's general machinery of *abelianization of nonabelian cohomology* [Bor98]. This latter allows one to pass from the nonabelian Galois cohomology set $\mathrm{H}^1(k,G)$ to a hypercohomology group of a 2-term complex of tori. We shall discuss this in Section 2 before proving Theorem A in Section 3. We also prove the following result, which reduces Conjecture B to the case of finite (étale) torsors.

Theorem B (Theorem 3.15). *Let k be a number field, X a smooth geometrically integral k-variety, and G a linear algebraic k-group. Suppose that $f : Y \to X$ is a torsor under G, with Y rationally connected (the existence of such a torsor requires in particular that X is rationally connected). Let G° denote the identity component of G and $Z := Y/G^\circ$, which gives a torsor $g : Z \to X$ under the finite k-group G/G°. Then*

$$\overline{\bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)})} = \overline{\bigcup_{[\tau]\in \mathrm{H}^1(k,G/G^\circ)} g^\tau(Z^\tau(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z^\tau)})}$$

in $X(k_\Omega)$. In particular, Conjecture B holds for the torsor $Y \to X$ if it holds for $Z \to X$.

What we shall show is in fact slightly stronger. Theorems A and B hold true even when we do not have a torsor *a priori*, but only a *descent type* (see Theorems 3.12 and 3.14). This is a definition introduced in the recent work [Lin25], which unifies the various notions

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of type by Colliot-Thélène–Sansuc, Harari–Skorobogatov, and Harpaz–Wittenberg. This new notion enables descent theory in the case where $X(k)$ is not necessarily nonempty (let us say, when X is a homogeneous space of a linear algebraic group), which is useful for the study of the property (**BMHP**).

As an application, we give a new proof for Borovoi's theorem, which states that the property (**BMHP**) and (**BMWA**) hold for homogeneous spaces of connected linear algebraic groups with connected geometric stabilizers (*cf.* Corollary 3.13), using the theorem of Sansuc for *principal* homogeneous spaces.

1.2. Analogue for zero-cycles

In Section 4, we consider the Brauer–Manin obstruction to the Hasse principle and to weak approximation for *zero-cycles*. Let X be a smooth *proper* geometrically integral variety over a number field k. Let $\mathrm{Z}_0(X)$ be the free abelian groups of zero-cycles on X, whose elements are (finite) formal sum $x = \sum_P n_P[P]$, where P runs through the closed points of X, and $n_P \in \mathbb{Z}$. The *degree* of such a cocycle is $\deg_k(x) := \sum_P n_P[k(P) : k]$. For any $\alpha \in \mathrm{Br}(X)$ and place v of k, we may evaluate α at the closed points $P_v \in X_{k_v}$ to obtain $\alpha(P_v) \in \mathrm{Br}(k_v(P_v)) \hookrightarrow \mathbb{Q}/\mathbb{Z}$. By linearity, we obtain a local pairing

$$\langle -, - \rangle_v : \mathrm{Z}_0(X_{k_v}) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}.$$

Let $\mathrm{CH}_0(X)$ denote the quotient of $\mathrm{Z}_0(X)$ by rational equivalence (see *e.g.* [Ful84, Chapter 1]). Since X is proper, the above pairing factors through a pairing $\mathrm{CH}_0(X_{k_v}) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$. If v is an Archimedean place, this latter also factors through $\mathrm{CH}_0'(X_{k_v}) := \mathrm{Coker}(\mathrm{N}_{\bar{k}_v/k_v})$, where $\mathrm{N}_{\bar{k}_v/k_v} : \mathrm{CH}_0(X_{\bar{k}_v}) \to \mathrm{CH}_0(X_{k_v})$ denotes the norm (or corestriction) map. By reassembling these local pairings, we obtain a Brauer–Manin type pairing

$$\langle -, - \rangle_{\mathrm{BM}} : \mathrm{CH}_{0,\Omega}(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

where $\mathrm{CH}_{0,\Omega}(X) := \prod_{v\nmid\infty} \mathrm{CH}_0(X_v) \times \prod_{v|\infty} \mathrm{CH}_0'(X_v)$. The pairing $\langle -, - \rangle_{\mathrm{BM}}$ vanishes on the image of the localization (or restriction) map $\mathrm{CH}_0(X) \to \mathrm{CH}_{0,\Omega}(X)$. Hence, we obtain a complex

$$\mathrm{CH}_0(X) \to \mathrm{CH}_{0,\Omega}(X) \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Since the group $\mathrm{Br}(X)$ is torsion, we have a complex

$$\varprojlim_n (\mathrm{CH}_0(X)/n) \to \varprojlim_n (\mathrm{CH}_{0,\Omega}(X)/n) \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{E}$$

Conjecture C (Conjecture (E)). *The complex* (E) *is exact for any smooth proper geometrically integral variety X over a number field k.*

The above conjecture is due to Colliot-Thélène, Sansuc, Kato and Saito [CTS81, KS86, CT95], and has been reformulated into this form by van Hamel [vH03] and Wittenberg [Wit12]. It entails two previous conjectures, namely Conjecture (E_0) and Conjecture (E_1).

Conjecture D (Conjecture (E_1)). *Let X be a smooth proper geometrically integral variety over a number field k. If there exists a family of local zero-cycles of degree* 1 *on X that*

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is orthogonal to $\mathrm{Br}(X)$ *relative to the pairing* $\langle -,-\rangle_{\mathrm{BM}}$*, then there exists a zero-cycle of degree* 1 *on* X*.*

The property in the statement of Conjecture D can be seen as an analogue of (**BMHP**).

Conjecture E (Conjecture (E_0))*.* *Let* X *be a smooth proper geometrically integral variety over a number field* k*. Let* $A_0(X) \subseteq \mathrm{CH}_0(X)$ *be the subgroup of rationally equivalence classes of zero-cycles of degree* 0*. Then, we have an exact sequence*

$$\varprojlim_n (A_0(X)/n) \to \prod_v \left(\varprojlim_n (A_0(X_{k_v})/n) \right) \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{E_0}$$

As was pointed out by Wittenberg in [Wit12, Remarques 1.1 (ii)], we have

$$\varprojlim_n (A_0(X_{k_v})/n) = \mathrm{Coker}(\mathrm{N}_{\bar{k}_v/k_v} : A_0(X_{k_v}) \to A_0(X_{\bar{k}_v}))$$

for Archimedean v. Thus, exactness of (E) does indeed imply that of (E_0). Liang has shown that the exactness of (E_0) and the existence of a global zero-cycle of degree 1 imply some sort Brauer–Manin obstruction to weak approximation for zero-cycles [Lia13, Proposition 2.2.1] (as soon as a suitable topology on the groups $\mathrm{Z}_0(X_{k_v})$ is defined, see Subsection 4.1 below). Thus, the property in Conjecture E is an analogue of (**BMWA**) (assuming the existence of a k-rational point).

In the recent work [BB24], Balestrieri and Berg introduced a formalism for the descent theory for zero-cycles. This allows us to formulate an analogue of Conjecture B in this context. It turns out that, under the rationally connected assumption, this version of the descent conjecture is true for torsors under an arbitrary linear algebraic group.

Theorem C (Theorem 4.1)*.* *Let* k *be a number field,* X *a smooth geometrically integral* k*-variety, and* G *a linear algebraic* k*-group. Suppose that* $Y \to X$ *is a torsor under* G*, with* Y *rationally connected (in particular,* X *is rationally connected). If Conjecture C holds for the smooth compactifications of the twisted torsors* $Y_K^\sigma \to X_K$*, for all finite extensions* K/k *and all* $[\sigma] \in \mathrm{H}^1(K,G)$*, then it also holds for the smooth compactifications of* X*.*

The proof of Theorem C combines Liang's method [Lia13], the fibration theorems of Harpaz and Wittenberg [HW16] and Theorem A on 'connected descent'.

1.3. Notations and conventions

The following notations shall be deployed throughout the article.

Abelian groups. When A is a topological abelian group which is either discrete torsion, profinite, or of finite exponent, we denote $A^D := \mathrm{Hom}_{\mathrm{cts}}(A, \mathbb{Q}/\mathbb{Z})$ its Pontrjagin dual.

Fields. When k is a field, we use $\bar{k}$ to denote a fixed *separable* closure of k and $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ to denote its absolute Galois group. By convention, a *k-variety* is a separated k-scheme of finite type. We usually denote $\bar{X} := X \times_k \bar{k}$, but $\bar{X}$ does not always mean the base change to $\bar{k}$ of some k-variety. A *smooth compactification* of X is a smooth proper variety containing X as a dense open subset. In characteristic 0, such a compactification (assuming X smooth) always exists by resolution of singularities [Hir64].

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A *local field* is, by definition, the field $\mathbb{R}$, $\mathbb{C}$ or a field k which is complete with respect to a discrete valuation, with finite residue field. If k is a number field, Ω_k (resp. Ω_k^{f}, resp. Ω_k^{∞}) denotes the set of places (resp. finite places, resp. Archimedean places) of k. For $v \in \Omega_k$, k_v denotes the v-adic completion of k, and $\mathcal{O}_v$ denotes the ring of integers of k_v if $v \in \Omega_k^{\mathrm{f}}$. If X is a k-variety, let $X_v := X \times_k k_v$, and write $\mathrm{loc}_v : \mathrm{H}^i(X,-) \to \mathrm{H}^i(X_v,-)$ for the localization maps in cohomology. Equip the set $X(k_v)$ with the natural topology induced by the topology on k_v (see *e.g.* [Con12, Proposition 3.1]), and the set $X(k_\Omega) := \prod_{v\in\Omega_k} X(k_v)$ with the product topology. On the subset $X(\mathbb{A}_k) \subseteq X(k_\Omega)$ of *adelic points* of X, we always consider the topology induced by the product topology on $X(k_\Omega)$ (there is also the *adelic* topology, related to strong approximation, but this does not concern us.).

Cohomology. Unless stated otherwise, all (hyper-)cohomology groups will be étale or Galois. If X is a scheme, $\mathcal{D}^+(X)$ denotes the derived category of bounded below complexes of sheaves of abelian groups on $X_{\text{ét}}$. If k is a field, $\mathcal{D}^+(k) := \mathcal{D}^+(\mathrm{Spec}(k))$ is equivalent to the bounded below derived category of Γ_k-modules.

Let k be a number field, and let C be a bounded below complex of either Γ_k-modules which are finitely generated (as abelian groups) or commutative algebraic k-groups. Let $\mathcal{C}$ be a complex of commutative group schemes over $\mathrm{Spec}(\mathcal{O}_{k,\Sigma})$ extending C, where $\Sigma \subseteq \Omega_k$ is some finite set containing Ω_k^{∞}. For $i \geq 0$, the *restricted product* $\mathbb{P}^i(k,C)$ is the subset of $\prod_{v\in\Omega_k} \mathrm{H}^i(k_v,C)$ consisting of families $(c_v)_{v\in\Omega_k}$ for which c_v comes from $\mathrm{H}^i(\mathcal{O}_v,\mathcal{C})$ for all but finitely many $v \notin \Sigma$. We define the *ith Tate–Shafarevich group* $Ш^i(k,C)$ to be the kernel of the localization $\mathrm{H}^i(k,C) \to \mathbb{P}^i(k,C)$.

Nonabelian cohomology. Let X be a scheme and let $G \to X$ be a *smooth* (not necessarily commutative) X-group scheme. A (right) *X-torsor under G* is, by definition, an fppf X-scheme Y equipped with a right X-action $Y \times_X G \to Y$ of G, such that the morphism

$$Y \times_X G \to Y \times_X Y, \quad (y,g) \mapsto (y, y\cdot g)$$

is an isomorphism. When G is affine, these torsors are classified by the nonabelian Čech cohomology (pointed) set $\mathrm{H}^1(X,G)$ (see [Mil80, Chapter III, §4] or [Sko01, p. 18]). The class in $\mathrm{H}^1(X,G)$ of such an X-torsor Y under G is denoted by $[Y]$.

Twisting. Suppose that G is a smooth algebraic group over a perfect field k. If $\sigma \in \mathrm{Z}^1(k,G)$ is a Galois cocycle (that is, a continuous map $\sigma : \Gamma_k \to G(\bar{k})$ satisfying $\sigma_{st} = \sigma_s {}^s\sigma_t$ for all $s,t \in \Gamma_k$), we may *twist* G by it to obtain an inner k-form G^σ, which can be defined by Galois descent as follows. As a $\bar{k}$-group, $\bar{G}^\sigma := \bar{G}$, but the twisted Galois action on $G^\sigma(\bar{k}) = G(\bar{k})$ is given by

$$\Gamma_k \times G^\sigma(\bar{k}) \to G^\sigma(\bar{k}), \quad (s,g) \mapsto \sigma_s {}^s g \sigma_s^{-1}.$$

We may also twist the *k-variety* G to obtain a k-form G_σ, whose Galois action is given by

$$\Gamma_k \times G_\sigma(\bar{k}) \to G_\sigma(\bar{k}), \quad (s,g) \mapsto {}^s g \sigma_s^{-1}.$$

Then, one has $G_\sigma(k) \neq \varnothing$ if and only if $[\sigma] = 1$, the distinguished element of $\mathrm{H}^1(k,G)$. Equipped with the obvious translation actions (which are *a priori* defined over $\bar{k}$), the variety G_σ is a *left* torsor under G and a right torsor under G^σ. When X is a k-variety

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and $f : Y \to X$ is a torsor under G (more precisely, under $G_X := G \times_k X$), we define the twisted torsor $f^\sigma : Y^\sigma \to X$ to be the contracted product $Y \times_k^G G_\sigma$, *i.e.*, the quotient of $Y \times_k G_\sigma$ by the left action $g \cdot (y,h) := (y \cdot g^{-1}, gh)$ of G. This is a right torsor under G^σ. Thus, we obtain a twisting map

$$\theta_\sigma : \mathrm{H}^1(X,G) \to \mathrm{H}^1(X,G^\sigma), \quad [Y] \mapsto [Y^\sigma].$$

For $X = \mathrm{Spec}(k)$, the map θ_σ takes $[\sigma]$ to 1 and 1 to $[G_\sigma]$. In general, we may 'evaluate' Y at any point $x \in X(k)$ to obtain a class $[Y](x) := [Y_x] = [f^{-1}(x)] \in \mathrm{H}^1(k,G)$. Then, we have $x \in f^\sigma(Y^\sigma(k))$ if and only if $[Y^\sigma](x) = 1$ in $\mathrm{H}^1(k,G^\sigma)$, or $[Y](x) = [\sigma]$ in $\mathrm{H}^1(k,G)$. If $\tau \in \mathrm{Z}^1(k,G)$ is another cocycle, then $G^\tau = (G^\sigma)^{\theta_\sigma(\tau)}$, $Y^\tau = (Y^\sigma)^{\theta_\sigma(\tau)}$ and $f^\tau = (f^\sigma)^{\theta_\sigma(\tau)}$. If G is commutative, then $G^\sigma = G$, $[G_\sigma] = -[\sigma]$ in $\mathrm{H}^1(k,G)$ and $[Y^\sigma] = [Y] - p^*[\sigma]$ in $\mathrm{H}^1(X,G)$, where $p : X \to \mathrm{Spec}(k)$ is the structure morphism.

Finally, if k is a number field, we define the pointed set $\mathbb{P}^1(k,G) \subseteq \prod_{v \in \Omega_k} \mathrm{H}^1(k_v,G)$ as a restricted product in a similar way as in the commutative case. We also put

$$Ш(G) := \mathrm{Ker}(\mathrm{H}^1(k,G) \to \mathbb{P}^1(k,G)).$$

This means, for any k-torsor P under G, one has $[P] \in Ш(G)$ if and only if $P(\mathbb{A}_k) \neq \varnothing$.

Cartier duality. Let k be a field. If G is an algebraic k-group, we denote by $\widehat{G} := \mathscr{H}om_k(G,\mathbb{G}_m)$ its Galois module of geometric characters, that is, $\widehat{G} = \mathrm{Hom}_{\bar{k}}(\bar{G},\mathbb{G}_{m,\bar{k}})$ as a (finitely generated) abelian group, equipped with the action of Γ_k defined by $({}^s\chi)(g) := {}^s\chi({}^{s^{-1}}g)$ for all $s \in \Gamma_k, \chi \in \widehat{G}$ and $g \in \bar{G}$. A *k-group of multiplicative type* is a k-form of $\mathbb{G}_m^n \times_k \prod_{i=1}^m \mu_{n_i}$, where $n \geq 0$ and $n_i \geq 1$. A *k-torus* is a connected k-group of multiplicative type (*i.e.*, a k-form of $\mathbb{G}_m^n$). The formation $G \mapsto \widehat{G}$ is an additive exact anti-equivalence of categories between k-groups of multiplicative type and finitely generated abelian groups equipped with a continuous action of Γ_k, called *Cartier duality*. This extends to a duality between groups of multiplicative type and twisted constant group schemes over an arbitrary base. We refer to [Gro70a, Gro70b, Gro70c] for a complete exposition.

A k-torus T is said to be *quasi-trivial* if it is isomorphic to $\mathrm{Res}_{A/k}(\mathbb{G}_{m,A})$ for some étale k-algebra A, where $\mathrm{Res}_{A/k}$ denotes the restriction of scalars *à la Weil*. This is equivalent to saying that $\widehat{T}$ is a Γ_k-permutation module, *i.e.*, it has a $\mathbb{Z}$-basis which is Γ_k-stable. In this case, one has $\mathrm{H}^1(k,T) = 0$ by Shapiro's lemma and Hilbert's Theorem 90.

Brauer group. By convention, the Brauer group of a scheme X is always the Brauer–Grothendieck group $\mathrm{Br}(X) := \mathrm{H}^2(X,\mathbb{G}_m)$. For a smooth integral variety X over a field k of characteristic 0, Grothendieck's purity theorem [CTS21, Theorem 3.7.7] asserts that

$$\mathrm{Br}(X) = \bigcap_{\xi \in X^{(1)}} \mathrm{Ker}(\partial_\xi : \mathrm{Br}(k(X)) \to \mathrm{H}^1(k(\xi),\mathbb{Q}/\mathbb{Z})), \tag{1.1}$$

where ∂_ξ are the *residue* maps, and $X^{(1)}$ is the set of points of codimension 1 of X. In particular, $\mathrm{Br}(X)$ is torsion. The *unramified* Brauer group $\mathrm{Br}_{\mathrm{nr}}(X)$ is the subgroup of $\mathrm{Br}(k(X))$ consisting of elements α whose residue $\partial_{\mathcal{O}}(\alpha)$ at every discrete valuation ring $\mathcal{O} \supseteq k$ with $\mathrm{Frac}(\mathcal{O}) = k(X)$ vanishes (in particular, $\mathrm{Br}_{\mathrm{nr}}(X) \subseteq \mathrm{Br}(X)$). If X^c is a smooth

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compactification of X, then $\mathrm{Br}_{\mathrm{nr}}(X) = \mathrm{Br}(X^c)$. The group $\mathrm{Br}_{\mathrm{nr}}$ is a stable birational invariant of smooth integral varieties [CTS21, Proposition 6.2.7, Corollaries 6.2.10 and 6.2.11].

When k is an (Archimedean or nonarchimedean) local field, we identify $\mathrm{Br}(k)$ to a subgroup of $\mathbb{Q}/\mathbb{Z}$ using the local invariant map (this group is $\mathbb{Q}/\mathbb{Z}$ in the nonarchimedean case, $\{0\}$ if $k = \mathbb{C}$, and $\mathbb{Z}/2$ if $k = \mathbb{R}$). If K/k is a finite extension, the local invariants of K and k are compatible with the corestriction map $\mathrm{cor}_{K/k} : \mathrm{Br}(K) \to \mathrm{Br}(k)$ (thus, the restriction $\mathrm{res}_{K/k} : \mathrm{Br}(k) \to \mathrm{Br}(K)$ corresponds to the multiplication by $[K : k]$ in $\mathbb{Q}/\mathbb{Z}$).

The *algebraic* Brauer group of X is $\mathrm{Br}_1(X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(\bar{X}))$ (recall that $\bar{X} = X \times_k \bar{k}$). If k is a number field, we define $Б(X)$ to be the subgroup of $\mathrm{Br}_1(X)$ consisting of 'everywhere locally constant' elements, *i.e.*,

$$Б(X) := \mathrm{Ker}\left(\mathrm{Br}_1(X) \to \prod_{v\in\Omega_k} \frac{\mathrm{Br}_1(X_v)}{\mathrm{Im}(\mathrm{Br}(k_v) \to \mathrm{Br}(X_v))}\right). \tag{1.2}$$

A result of Harari [Har94] assures that $Б(X) \subseteq \mathrm{Br}_{\mathrm{nr}}(X)$ (for X smooth geometrically integral). A variant of the product Brauer–Manin pairing ($\mathrm{BM}_{\mathrm{nr}}$) is the *adelic Brauer–Manin pairing*

$$\langle -,-\rangle_{\mathrm{BM}} : X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_{v\in\Omega_k}, \alpha) \mapsto \sum_{v\in\Omega_k} \alpha(x_v), \tag{$\mathrm{BM}_{\mathbb{A}}$}$$

For any subgroup $A \subseteq \mathrm{Br}(X)$, we denote by $X(\mathbb{A}_k)^A$ the subset of adelic points orthogonal to A relative to the pairing ($\mathrm{BM}_{\mathbb{A}}$). By the global reciprocity law, every adelic point is orthogonal to $\mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$, and $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. When X is proper, one has $X(\mathbb{A}_k) = X(k_\Omega)$ by the valuative criterion for properness, $\mathrm{Br}_{\mathrm{nr}}(X) = \mathrm{Br}(X)$, and the pairings ($\mathrm{BM}_{\mathrm{nr}}$) and ($\mathrm{BM}_{\mathbb{A}}$) coincide.

2. Preliminary remarks

2.1. Nonabelian descent types

The notion of (nonabelian) descent types enables descent theory in the context where no torsors are given *a priori*. An example where the need of this object arises is as follows. If G is a linear algebraic group over a field k, $H \subseteq G$ is a Zariski closed subgroup, and $X := H\backslash G$, then the projection $G \to X$ is a torsor under H. But if X is only a homogeneous space of G (with $X(k) = \varnothing$), then we do not have a torsor $G \to X$. Worse, the stabilizer $\bar{H}$ of a point $x \in X(\bar{k})$ need not be defined over k.

A complete theory of nonabelian descent types is available in [Lin25], which builds on the previous works of various authors on *kernels* (or *lien*, or *band*) and *nonabelian Galois 2-cohomology*. We refer to [Bor93, §1 and §2], [FSS98, §1] and [DLA19, §2.2] for this subject. To summarize, a k-kernel (where k is a perfect field) is a smooth algebraic $\bar{k}$-group $\bar{G}$ equipped with an *outer semilinear action* κ of Γ_k (*grosso modo*, it is a semilinear Galois action modulo conjugation) satisfying certain continuity and algebraicity conditions. If $L = (\bar{G}, \kappa)$ is a k-kernel, the set $\mathrm{H}^2(k, L)$ of nonabelian Galois

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2-cohomology with coefficients in L classifies the extensions of topological groups

$$1 \to \bar{G}(\bar{k}) \to E \to \Gamma_k \to 1$$

bound by L, that is, the outer action of Γ_k on $\bar{G}(\bar{k})$ induced by this extension coincides with the one induced by κ. Here, we require that $\bar{G}(\bar{k})$ is discrete in E and that the map $E \to \Gamma_k$ is open. If such an extension E has a (continuous) section ς, we say that its class $[E] \in \mathrm{H}^2(k,L)$ is *neutral*. The section ς defines a semilinear action on $\bar{G}$, hence also a k-form G of $\bar{G}$ (by Galois descent). In this case, we call G a k-form of L, and write $L = \mathbf{lien}(G)$. It might happen that the set $\mathrm{H}^2(k,L)$ does not have any neutral class; and a neutral class is not necessarily unique if it exists. If $\bar{G}$ is commutative, then the outer action κ is in fact an action, hence L has unique k-form G. In this case, the set $\mathrm{H}^2(k,G) := \mathrm{H}^2(k,\mathbf{lien}(G))$ is the usual Galois 2-cohomology group with coefficients in G, and its unique neutral class is 0.

Let X be a nonempty k-variety. If $\pi : \bar{Y} \to \mathrm{Spec}(\bar{k})$ is a nonempty $\bar{k}$-variety equipped with a morphism $\bar{Y} \to \bar{X} := X_{\bar{k}}$, we say that an automorphism $\alpha \in \mathrm{Aut}(\bar{Y}/X)$ is *semilinear* if there exists $s \in \Gamma_k$ such that $\pi \circ \alpha = \mathrm{Spec}(s^{-1}) \circ \pi$. Such an s is necessarily unique. If $\mathrm{SAut}(\bar{Y}/X) \subseteq \mathrm{Aut}(\bar{Y}/X)$ denotes the subgroup of semilinear X-automorphisms of $\bar{Y}$, then we have an exact sequence

$$1 \to \mathrm{Aut}(\bar{Y}/\bar{X}) \to \mathrm{SAut}(\bar{Y}/X) \to \Gamma_k. \tag{2.1}$$

Let $L = (\bar{G},\kappa)$ be a k-kernel. A *descent type on X bound by L* [Lin25, Definition 2.1] is the equivalent class $\lambda = [\bar{Y},E]$ of a pair $(\bar{Y},E)$, where $\bar{Y} \to \bar{X}$ is a torsor under $\bar{G}$ and $E \subseteq \mathrm{SAut}(\bar{Y}/X)$ is a subgroup fitting in an exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \bar{G}(\bar{k}) & \longrightarrow & E & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{Aut}(\bar{Y}/\bar{X}) & \longrightarrow & \mathrm{SAut}(\bar{Y}/X) & \longrightarrow & \Gamma_k & &
\end{array}$$

of topological groups, where the top row is bound by L and the bottom row is (2.1). Here, we impose that two such pairs $(\bar{Y},E)$ and $(\bar{Y}',E')$ are equivalent if there exists a $\bar{G}$-equivariant $\bar{X}$-isomorphism $\iota : \bar{Y} \to \bar{Y}'$ such that $\iota \circ \alpha \circ \iota^{-1} \in E'$ for all $\alpha \in E$. The set of descent types on X bound by L is denoted by $\mathrm{DType}(X,L)$.

Thus, a descent type $\lambda \in \mathrm{DType}(X,L)$ entails not only a torsor $\bar{Y} \to \bar{X}$ under $\bar{G}$, but also a class $\partial(\lambda) := [E] \in \mathrm{H}^2(k,L)$. If $Y \to X$ is a torsor under a k-form G of L, it defines a descent type $\mathbf{dtype}(Y) \in \mathrm{DType}(X,G) := \mathrm{DType}(X,\mathbf{lien}(G))$, whose attached 2-cohomology class in $\mathrm{H}^2(k,G)$ is neutral. The converse also holds.

Lemma 2.1. *Let X be a quasi-projective, reduced and geometrically connected variety over a perfect field k. Let $L = (\bar{G},\kappa)$ be a k-kernel, and $\lambda = [\bar{Y},E] \in \mathrm{DType}(X,L)$.*

(i) *The class $[E] \in \mathrm{H}^2(k,L)$ is neutral if and only if there exists a torsor $Y \to X$ under a k-form G of L, of descent type $\mathbf{dtype}(Y) = \lambda$.*

(ii) *If $X(k) \neq \varnothing$, then the class $[E]$ is neutral.*

(iii) *Let $Y \to X$ be a torsor under a k-form G of L, of descent type $\mathbf{dtype}(Y) = \lambda$. If $Y' \to X$ is a torsor under a k-form G' of L, then $\mathbf{dtype}(Y') = \lambda$ if and only if*

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there exists a cocycle $\sigma \in \mathrm{Z}^1(k,G)$ such that $G' = G^\sigma$ and the X-torsors Y' and Y^σ under G^σ are isomorphic.

Proof. See [Lin25, Proposition 2.4]. □

The set $\mathrm{DType}(X,L)$ enjoys a functoriality property in X (see [Lin25, Construction 2.5]) as follows. Let $f : X' \to X$ be a morphism of nonempty k-varieties. If $\lambda = [\bar{Y},E] \in \mathrm{DType}(X,L)$, we may define the pullback type $f^*\lambda = [\bar{Y}',E'] \in \mathrm{DType}(X',L)$, where $\bar{Y}' := \bar{Y} \times_{\bar{X}} \bar{X}'$ and $\partial(f^*\lambda) = [E'] = [E] = \partial(\lambda)$ in $\mathrm{H}^2(k,L)$. Furthermore, for any overfield K/k, the k-kernel L restricts to a K-kernel L_K, and there is a restriction map

$$\mathrm{res}_{K/k} : \mathrm{DType}(X,L) \to \mathrm{DType}(X_K,L_K), \quad \lambda \mapsto \lambda_K,$$

which is compatible with the restriction map $\mathrm{H}^2(k,L) \to \mathrm{H}^2(K,L_K)$ on 2-cohomology.

There is also a partial functoriality with respect to L, which allows some sort dévissage argument on descent types. Let $L' = (\bar{G}',\kappa')$ be a second k-kernel, and $\varphi : L \to L'$ a surjective morphism of k-kernels (that is, a surjective morphism $\bar{G} \to \bar{G}'$ of algebraic $\bar{k}$-groups which is compatible with the outer Galois actions κ and κ'). Let $\bar{H} := \mathrm{Ker}(\bar{G} \xrightarrow{\varphi} \bar{G}')$.

Lemma 2.2. *With the above notations, the following are true.*

(i) *There is a canonical pushforward map $\varphi_* : \mathrm{DType}(X,L) \to \mathrm{DType}(X,L')$. If a descent type $\lambda \in \mathrm{DType}(X,L)$ has the corresponding torsor $\bar{Y} \to \bar{X}$ under $\bar{G}$, then the $\bar{X}$-torsor under $\bar{G}'$ associated with $\varphi_*\lambda \in \mathrm{DType}(X,L')$ is $\bar{Y}/\bar{H}$.*

(ii) *Let $\lambda = [\bar{Y},E] \in \mathrm{DType}(X,L)$. Then, every torsor $Z \to X$ of descent type $\mathbf{dtype}(Z) = \varphi_*\lambda$ gives rise to a k-kernel $L_Z = (\bar{H},\kappa_Z)$ and a descent type $\lambda_Z = [\bar{Y},E_Z] \in \mathrm{DType}(Z,L_Z)$.*

(iii) *Let Z, G', L_Z, and λ_Z be as in (ii). Then, for any torsor $Y \to Z$ under a k-form H of L_Z, of descent type λ_Z, the composite $Y \to Z \to X$ is a torsor under a k-form G of L, of descent type λ. In particular, one has $G' = G/H$ and $Z = Y/H$.*

Proof. See [Lin25, Construction 2.6, Lemma 2.7, Proposition 2.11]. □

For abelian descent types, the set DType has a very nice description.

Lemma 2.3. *Let G be a commutative algebraic k-group. Then, the set $\mathrm{DType}(X,G)$ is equipped with a structure of abelian group, functorial in X and G (even for nonsurjective morphisms $G \to G'$ of commutative algebraic k-groups), and fits in a complex*

$$\mathrm{H}^1(k,G) \to \mathrm{H}^1(X,G) \xrightarrow{\mathbf{dtype}} \mathrm{DType}(X,G) \xrightarrow{\partial} \mathrm{H}^2(k,G) \to \mathrm{H}^2(X,G). \tag{2.2}$$

If X is quasi-projective, reduced and geometrically connected, then (2.2) is exact.

Proof. See [Lin25, Construction 2.6, Lemma 2.8, Theorem 3.8]. □

We note that if G is a k-group of multiplicative type and X is a smooth quasi-projective geometrically integral k-variety, then (2.2) is equivalent to the

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'fundamental exact sequence' for *extended types* by Harari–Skorobogatov [HS13, Proposition 8.1] (see [Lin25, Theorem 4.1]).

If X is geometrically connected, a *finite descent type* on X in the sense of Harpaz–Wittenberg [HW24, Definition 2.1] is a connected étale cover $\bar{Y}$ of $\bar{X}$ such that the composite $\bar{Y} \to \bar{X} \to X$ is Galois (*i.e.*, the field extension $\bar{k}(\bar{Y})/k(X)$ is Galois). In this case, the homotopy exact sequence

$$1 \to \mathrm{Aut}(\bar{Y}/\bar{X}) \to \mathrm{Aut}(\bar{Y}/X) \to \Gamma_k \to 1$$

yields an outer action κ of Γ_k on $\bar{G} := \mathrm{Aut}(\bar{Y}/\bar{X})$ (viewed as a finite constant algebraic $\bar{k}$-group), hence a k-kernel $L = (\bar{G},\kappa)$, as well as a descent type $[\bar{Y}, \mathrm{Aut}(\bar{Y}/X)]$ on X bound by L.

2.2. Abelianization of nonabelian cohomology

The proof of Theorem A (*i.e.*, Conjecture B for torsors under connected linear algebraic groups) requires the machinery of abelianization of nonabelian cohomology. Fix a base scheme X, which, for simplicity, is assumed to have residue characteristics 0. Let us start with the following remark on 2-term complexes of tori. Let $C = [T \xrightarrow{\rho} S]$ be a complex of X-tori, with T in degree -1. Let $\widehat{T} := \mathscr{H}om_X(T,\mathbb{G}_m)$, $\widehat{S} := \mathscr{H}om_X(S,\mathbb{G}_m)$, and let $\widehat{C} = [\widehat{S} \xrightarrow{\rho^*} \widehat{T}]$ (with $\widehat{S}$ in degree -1) be the 'Cartier dual' of C. Following Demarche's construction [Dem11b, p. 4], we have a pairing

$$C \otimes^{\mathbb{L}} \widehat{C} \to \mathbb{G}_m[1] \tag{2.3}$$

in $\mathcal{D}^+(X)$, as follows. Represent the derived tensor product $C \otimes^{\mathbb{L}} \widehat{C}$ by the 'total complex'

$$\mathrm{Tot}(C \otimes \widehat{C}) := [T \otimes \widehat{S} \to (T \otimes \widehat{T}) \oplus (S \otimes \widehat{S}) \to S \otimes \widehat{T}],$$

concentrated in degrees $-2, -1, 0$, where the first differential is $(\mathrm{id}_T \otimes \rho^*, -\rho \otimes \mathrm{id}_{\widehat{S}})$ and the second one is $\rho \otimes \mathrm{id}_{\widehat{T}} + \mathrm{id}_S \otimes \rho^*$. The composite

$$(T \otimes \widehat{T}) \oplus (S \otimes \widehat{S}) \to \mathbb{G}_m \oplus \mathbb{G}_m \to \mathbb{G}_m,$$

where the first arrow is just the direct sum of the canonical pairings and the second one is the product map, gives rise to a morphism of complexes $\mathrm{Tot}(C \otimes \widehat{C}) \to \mathbb{G}_m[1]$, hence a pairing as in (2.3). From its construction, this pairing is compatible with the canonical pairings between the tori T, S and their respective Cartier duals. Since the pairing $T \otimes \widehat{T} \to \mathbb{G}_m$ induces an isomorphism $T \cong \mathscr{H}om_X(\widehat{T},\mathbb{G}_m)$, and $\mathscr{E}xt^i_X(\widehat{T},\mathbb{G}_m) = 0$ for all $i > 0$ [CTS87, Lemme 1.3.3 (ii)] (and similarly for S), we deduce, using a dévissage argument, that (2.3) induces a quasi-isomorphism

$$C \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_X(\widehat{C},\mathbb{G}_m)[1]. \tag{2.4}$$

In particular, taking hypercohomology yields $\mathrm{H}^i(X,C) \cong \mathrm{Ext}^{i+1}_X(\widehat{C},\mathbb{G}_m)$ for all $i \geq -1$.

A *reductive X-group scheme* is a smooth affine group scheme $G \to X$ with connected reductive fibres [Dem70a, Définition 2.7]. We use G^{ss} to denote its derived subgroup (whose fibres are connected semisimple), $G^{\mathrm{tor}} = G/G^{\mathrm{ss}}$ its coradical torus [Dem70b, Définition §6.2], and G^{sc} the universal covering of G^{ss} (whose fibres are simply connected

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semisimple) [Con11, Exercise 1.6.13 (ii)]. The morphism $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$ is a central isogeny, whose kernel $\mu := \pi_1(G^{\mathrm{ss}})$ is a finite étale commutative X-group scheme (the *algebraic fundamental group of* G^{ss}).

Associated with G is a 2-term complex of X-tori C concentrated in degrees -1 and 0, which is canonically defined up to quasi-isomorphism, called the *algebraic fundamental complex of* G. It can be defined in terms of *t-resolutions* [BGA14, Definition 2.1, Propositions 2.2 and 4.1] as follows. A t-resolution of G is a central extension

$$1 \to T \to H \to G \to 1,$$

where T is an X-torus and H is a reductive X-group scheme such that $H^{\mathrm{ss}} = H^{\mathrm{sc}}$. The complex C is then represented by the cone of the composite $T \hookrightarrow H \twoheadrightarrow S := H^{\mathrm{tor}}$, that is, the complex $[T \to S]$ (with T in degree -1). The cohomology of C is $\mathscr{H}^{-1}(C) = \mu$ and $\mathscr{H}^0(C) = S/T = G^{\mathrm{tor}}$.

The *dual algebraic fundamental complex* $\pi_1^D(G)$ of G is the complex $\widehat{C} := [\widehat{S} \xrightarrow{\rho^*} \widehat{T}]$, with $\widehat{S}$ in degree -1. Its cohomology is $\mathscr{H}^{-1}(\widehat{C}) = \widehat{G^{\mathrm{tor}}} =: \widehat{G}$ and $\mathscr{H}^0(\widehat{C}) = \widehat{\mu}$. If G has a maximal torus $T^{\max}$ (*e.g.*, when $X = \mathrm{Spec}(k)$, where k is a field of characteristic 0), let $T := T^{\max} \times_G G^{\mathrm{sc}}$ (which is maximal torus of G^{sc}). Then, C is represented by the complex $[T \to T^{\max}]$, with T in degree -1 (see [BGA14, Lemma 3.9]). There are also the *abelianization maps*

$$\mathrm{ab}^i : \mathrm{H}^i(X,G) \to \mathrm{H}^i(X,C),$$

defined for $i = 0,1$. When $X = \mathrm{Spec}(k)$, they were introduced by Borovoi in [Bor98]. This was later generalized to arbitrary base schemes by González-Avilés (combine [GA12, (3.12)] and [BGA14, Proposition 4.1]). There are two extreme cases where the maps ab^i have a simple description.

- If $G^{\mathrm{ss}} = G^{\mathrm{sc}}$, then $C \simeq G^{\mathrm{tor}}$, and $\mathrm{ab}^i : \mathrm{H}^i(X,G) \to \mathrm{H}^i(X,G^{\mathrm{tor}})$ are the pushforward maps.
- If $G = G^{\mathrm{ss}}$, then $C \simeq \mu[1]$, and $\mathrm{ab}^i : \mathrm{H}^i(X,G) \to \mathrm{H}^{i+1}(X,\mu)$ are the connecting maps induced by the central extension $1 \to \mu \to G^{\mathrm{sc}} \to G \to 1$.

Proposition 2.4. *With the above notations, the following are true.*

(i) *(Breen's exact sequence) We have a functorial exact sequence*

$$1 \to \mu(X) \to G^{\mathrm{sc}}(X) \xrightarrow{\rho} G(X) \xrightarrow{\mathrm{ab}^0} \mathrm{H}^0(X,C) \to \mathrm{H}^1(X,G^{\mathrm{sc}}) \xrightarrow{\rho_*} \mathrm{H}^1(X,G) \xrightarrow{\mathrm{ab}^1} \mathrm{H}^1(X,C)$$

of pointed sets. Furthermore, ab^0 *is a group homomorphism which induces an injection*

$$\mathrm{H}^0(X,C)/\mathrm{ab}^0(G(X)) \hookrightarrow \mathrm{H}^1(X,G^{\mathrm{sc}}).$$

(ii) *If* $X = \mathrm{Spec}(k)$*, where* k *is a local field of characteristic* 0*, then* $\mathrm{ab}^1 : \mathrm{H}^1(k,G) \to \mathrm{H}^1(k,C)$ *is surjective. It is bijective if* k *is nonarchimedean.*

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(iii) *If $X = \mathrm{Spec}(k)$, where k is a number field, then there is a Cartesian square*

$$\begin{array}{ccc} \mathrm{H}^1(k,G) & \xrightarrow{\mathrm{ab}^1} & \mathrm{H}^1(k,C) \\ \downarrow & & \downarrow \\ \prod\limits_{v\in\Omega_k^\infty} \mathrm{H}^1(k_v,G) & \xrightarrow{\mathrm{ab}^1} & \prod\limits_{v\in\Omega_k^\infty} \mathrm{H}^1(k_v,C) \end{array}$$

with surjective arrows. In particular, ab^1 *induces a bijection* $Ш(G) \xrightarrow{\cong} Ш^1(k,C)$.

Proof. For (i), see [GA12, Theorem 1.1, Proposition 3.14 (a)]. For (ii) (resp. (iii)), we refer to Theorem 5.4 and Corollary 5.4.1 (resp. Theorems 5.12 and 5.13) in [Bor98]. □

Proposition 2.5. *Let G be a reductive X-group scheme and $f : Y \to X$ a right torsor under G. Let $P \to X$ be a left torsor under G and let G^P be the corresponding twisted X-group (cf. [Gir71, Chapitre III, §2.3]; in particular, $P \to X$ is also a right torsor under G^P). Let $f^P : Y^P \to X$ be the twist of Y by P (that is, Y^P is the quotient of $Y \times_X P$ by the left action $(g,y,p) \mapsto (y \cdot g^{-1}, g \cdot p)$ of G), which is a right torsor under G^P. Denote by C the algebraic fundamental complex of G, and by $\widehat{C}$ the dual algebraic fundamental complex $\pi_1^D(G)$.*

(i) *The constructions $G \mapsto C$ and $G \mapsto \widehat{C}$ commute with base change.*

Assume that G admits a maximal torus $T^{\max}$. Then, the following are true.

(ii). *The algebraic fundamental complex (resp. the dual algebraic fundamental complex) of G^P is canonically isomorphic to C (resp. $\widehat{C}$).*

(iii). *We have* $\mathrm{ab}^1([Y^P]) = \mathrm{ab}^1([Y]) - \mathrm{ab}^1([P])$ *in* $\mathrm{H}^1(X,C)$.

Proof.

(i) This follows from the fact that t-resolutions are preserved by base change.

(ii) It suffices to combine [GA12, Definition 3.1] and [BGA14, Proposition 4.1], noting that the centers of G and G^{sc} remains unchanged under twisting.

(iii) See [GA12, Proposition 3.11]. □

2.3. The relative units-Picard complex

Let $f : Y \to X$ be a morphism of schemes. Then, we have a canonical morphism $\mathbb{G}_{m,X} \to f_*\mathbb{G}_{m,Y}$ of sheaves on $X_{\text{ét}}$. We define the *sheaf of relative units* of Y/X to be

$$\mathrm{U}_{Y/X} := \mathrm{Coker}(\mathbb{G}_{m,X} \to f_*\mathbb{G}_{m,Y}).$$

We also recall the *relative Picard sheaf* $\mathrm{Pic}_{Y/X} := \mathbb{R}^1 f_*\mathbb{G}_{m,Y}$. Following González-Avilés [GA18, (3.15)], we define the *relative units-Picard complex* $\mathrm{UPic}_{Y/X}$ to be the cone of the morphism

$$\mathbb{G}_{m,X}[1] \to (\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y})[1].$$

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It is represented by a complex concentrated in degrees -1 and 0, with cohomology $\mathscr{H}^{-1}(\mathrm{UPic}_{Y/X}) = \mathrm{U}_{Y/X}$ and $\mathscr{H}^{0}(\mathrm{UPic}_{Y/X}) = \mathrm{Pic}_{Y/X}$. By definition, we have a distinguished triangle

$$\mathbb{G}_{m,X} \to \tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y} \to \mathrm{UPic}_{Y/X}[-1] \to \mathbb{G}_{m,X}[1], \tag{2.5}$$

which splits if f has a section [GA18, Lemma 3.8]. The class in $\mathrm{Ext}^2_X(\mathrm{UPic}_{Y/X},\mathbb{G}_m)$ of the morphism $\mathrm{UPic}_{Y/X} \to \mathbb{G}_{m,X}[2]$ in $\mathcal{D}^+(X)$ associated with (2.5) is then an obstruction to the existence of sections of f.

Proposition 2.6. *Let X be a locally Noetherian regular scheme, G a reductive X-group scheme, and $Y \to X$ a torsor under G. There exists an isomorphism $\varphi_Y : \mathrm{UPic}_{Y/X} \xrightarrow{\simeq} \pi_1^D(G)$ in $\mathcal{D}^+(X)$, functorial in Y and G. In particular, $\mathrm{U}_{Y/X} \cong \widehat{G}$ and $\mathrm{Pic}_{Y/X} \cong \widehat{\mu}$, where $\mu := \mathrm{Ker}(G^{\mathrm{sc}} \to G^{\mathrm{ss}})$.*

Proof. The case where $Y = G$ is the trivial torsor was dealt with by González-Avilés in [GA19, Theorem 1.1]. The case where $X = \mathrm{Spec}(k)$ is due to Borovoi–van Hamel [BvH09, Lemma 5.2]. The proof in the general case is similar; we shall deploy Sansuc's argument, using the additivity theorem for UPic. More precisely, the projections $p_Y : Y \times_X G \to Y$ and $p_G : Y \times_X G \to G$ induce a canonical morphism

$$\varsigma_Y : \mathrm{UPic}_{Y/X} \oplus \mathrm{UPic}_{G/X} \to \mathrm{UPic}_{(Y\times_X G)/X}$$

in $\mathcal{D}^+(X)$. By [GA18, Proposition 4.4, Remark 5.2 (a)], ς_Y is an isomorphism. Note that the existence of an 'étale quasi-section' of $Y \times_X G \to X$ as in (i) of *loc. cit.* is guaranteed by [Gro67, Corollaires 17.16.2 and 17.16.3 (ii)], since this morphism is smooth [Mil80, Chapter III, Proposition 4.2] and surjective, and that for every generic point $\eta \in X$, the 'étale index' [GA18, Definition 2.3] of G_η is 1 since it has a $\kappa(\eta)$-rational point. Let φ_Y denote the composite

$$\mathrm{UPic}_{Y/X} \xrightarrow{\alpha^*} \mathrm{UPic}_{(Y\times_X G)/X} \xrightarrow{\varsigma_Y^{-1}} \mathrm{UPic}_{Y/X} \oplus \mathrm{UPic}_{G/X} \xrightarrow{\pi_G} \mathrm{UPic}_{G/X},$$

where $\alpha : Y \times_X G \to Y$ is the X-action of G on Y, and π_G is the second projection. According to its construction, φ_Y is functorial in Y and G. Since the result is known for the trivial torsor $G \to X$, it remains to show that φ_Y is a quasi-isomorphism, that is, $\mathscr{H}^{-1}(\varphi_Y) : \mathrm{U}_{Y/X} \to \mathrm{U}_{G/X}$ and $\mathscr{H}^{0}(\varphi_Y) : \mathrm{Pic}_{Y/X} \to \mathrm{Pic}_{G/X}$ are isomorphisms of étale sheaves over X. This follows from the units-Picard-Brauer sequence, see [GA19, Lemmata 2.5 and 3.2, Proposition 3.6, Corollary 3.8] and their proofs. □

Let X be a locally Noetherian regular scheme, G a reductive X-group scheme, and $f : Y \to X$ a torsor under G. Let C (resp. $\widehat{C}$) denote the fundamental algebraic complex (resp. the dual fundamental algebraic complex) of G. In view of Proposition 2.6, triangle (2.5) becomes

$$\mathbb{G}_{m,X} \to \tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y} \to \widehat{C}[-1] \to \mathbb{G}_{m,X}[1]. \tag{2.6}$$

Our aim is to relate the class in $\mathrm{Ext}^2_X(\widehat{C},\mathbb{G}_m)$ of the corresponding morphism $\widehat{C} \to \mathbb{G}_{m,X}[2]$ in $\mathcal{D}^+(X)$ and the abelianized class $\mathrm{ab}^1([Y]) \in \mathrm{H}^1(X,C)$ under the identification (2.4).

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Proposition 2.7. *With the above notations, the isomorphism* $\mathrm{H}^1(X,C) \xrightarrow{\cong} \mathrm{Ext}^2_X(\widehat{C},\mathbb{G}_m)$ *induced by (2.4) takes* $\mathrm{ab}^1([Y])$ *to the inverse class of the morphism* $\widehat{C} \to \mathbb{G}_{m,X}[2]$ *from (2.6), in the following cases.*

(i) $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ *(for example, when* G *is a torus).*

(ii) *The torsor* $Y \to X$ *is isotrivial, that is, there exists a finite étale surjective morphism* $U \to X$ *such that* $Y_U \to U$ *is a trivial torsor under* G_U *(for example, if* $X = \mathrm{Spec}(k)$*, where* k *is a field of characteristic* 0*).*

Proof.

(i) When $G^{\mathrm{ss}} = G^{\mathrm{sc}}$, one has $C \simeq G^{\mathrm{tor}}$ and $\widehat{C} \simeq \widehat{G^{\mathrm{tor}}}[1] = \widehat{G}[1]$, and the abelianization map $\mathrm{ab}^1 : \mathrm{H}^1(X,G) \to \mathrm{H}^1(X,C)$ is simply the pushfoward along the projection $G \to G^{\mathrm{tor}}$. By functoriality, it is enough to prove the claim when $G = T$ is a torus. In this case, $\mathrm{Pic}_{Y/X} = 0$, and it was shown by Colliot-Thélène and Sansuc that triangle (2.6) is quasi-isomorphic to the short exact sequence

$$0 \to \mathbb{G}_{m,X} \to f_*\mathbb{G}_{m,Y} \to \widehat{T} \to 0,$$

provided by the relative version of Rosenlicht's lemma, whose class in $\mathrm{Ext}^1_X(\widehat{T},\mathbb{G}_m)$ turns out to be precisely $-[Y]$ (see [CTS87, Propositions 1.4.2 and 1.4.3]).

(ii) We follow Kottwitz's argument in [Kot86, p. 369]. Suppose that $\pi : U \to X$ is a surjective finite étale morphism trivializing Y. Take any t-resolution

$$1 \to T \to H \to G \to 1$$

of G, and consider the sheaf $T_1 := \pi_* T_U$ (which is representable by an X-torus). Let $\partial : \mathrm{H}^1(X,G) \to \mathrm{H}^2(X,T)$ denote the connecting map. Then, the restriction of $\partial([Y])$ to $\mathrm{H}^2(U,T)$ is 0. Now, by the global version of Shapiro's lemma (which can be obtained by combining [Gro73, Corollaire 5.6] and the Leray spectral sequence), one has $\mathrm{H}^2(U,T) \cong \mathrm{H}^2(X,T_1)$. Hence, the map $i_* : \mathrm{H}^2(X,T) \to \mathrm{H}^2(X,T_1)$ (induced by the inclusion $i : T \hookrightarrow T_1$) takes $\partial([Y])$ to 0. Let H_1 denote the pushout of i and the inclusion $T \hookrightarrow H$. Then $H_1^{\mathrm{ss}} = H^{\mathrm{ss}} = H^{\mathrm{sc}}$ is simply connected, hence $1 \to T_1 \to H_1 \to G \to 1$ is also a t-resolution of G, and the corresponding connecting map $\delta : \mathrm{H}^1(X,G) \to \mathrm{H}^2(X,T_1)$ takes $[Y]$ to 0. It follows that there is a torsor $Z \to X$ under H_1 such that the map $\mathrm{H}^1(X,H_1) \to \mathrm{H}^1(X,G)$ takes $[Z]$ to $[Y]$. We know from (i) that the statement of the proposition holds for Z. By functoriality, it also holds for Y. □

Remarks 2.8. When $X = \mathrm{Spec}(k)$, the statement of Proposition 2.7 *without any restriction on* G *or* Y was proved by Borovoi and van Hamel in [BvH09, Theorem 5.5]. The author, unfortunately, does not know if this result holds true in full generality. Should it be the case, a possible approach would be to adapt the proof from the toric case by Colliot-Thélène and Sansuc [CTS87, Propositions 1.4.2 and 1.4.3] using Čech cocycles for étale hypercohomology. A catch there is that the quasi-isomorphism $\mathrm{UPic}_{Y/X} \simeq \pi_1^D(G)$

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from Proposition 2.6 is not explicit (it is constructed using the axioms of triangulated categories, *cf.* the proof of [GA19, Theorem 1.1]). This would make the computation rather cumbersome, and since the result in the isotrivial case is sufficient for further uses in the article, the author decides to refrain from including a proof for the general case (if it exists at all).

Corollary 2.9. *Let k be a field of characteristic 0, $p : X \to \mathrm{Spec}(k)$ a smooth geometrically integral k-variety, G a connected reductive linear algebraic k-group, and $f : Y \to X$ a torsor under G. Let C (resp. $\widehat{C}$) denote the algebraic fundamental complex (resp. dual algebraic fundamental complex) of G. Then, for every Galois cocycle $\sigma \in \mathrm{Z}^1(k,G)$, we have a distinguished triangle*

$$\mathbb{G}_{m,X} \to \tau_{\leq 1}\mathbb{R}f^{\sigma}_{*}\mathbb{G}_{m,Y^{\sigma}} \to \widehat{C}_X[-1] \to \mathbb{G}_{m,X}[1]$$

in $\mathcal{D}^{+}(X)$. If either $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ or the torsor $f : Y \to X$ is isotrivial (cf. Proposition 2.7 ii), then the map $\mathrm{H}^1(X,\widehat{C}) \to \mathrm{H}^3(X,\mathbb{G}_m)$ associated with the above triangle is given by $\widehat{c} \mapsto (-\mathrm{ab}^1([Y]) + p^*[\sigma]) \cup \widehat{c}$, where the cup-product is induced by the pairing (2.3).

Proof. This follows directly from Propositions 2.5 and 2.7. We note that in the case where $f : Y \to X$ is isotrivial, the twisted torsor $f^{\sigma} : Y^{\sigma} \to X$ is also isotrivial for any $\sigma \in \mathrm{Z}^1(k,G)$. This is because the class $[\sigma] \in \mathrm{H}^1(k,G)$ is trivialized by a finite extension of k. □

For further use in Section 3, we also require the following technical results.

Proposition 2.10. *Let X be an integral Noetherian regular scheme and let $1 \to T \to H \xrightarrow{p} G \to 1$ be a t-resolution of a reductive X-group scheme G. Let $Z \to X$ (resp. $Y \to X$) be a torsor under H (resp. G) such that $p_*[Z] = [Y]$ (i.e., there exists a p-equivariant X-morphism $\pi : Z \to Y$ making Z a Y-torsor under T). Then, the following are true.*

(i) *There is a distinguished triangle*

$$\mathrm{UPic}_{Y/X} \to \mathrm{UPic}_{Z/X} \to \widehat{T}[1] \to \mathrm{UPic}_{Y/X}[1]. \tag{2.7}$$

(ii) *Triangle (2.7) is functorial in the following sense. Let $1 \to T' \to H' \xrightarrow{p'} G' \to 1$ be a t-resolution of another reductive X-group scheme G'. Let $Z' \to X$ (resp. $Y' \to X$) be a torsor under H' (resp. G') such that there exists a p'-equivariant X-morphism $\pi' : Z' \to Y'$. Let $u : T' \to T$ be a morphism of X-tori and let $v : Z' \to Z$, $w : Y' \to Y$ be morphisms of X-schemes such that the diagram*

$$\begin{array}{ccccc} Z' \times_X T' & \xrightarrow{(z,t)\mapsto(z\cdot t)} & Z' & \xrightarrow{\pi'} & Y' \\ \downarrow{\scriptstyle v\times_X u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ Z \times_X T & \xrightarrow{(z',t')\mapsto(z'\cdot t')} & Z & \xrightarrow{\pi} & Y \end{array}$$

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commutes. Then, we have morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathrm{UPic}_{Y/X} & \longrightarrow & \mathrm{UPic}_{Z/X} & \longrightarrow & \widehat{T}[1] & \longrightarrow & \mathrm{UPic}_{Y/X}[1] \\
\downarrow{\scriptstyle w^*} & & \downarrow{\scriptstyle v^*} & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle w^*[1]} \\
\mathrm{UPic}_{Y'/X} & \longrightarrow & \mathrm{UPic}_{Z'/X} & \longrightarrow & \widehat{T'}[1] & \longrightarrow & \mathrm{UPic}_{Y'/X}[1].
\end{array}$$

Proof.

(i) We construct a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{UPic}_{Y/X} & \xrightarrow{\pi^*} & \mathrm{UPic}_{Z/X} & \longrightarrow & \widehat{T}[1] & \xrightarrow{\lambda_\pi[1]} & \mathrm{UPic}_{Y/X}[1] \\
\downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_Z} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \varphi_Y[1]} \\
\pi_1^D(G) & \longrightarrow & \pi_1^D(H) & \longrightarrow & \pi_1^D(T) & \longrightarrow & \pi_1^D(G)[1],
\end{array} \tag{2.8}$$

where the bottom row is the distinguished triangle associated with the given t-resolution of G (see [GA19, Proposition 4.12]). The quasi-isomorphisms φ_Y and φ_Z are given by Proposition 2.6. The unlabeled arrow on the top row of (2.8) is the composite

$$\mathrm{UPic}_{Z/X} \xleftarrow{\simeq} \mathrm{U}_{Z/X}[1] \xrightarrow{\chi_\pi[1]} \widehat{T},$$

where $\chi_\pi : \mathrm{U}_{Z/X} \to \widehat{T}$ is the map from the units-Picard-Brauer sequence (*cf.* [GA19, Corollary 3.8 (i)]), and where the inclusion $\mathrm{U}_{Z/X}[1] \hookrightarrow \mathrm{UPic}_{Z/X}$ is a quasi-isomorphism because $\mathrm{Pic}_{Z/X} = 0$ (note that $H^{\mathrm{sc}} = H^{\mathrm{ss}}$). The map $\lambda_\pi : \widehat{T} \to \mathrm{UPic}_{Y/X}[1]$ is the *extended type* of the torsor $Z \xrightarrow{\pi} Y$ under T, constructed in [HS13, Definition 8.2]. Let us verify that diagram (2.8) indeed commutes. To this end, we recall the construction of φ_Y and φ_Z from the proof of Proposition 2.6. The left square of (2.8) commutes because we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{UPic}_{Y/X} & \xrightarrow{\alpha_Y^*} & \mathrm{UPic}_{(Y\times_X G)/X} & \xleftarrow{\simeq} & \mathrm{UPic}_{Y/X} \oplus \mathrm{UPic}_{G/X} & \longrightarrow & \mathrm{UPic}_{G/X} \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle (\pi\times_X p)^*} & & \downarrow{\scriptstyle \pi^*\oplus p^*} & & \downarrow{\scriptstyle p^*} \\
\mathrm{UPic}_{Z/X} & \xrightarrow{\alpha_Z^*} & \mathrm{UPic}_{(Z\times_X H)/X} & \xleftarrow{\simeq} & \mathrm{UPic}_{Z/X} \oplus \mathrm{UPic}_{H/X} & \longrightarrow & \mathrm{UPic}_{H/X},
\end{array}$$

where $\alpha_Y : Y \times_X G \to Y$ and $\alpha_Z : Z \times_X H \to Z$ denote the respective action morphisms, and where right horizontal arrows are the second projections. Indeed, the left square commutes because $\alpha_Y \circ (\pi \times_X p) = \pi \circ \alpha_Z$ (the morphism π being p-equivariant), and the other two squares obviously commute.

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The middle square of (2.8) commutes because it corresponds to a commutative square

$$\begin{array}{ccc} \mathrm{U}_{Z/X} & \xrightarrow{\chi_\pi} & \widehat{T} \\ \downarrow{\scriptstyle \mathscr{H}^{-1}(\varphi_Z)} & & \| \\ \widehat{H} & \longrightarrow & \widehat{T}. \end{array}$$

Indeed, $\mathscr{H}^{-1}(\varphi_Z)$ is just the map from the units-Picard-Brauer sequence for Z (*cf.* [GA19, Corollary 3.8 (i)], and hence the commutativity of the above square follows from the functoriality of the same sequence (see *loc. cit.*, Remarks 3.7 (a)). As for the right square of (2.8), since T is central in H, we have a commutative diagram

$$\begin{array}{ccccc} Z \times_X H \times_X T \times_X T & \xrightarrow{(z,h,t_1,h_2)\mapsto(z\cdot t_1, ht_2)} & Z \times_X H & \xrightarrow{\pi \times_X p} & Y \times_X G \\ \downarrow{\scriptstyle \alpha_Z \times_X \nabla} & & \downarrow{\scriptstyle \alpha_Z} & & \downarrow{\scriptstyle \alpha_Y} \\ Z \times_X T & \xrightarrow{(z,t)\mapsto(z\cdot t)} & Z & \xrightarrow{\pi} & Y, \end{array}$$

where $\nabla : T \times_X T \to T$ denotes the multiplication morphism. By functoriality of the extended type construction, we obtain the commutativity of the left square of the following diagram:

$$\begin{array}{ccccccc} \widehat{T} & \xrightarrow{\nabla^*} & \widehat{T \times_X T} & \xleftarrow{\simeq} & \widehat{T} \oplus \widehat{T} & \longrightarrow & \widehat{T} \\ \downarrow{\scriptstyle \lambda_\pi[-1]} & & \downarrow{\scriptstyle \lambda_{\pi \times_X p}[-1]} & & \downarrow{\scriptstyle (\lambda_\pi \oplus \lambda_p)[-1]} & & \downarrow{\scriptstyle \lambda_p[-1]} \\ \mathrm{UPic}_{Y/X} & \xrightarrow{\alpha_Y^*} & \mathrm{UPic}_{(Y \times_X G)/X} & \xleftarrow{\simeq} & \mathrm{UPic}_{Y/X} \oplus \mathrm{UPic}_{G/X} & \longrightarrow & \mathrm{UPic}_{G/X}, \end{array}$$

The other squares in the above diagram obviously commute. The composite of the top row is $\mathrm{id}_{\widehat{T}}$ (composing the first two arrows yields the diagonal map $\widehat{T} \to \widehat{T} \oplus \widehat{T}$), and that of the bottom row is φ_Y (when $\mathrm{UPic}_{G/X}$ is identified to $\pi_1^D(G)$). Furthermore, by unwinding the definition of the extended type in [HS13, Proposition 8.1, Definition 8.2] (see also [Lin25, §4.1, pp. 415—416], one checks that the composite $\widehat{T} \xrightarrow{\simeq} \mathrm{UPic}_{T/X} \to \mathrm{UPic}_{G/X}[1]$ is precisely the extended type λ_p of the torsor $p : H \to G$ under T. This shows that the right square of (2.8) commutes.

Since the bottom row of (2.8) is a distinguished triangle and since the vertical arrows there are quasi-isomorphisms, the top row is also a distinguished triangle.

(ii) The catch here is that the morphism $v : Z' \to Z$ need not correspond to a morphism $H' \to H$ of X-group schemes. Nevertheless, once the maps in (2.7) are explicitly constructed, our claims follow easily from the functoriality of the maps from the units-Picard-Brauer sequence as well as the extended types. □

Proposition 2.11. *Let X be an integral Noetherian regular scheme, G a reductive X-group scheme, and $f : Y \to X$ a right torsor under G. Let $P \to X$ be a left torsor under G and let G^P be the corresponding twisted X-group. Let $f^P : Y^P \to X$ be the twist of Y by*

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P (cf. Proposition 2.5). Let C (resp. $\widehat{C}$) denote the fundamental algebraic complex (resp. the dual fundamental algebraic complex) of G and assume either of the following.

(i) *$G^{\mathrm{ss}} = G^{\mathrm{sc}}$ (for example, when G is a torus).*

(ii) *The X-torsors Y and P are isotrivial (cf. Proposition 2.7 (ii)).*

Then, we have a commutative square

$$\begin{array}{ccc} \mathrm{UPic}_{Y^P/X} & \xrightarrow{\simeq} & \widehat{C} \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \Delta} \\ \mathrm{UPic}_{(Y\times_X P)/X} & \xrightarrow{\simeq} & \widehat{C}\oplus\widehat{C}. \end{array}$$

in $\mathcal{D}^+(X)$, where the horizontal maps are the quasi-isomorphisms from Proposition 2.6 (see also Proposition 2.5 (ii)), where $\pi : Y \times_X P \to Y^P$ is the canonical projection, and where Δ is the diagonal map.

Proof. The main difficulty here is that if G is not commutative, the morphism π is not equivariant with respect to any morphism $G \times_X G^P \to G^P$ of X-group schemes. We start with the case where $G = T$ is a torus. In this case, one has $G^P = T$, $\widehat{C} \simeq \widehat{T}[1]$ and π is ∇-equivariant, where $\nabla : T \times_X T \to T$ is the multiplication morphism. Since ∇ induces the diagonal map $\Delta : \widehat{T} \to \widehat{T} \oplus \widehat{T}$ on character modules, the claim in the toric case follows.

Next, we consider the case where $G^{\mathrm{ss}} = G^{\mathrm{sc}}$. Then $\widehat{C} \simeq \widehat{G^{\mathrm{tor}}}[1]$. Let Z (resp. Q) denote the pushforward of Y (resp. P) along the projection $u : G \to G^{\mathrm{tor}}$, which are torsors under the X-torus G^{tor}. Then, the projections $u^P : G^P \to (G^P)^{\mathrm{tor}} = G^{\mathrm{tor}}$ and $u \times_X u^P : G \times_X G^P \to (G \times_X G^P)^{\mathrm{tor}} = G^{\mathrm{tor}} \times_X G^{\mathrm{tor}}$ have the respective properties that $\pi^P_* Y^P = Z^Q$ and $(\pi \times_X \pi^P)_*(Y \times_X P) = Z \times_X Q$. Let $\varpi : Z \times_X Q \to Z^Q$ be the canonical projection. The commutativity of the square

$$\begin{array}{ccc} Y \times_X P & \longrightarrow & Z \times_X Q \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \varpi} \\ Y^P & \longrightarrow & Z^Q \end{array}$$

yields the commutativity of the left square in the following diagram:

$$\begin{array}{ccccc} \mathrm{UPic}_{Y^P/X} & \longrightarrow & \mathrm{UPic}_{Z^Q/X} & \xrightarrow{\simeq} & \widehat{G^{\mathrm{tor}}}[1] \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \varpi^*} & & \downarrow{\scriptstyle \Delta} \\ \mathrm{UPic}_{(Y\times_X P)/X} & \longrightarrow & \mathrm{UPic}_{(Z\times_X Q)/X} & \xrightarrow{\simeq} & (\widehat{G^{\mathrm{tor}}} \oplus \widehat{G^{\mathrm{tor}}})[1]. \end{array}$$

The right square commutes thanks to the toric case which was dealt with above.

Now, we consider the case where the torsors $Y \to X$ and $P \to X$ are isotrivial. Take a surjective finite étale morphism $U \to X$ trivializing both Y and P. Then, by repeating the argument from the proof of Proposition Proposition 2.7 (ii), we obtain a t-resolution

$$1 \to T \to H \xrightarrow{u} G \to 1$$

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such that $Y = u_*Z$ and $P = u_*Q$ for some right torsor $Z \to X$ and some left torsor $Q \to X$ under H. Note that

$$1 \to T \to H^Q \xrightarrow{u^Q} G^P \to 1 \quad \text{and} \quad 1 \to T \times_X T \to H \times_X H^Q \xrightarrow{u \times_X u^Q} G \times_X G^P \to 1$$

are also t-resolutions (recall that T is central in H hence remains unchanged under twisting), and that we have $u^Q_* Z^Q = Y^P$ and $(u \times_X u^Q)_*(Z \times_X Q) = Y \times_X P$. Let $\varpi : Z \times_X Q \to Z^Q$ (resp. $\nabla : T \times_X T \to T$) denote the canonical projection (resp. the multiplication morphism). One easily checks that the diagram

$$\begin{array}{ccccc} Z \times_X Q \times_X T \times_X T & \xrightarrow{(z,q,t_1,h_2) \mapsto (z \cdot t_1, q \cdot t_2)} & Z \times_X Q & \longrightarrow & Y \times_X P \\ \downarrow{\scriptstyle \varpi \times_X \nabla} & & \downarrow{\scriptstyle \varpi} & & \downarrow{\scriptstyle \pi} \\ Z^Q \times_X T & \xrightarrow{(z',t) \mapsto (z' \cdot t)} & Z^Q & \longrightarrow & Y, \end{array}$$

commutes. By Proposition 2.10, we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathrm{UPic}_{Y^P/X} & \longrightarrow & \mathrm{UPic}_{Z^Q/X} & \longrightarrow & \widehat{T}[1] & \longrightarrow & \mathrm{UPic}_{Y^P/X}[1] \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \varpi^*} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \pi^*[1]} \\ \mathrm{UPic}_{(Y \times_X P)/X} & \longrightarrow & \mathrm{UPic}_{(Z \times_X Q)/X} & \longrightarrow & (\widehat{T} \oplus \widehat{T})[1] & \longrightarrow & \mathrm{UPic}_{(Y \times_X P)/X}[1]. \end{array}$$

Now, since $H^{\mathrm{ss}} = H^{\mathrm{sc}}$, we deduce from the result in the previous case that the second column in the above diagram can be replaced by the map $\Delta[1] : \widehat{S}[1] \to (\widehat{S} \oplus \widehat{S})[1]$, where $S := H^{\mathrm{tor}}$. Since $\widehat{C}$ is by definition the cone of the morphism $\widehat{S} \to \widehat{T}$, we obtain the claimed result using the functoriality of the quasi-isomorphisms constructed from Proposition 2.6. □

3. Torsors under connected linear algebraic groups

This section is devoted to the proof of Theorems A and B.

3.1. Some calculations

In this subsection, we fix a field k of characteristic 0, a smooth geometrically integral k-variety $p : X \to \mathrm{Spec}(k)$ and a connected reductive linear algebraic group G over k. Let C (resp. $\widehat{C}$) denote the algebraic fundamental complex (resp. dual algebraic fundamental complex) of G. Let $f : Y \to X$ be a torsor under G. Taking hypercohomology of the distinguished triangle (2.6), we obtain an exact sequence

$$\mathrm{Br}(X) \to \mathrm{H}^2(X, \tau_{\leq 1} \mathbb{R} f_* \mathbb{G}_{m,Y}) \to \mathrm{H}^1(X, \widehat{C}) \to \mathrm{H}^3(X, \mathbb{G}_m), \tag{3.1}$$

where the last arrow is given by $\widehat{c} \mapsto -\mathrm{ab}^1([Y]) \cup \widehat{c}$ if either $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ or the torsor $Y \to X$ is isotrivial (Corollary 2.9). Let us investigate the two middle terms of (3.1). First, we need some lemmata.

Lemma 3.1. *If G is split, then $\mathrm{Br}_{\mathrm{nr}}(Y) \subseteq f^*(\mathrm{Br}(X))$.*

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Proof. Take a split maximal torus S of G, and let T be its preimage in G^{sc}. Then $[\widehat{S} \to \widehat{T}]$ (where $\widehat{S}$ is in degree -1) is a complex of split tori representing $\widehat{C}$. Let $K := k(X)$ and let $V \to \mathrm{Spec}(K)$ be the generic fibre of f. By [BK00, Corollary B], the group $\mathrm{Br}_{\mathrm{nr}}(V)/f^*(\mathrm{Br}(K))$ injects into the subgroup $Ш^1_\omega(K,\widehat{C})$ of $\mathrm{H}^1(K,\widehat{C})$ consisting of elements that become trivial after restriction to every procyclic subgroup of Γ_K. Let $r := \dim(T) = \dim(S)$, then $\widehat{T} \cong \widehat{S} \cong \mathbb{Z}^r$ as Γ_K-modules. In particular, one has $\mathrm{H}^1(K,\widehat{T}) = 0$. It follows that $\mathrm{H}^1(K,\widehat{C})$ injects into $\mathrm{H}^2(K,\widehat{S})$. On the other hand, there is an exact sequence

$$0 \to \widehat{S} \to \mathbb{Q}^r \to (\mathbb{Q}/\mathbb{Z})^r \to 0 \tag{3.2}$$

of abelian groups equipped with the trivial actions of Γ_K. The term $\mathbb{Q}^r$ is uniquely divisible, hence *cohomologically trivial*. It follows that $\mathrm{H}^2(K,\widehat{S}) = \mathrm{H}^1(K,(\mathbb{Q}/\mathbb{Z})^r) = \mathrm{Hom}_{\mathrm{cts}}(\Gamma_K,(\mathbb{Q}/\mathbb{Z})^r)$, and similarly for any procyclic subgroup of Γ_K. It follows that $Ш^2_\omega(K,\widehat{S}) = Ш^1_\omega(K,(\mathbb{Q}/\mathbb{Z})^r) = 0$, *a fortiori* $Ш^1_\omega(K,\widehat{C}) = 0$, and hence the map $f^* : \mathrm{Br}(K) \to \mathrm{Br}_{\mathrm{nr}}(V)$ is surjective.

Viewing that $k(Y) = K(V)$, we have $\mathrm{Br}(Y) \subseteq \mathrm{Br}(V) \subseteq \mathrm{Br}(k(Y))$. Let $\alpha \in \mathrm{Br}_{\mathrm{nr}}(Y) \subseteq \mathrm{Br}_{\mathrm{nr}}(V)$. From what has been established above, $\alpha = f^*\beta$ for some $\beta \in \mathrm{Br}(K)$. It remains to show that $\beta \in \mathrm{Br}(X)$. Indeed, let $\xi \in X^{(1)}$ be any codimension 1 point. Let η be the generic point of the fibre $f^{-1}(\xi) \to \mathrm{Spec}(k(\xi))$, which is a codimension 1 point of Y. Since $\alpha \in \mathrm{Br}(Y)$, the residue $\partial_\eta(\alpha) \in \mathrm{H}^1(k(\eta),\mathbb{Q}/\mathbb{Z})$ vanishes (*cf.* (1.1)). Since $f^{-1}(\xi)$ is geometrically integral, the map $\mathrm{H}^1(k(\xi),\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k(\eta),\mathbb{Q}/\mathbb{Z})$ is injective. By functoriality of the residue map [CTS21, Theorem 3.7.5], we have $\partial_\xi(\beta) = 0$. It follows that $\beta \in \mathrm{Br}(X)$, which concludes the proof. □

Following Harpaz and Wittenberg [HW20, §2.2], we define the *relative algebraic Brauer group*

$$\mathrm{Br}_1(Y/X) := \mathrm{Ker}(\mathrm{Br}(Y) \to \mathrm{Br}(\bar{Y})/f^*(\mathrm{Br}(\bar{X}))).$$

It is obvious that $f^*(\mathrm{Br}(X)) \subseteq \mathrm{Br}_1(Y/X)$. By Lemma 3.1, one has $\mathrm{Br}_{\mathrm{nr}}(Y) \subseteq \mathrm{Br}_1(Y/X)$.

Lemma 3.2. *The group* $\mathrm{H}^2(X,\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y})$ *is identified to*

$$\mathrm{Ker}(\mathrm{Br}(Y) \to \mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y})),$$

and it contains $\mathrm{Br}_1(Y/X)$.

Proof. Taking hypercohomology of the distinguished triangle

$$\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y} \to \tau_{\leq 2}\mathbb{R}f_*\mathbb{G}_{m,Y} \to \mathbb{R}^2f_*\mathbb{G}_{m,Y}[-2] \to (\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y})[1]$$

in $\mathcal{D}^+(X)$ yields an exact sequence

$$0 \to \mathrm{H}^2(X,\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y}) \to \mathrm{H}^2(X,\tau_{\leq 2}\mathbb{R}f_*\mathbb{G}_{m,Y}) \to \mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y}). \tag{3.3}$$

On the other hand, by taking cohomology of the distinguished triangle

$$\tau_{\leq 2}\mathbb{R}f_*\mathbb{G}_{m,Y} \to \mathbb{R}f_*\mathbb{G}_{m,Y} \to \tau_{\geq 3}\mathbb{R}f_*\mathbb{G}_{m,Y} \to (\tau_{\leq 2}\mathbb{R}f_*\mathbb{G}_{m,Y})[1],$$

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we have $\mathrm{H}^2(X,\tau_{\leq 2}\mathbb{R}f_*\mathbb{G}_{m,Y}) = \mathrm{H}^2(X,\mathbb{R}f_*\mathbb{G}_{m,Y}) = \mathrm{H}^2(Y,\mathbb{G}_m) = \mathrm{Br}(Y)$, noting that $\tau_{\geq 3}\mathbb{R}f_*\mathbb{G}_{m,Y}$ is acyclic in degrees 0, 1, 2, and that $\mathbb{H}(X,-)\circ\mathbb{R}f_* \simeq \mathbb{H}(Y,-)$ [Wei94, Corollary 10.8.3]. It follows that (3.3) yields an identification

$$\mathrm{H}^2(X,\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y}) = \mathrm{Ker}(\mathrm{Br}(Y) \to \mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y})).$$

Let us check that this group contains $\mathrm{Br}_1(Y/X)$. First, note that the composite

$$\mathrm{Br}(\bar{X}) \xrightarrow{f^*} \mathrm{Br}(\bar{Y}) \to \mathrm{H}^0(\bar{X},\mathbb{R}^2f_*\mathbb{G}_{m,\bar{Y}})$$

is 0 because f^* factors through the map

$$\mathrm{Br}(\bar{X}) \to \mathrm{H}^2(\bar{X},\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,\bar{Y}}) = \mathrm{Ker}(\mathrm{Br}(\bar{Y}) \to \mathrm{H}^0(\bar{X},\mathbb{R}^2f_*\mathbb{G}_{m,\bar{Y}}))$$

from (3.3). In view of the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^2(X,\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y}) & \longrightarrow & \mathrm{Br}(Y) & \longrightarrow & \mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^2(\bar{X},\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,\bar{Y}}) & \longrightarrow & \mathrm{Br}(\bar{Y}) & \longrightarrow & \mathrm{H}^0(\bar{X},\mathbb{R}^2f_*\mathbb{G}_{m,\bar{Y}}),
\end{array}$$

we see that the inclusion $\mathrm{Br}_1(Y/X) \subseteq \mathrm{H}^2(X,\tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,Y})$ would follow from the injectivity of the canonical map $\mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y}) \to \mathrm{H}^0(\bar{X},\mathbb{R}^2f_*\mathbb{G}_{m,\bar{Y}})$. Indeed, since $\mathbb{R}^2f_*\mathbb{G}_{m,Y}$ is the sheaf on $X_{\text{ét}}$ associated to the presheaf $(X' \to X) \mapsto \mathrm{Br}(Y')$ (where $Y' := Y \times_X X'$), any element $a \in \mathrm{H}^0(X,\mathbb{R}^2f_*\mathbb{G}_{m,Y})$ comes from some element $\alpha' \in \mathrm{Br}(Y')$ for some étale cover $X' \to X$. If the image of a in $\mathrm{H}^0(\bar{X},\mathbb{R}^2f_*\mathbb{G}_{m,\bar{Y}})$ is 0, then there is a finite field extension K/k and an étale cover $X'' \to X'_K$ such that the image of α' in $\mathrm{Br}(Y'')$ is 0, where $Y'' := Y_K \times_{X_K} X'' = Y \times_X X''$. This means α' is trivialized by the étale cover $X'' \to X$ refining X', hence $a = 0$. □

Lemma 3.3. *We have* $\mathrm{H}^1(k,\widehat{C}) = \mathrm{Ker}(\mathrm{H}^1(X,\widehat{C}) \to \mathrm{H}^1(\bar{X},\widehat{C}))$.

Proof. Represent C by a complex $[T \to S]$ of k-tori (with T in degree -1), then $\widehat{C} \simeq [\widehat{S} \to \widehat{T}]$ (with $\widehat{S}$ in degree -1). We have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}^1(k,\widehat{S}) & \longrightarrow & \mathrm{H}^1(k,\widehat{T}) & \longrightarrow & \mathrm{H}^1(k,\widehat{C}) & \longrightarrow & \mathrm{H}^2(k,\widehat{S}) & \longrightarrow & \mathrm{H}^2(k,\widehat{T}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(X,\widehat{S}) & \longrightarrow & \mathrm{H}^1(X,\widehat{T}) & \longrightarrow & \mathrm{H}^1(X,\widehat{C}) & \longrightarrow & \mathrm{H}^2(X,\widehat{S}) & \longrightarrow & \mathrm{H}^2(X,\widehat{T}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(\bar{X},\widehat{S}) & \longrightarrow & \mathrm{H}^1(\bar{X},\widehat{T}) & \longrightarrow & \mathrm{H}^1(\bar{X},\widehat{C}) & \longrightarrow & \mathrm{H}^2(\bar{X},\widehat{S}) & \longrightarrow & \mathrm{H}^2(\bar{X},\widehat{T})
\end{array}$$

with exact rows. Since $\widehat{S} \cong \mathbb{Z}^{\dim(S)}$ over $\bar{k}$, we have $\mathrm{H}^1(\bar{X},\widehat{S}) = 0$. By the Hochschild–Serre spectral sequence $\mathrm{H}^p(k,\mathrm{H}^q(\bar{X},\widehat{S})) \Rightarrow \mathrm{H}^{p+q}(X,\widehat{S})$, one has $\mathrm{H}^1(k,\widehat{S}) = \mathrm{H}^1(X,\widehat{S})$ and $\mathrm{H}^2(k,\widehat{S}) = \mathrm{Ker}(\mathrm{H}^2(X,\widehat{S}) \to \mathrm{H}^2(\bar{X},\widehat{S}))$. Similarly, we have $\mathrm{H}^1(\bar{X},\widehat{T}) = 0$, $\mathrm{H}^1(k,\widehat{T}) = \mathrm{H}^1(X,\widehat{T})$, and $\mathrm{H}^2(k,\widehat{T}) = \mathrm{Ker}(\mathrm{H}^2(X,\widehat{T}) \to \mathrm{H}^2(\bar{X},\widehat{T}))$. By a diagram chasing, we check

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that the map $\mathrm{H}^1(k,\widehat{C}) \to \mathrm{H}^1(X,\widehat{C})$ is injective, and that its image is precisely the kernel of the map $\mathrm{H}^1(X,\widehat{C}) \to \mathrm{H}^1(\bar{X},\widehat{C})$. □

In view of Lemmata 3.2 and 3.3, we extract from (3.1) an exact sequence

$$\mathrm{Br}(X) \xrightarrow{f^*} \mathrm{Br}_1(Y/X) \xrightarrow{\varphi^1} \mathrm{H}^1(k,\widehat{C}) \xrightarrow{\delta} \mathrm{H}^3(X,\mathbb{G}_m), \tag{3.4}$$

which lies in the heart of the proof of Theorem A. By Corollary 2.9, if either $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ or the torsor $Y \to X$ is isotrivial, then the map δ in (3.4) is given by $\delta(\widehat{c}) = -\mathrm{ab}^1([Y]) \cup p^*\widehat{c}$ (the cup-product being induced by the pairing (2.3)).

3.2. The method of Harpaz and Wittenberg

Having established the exact sequence (3.4), we shall now prove in this subsection the following generalization of [HW20, Théorème 2.1], which would imply Theorem A.

Theorem 3.4. *Let G be a connected reductive linear algebraic group over a number field k. Let X be a smooth geometrically integral k-variety and $f : Y \to X$ a torsor under G. Let $A \subseteq \mathrm{Br}(X)$ denote the preimage of $\mathrm{Br}_{\mathrm{nr}}(Y)$ by the map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$. Then*

$$X(\mathbb{A}_k)^A = \bigcup_{[\sigma] \in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}).$$

We keep the notations k, G, X, Y, p, f, C, and $\widehat{C}$ from the previous subsection.

Lemma 3.5. *Let $Z \to \mathrm{Spec}(k)$ be a torsor under G. Consider the exact sequence (3.4) for Z:*

$$\mathrm{Br}(k) \to \mathrm{Br}_1(Z) \xrightarrow{\varphi} \mathrm{H}^1(k,\widehat{C}) \to \mathrm{H}^3(k,\mathbb{G}_m).$$

Then, for any field extension K/k and any $z \in Z(K)$, $g \in G(K)$, $\alpha \in \mathrm{Br}_1(Z)$, one has

$$\alpha(z \cdot g) = \alpha(z) + \mathrm{ab}^0(g) \cup \mathrm{res}_{K/k}(\varphi(\alpha))$$

in $\mathrm{H}^2(K,\mathbb{G}_m) = \mathrm{Br}(K)$. Here, the cup-product is induced by the pairing (2.3).

Proof. If G is a torus, this is [HW20, Lemme 2.4]. When $G^{\mathrm{ss}} = G^{\mathrm{sc}}$, we have $C \simeq G^{\mathrm{tor}}$, and the map ab^0 is just the homomorphism $G(K) \to G^{\mathrm{tor}}(K)$. By functoriality and the toric case, the result also holds for the case where $G^{\mathrm{ss}} = G^{\mathrm{sc}}$.

In the general case, it suffices to repeat the argument in [Kot86, p. 369]. Take any t-resolution

$$1 \to T \to H \to G \to 1$$

of G and let $\partial : \mathrm{H}^1(k,G) \to \mathrm{H}^2(k,T)$ denote the connecting map. Let K/k be a finite extension trivializing the class $\partial([Z]) \in \mathrm{H}^2(k,T)$ and let $T_1 := \mathrm{Res}_{K/k}(T_K)$ (where Res denotes the restriction of scalars *à la* Weil). Let $i : T \hookrightarrow T_1$ denote the canonical inclusion. By Shapiro's lemma and [NSW08, Proposition 1.6.5], the map $i_* : \mathrm{H}^2(k,T) \to \mathrm{H}^2(k,T_1) \cong \mathrm{H}^2(K,T)$ takes $\partial([Z])$ to 0. Thus, we obtain a t-resolution

$$1 \to T_1 \to H_1 \xrightarrow{\pi} G \to 1,$$

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where H_1 is the pushforward of H along i, with the property that the connecting map $\mathrm{H}^1(k,G) \to \mathrm{H}^2(k,T_1)$ takes $[Z]$ to 0, that is, there exists a k-torsor W under H_1 and a π-equivariant morphism $W \to Z$.

Now, we consider the torsors $H_1 \to G$ and $W \to Z$ under T_1, and the fibres $(H_1)_g \to \mathrm{Spec}(K)$, $W_z \to \mathrm{Spec}(K)$. Take a finite extension K'/K trivializing the classes $[(H_1)_g],[W_z] \in \mathrm{H}^1(K,T_1)$. Put $T_2 := \mathrm{Res}_{K'/K}((T_1)_{K'})$ and $T_3 := \mathrm{Res}_{K/k}(T_2) = \mathrm{Res}_{K'/k}(T_{K'})$, then $(T_3)_K \cong T_2^{[K:k]}$. Let $j : T_1 \hookrightarrow T_3$ denote the canonical inclusion. By Shapiro's lemma and [NSW08, Proposition 1.6.5], the map $j_* : \mathrm{H}^1(K,T_1) \to \mathrm{H}^1(K,T_3) \cong \mathrm{H}^1(K,T_2)^{\oplus[K:k]} \cong \mathrm{H}^1(K',T_1)^{\oplus[K:k]}$ takes $[(H_1)_g]$ and $[W_z]$ to 0. Thus, we obtain a t-resolution

$$1 \to T_3 \to H_3 \to G \to 1,$$

where H_3 denotes the pushforward of H_1 along j, with the following properties.

- There exists a k-torsor W' under H_3 such that the map $\mathrm{H}^1(k,H_3) \to \mathrm{H}^1(k,G)$ takes $[W']$ to $[Z]$.
- The point $g \in G(K)$ lifts to a point $h \in H_3(K)$.
- The point $z \in Z(K)$ lifts to a point $y \in W'(K)$.

Since the claim in the lemma holds for W' (because $H_3^{\mathrm{ss}} = H_3^{\mathrm{sc}}$), it also holds for Z by functoriality. □

Let $\sigma \in \mathrm{Z}^1(k,G)$ be a Galois cycle. The complexes C and $\widehat{C}$ remain unaffected after twisting by G (see Proposition 2.5 (ii)). By Lemma 3.1, we have

$$\mathrm{Br}_{\mathrm{nr}}(Y^\sigma) \subseteq \mathrm{Br}_1(Y^\sigma/X) := \mathrm{Ker}(\mathrm{Br}(Y^\sigma) \to \mathrm{Br}(\bar{Y})/f^*(\mathrm{Br}(\bar{X}))),$$

and the exact sequence (3.4) for the twisted torsor $f^\sigma : Y^\sigma \to X$ is

$$\mathrm{Br}(X) \xrightarrow{(f^\sigma)^*} \mathrm{Br}_1(Y^\sigma/X) \xrightarrow{\varphi^\sigma} \mathrm{H}^1(k,\widehat{C}) \xrightarrow{\delta} \mathrm{H}^3(X,\mathbb{G}_m). \tag{3.5}$$

The following is an analogue of [HW20, Proposition 2.3].

Proposition 3.6. *Let $Z := G_\sigma$ be the left torsor under G defined by σ (which is also a right torsor under G^σ), so that $Y^\sigma = Y \times_k^G Z$ (see Subsection 1.3). Then, we have an exact sequence (cf. (3.4))*

$$\mathrm{Br}(k) \to \mathrm{Br}_1(Z) \xrightarrow{\varphi} \mathrm{H}^1(k,\widehat{C}) \to \mathrm{H}^3(k,\mathbb{G}_m). \tag{3.6}$$

(i) *Suppose that $\mathrm{H}^3(k,\mathbb{G}_m) = 0$. Let $\pi : Y \times_k Z \to Y^\sigma$, $p_1 : Y \times_k Z \to Y$ and $p_2 : Y \times_k Z \to Z$ be the canonical projections. Then, for every $\alpha^\sigma \in \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$, there exist $\alpha \in \mathrm{Br}_1(Z)$ and $\alpha^1 \in \mathrm{Br}_{\mathrm{nr}}(Y)$ such that $\varphi^\sigma(\alpha^\sigma) = \varphi^1(\alpha^1) = \varphi(\alpha)$ in $\mathrm{H}^1(k,\widehat{C})$ and $p_1^*\alpha^1 + p_2^*\alpha = \pi^*\alpha^\sigma$ in $\mathrm{Br}_1((Y \times_k Z)/X)$.*

Also, the subgroup $\varphi^\sigma(\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)) \subseteq \mathrm{H}^1(k,\widehat{C})$ does not depend on σ.

(ii) *The preimage of $\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$ by $(f^\sigma)^* : \mathrm{Br}(X) \to \mathrm{Br}(Y^\sigma)$ does not depend on σ.*

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Proof.

(i) Since $\mathrm{H}^3(k,\mathbb{G}_m)=0$, the map φ from (3.6) is surjective; hence $\varphi^\sigma(\alpha^\sigma)=\varphi(\alpha)$ for some $\alpha\in \mathrm{Br}_1(Z)$. Take a nonempty open subset $U\subseteq X$ such that the restriction $V:=f^{-1}(U)\xrightarrow{f|_V} U$ is an isotrivial torsor under G. When we replace f^σ by $f^\sigma|_{V^\sigma}$ in the sequence (3.5), the map $\delta:\mathrm{H}^1(k,\widehat{C})\to \mathrm{H}^3(U,\mathbb{G}_m)$ is given by

$$\delta(\widehat{c})=-\mathrm{ab}^1([V])\cup p^*\widehat{c}+p^*([\sigma]\cup\widehat{c})=-\mathrm{ab}^1([V])\cup p^*\widehat{c},$$

thanks to Corollary 2.9 and the fact that $[\sigma]\cup\widehat{c}\in \mathrm{H}^3(k,\mathbb{G}_m)=0$ (the cup-product being induced by the pairing (2.3)). Thus, this map does not depend on σ. It follows that the subgroup $\varphi^\sigma(\mathrm{Br}_1(V^\sigma/U))\subseteq \mathrm{H}^1(k,\widehat{C})$ does not depend on σ. Hence, we may write $\varphi^\sigma(\alpha^\sigma)=\varphi^1(\alpha^1)$ for some $\alpha^1\in \mathrm{Br}_1(V/U)$. Let us now make the necessary modifications to α^1 and α so that the conditions in the statement are fulfilled. From now on, the argument is the same as in [HW20, Proposition 2.3 (i) and (ii)]. Let $q:Z\to \mathrm{Spec}(k)$ denote the structure morphism and let $h:=f\times_k q=f\circ p_1=f^\sigma\circ\pi$. Then $h:Y\times_k Z\to X$ is a torsor under $G\times_k G^\sigma$ (a connected reductive k-group with dual algebraic fundamental complex $\widehat{C}\oplus\widehat{C}$). Let us now consider the distinguished triangles (2.6) (deduced from (2.5) and Proposition 2.6) associated with the torsors $f^\sigma|_{V^\sigma}:V^\sigma\to U$, $h|_{V\times_k Z}:V\times_k Z\to U$, $f|_V:V\to U$, and $q:Z\to\mathrm{Spec}(k)$. By functoriality and by Proposition 2.11, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{G}_{m,U} & \longrightarrow & \tau_{\le 1}\mathbb{R}f^\sigma_*\mathbb{G}_{m,V^\sigma} & \longrightarrow & \widehat{C}_U[-1] & \longrightarrow & \mathbb{G}_{m,U}[1] \\
\| & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \Delta[-1]} & & \| \\
\mathbb{G}_{m,U} & \longrightarrow & \tau_{\le 1}\mathbb{R}h_*\mathbb{G}_{m,V^\sigma} & \longrightarrow & (\widehat{C}_U\oplus\widehat{C}_U)[-1] & \longrightarrow & \mathbb{G}_{m,U}[1] \\
\uparrow{\scriptstyle \nabla} & & \uparrow{\scriptstyle p_1^*+p_2^*} & & \| & & \uparrow{\scriptstyle \nabla} \\
\mathbb{G}_{m,U}\oplus\mathbb{G}_{m,U} & \longrightarrow & (\tau_{\le 1}\mathbb{R}f_*\mathbb{G}_{m,V})\oplus p^*(\tau_{\le 1}\mathbb{R}q_*\mathbb{G}_{m,Z}) & \longrightarrow & (\widehat{C}_U\oplus\widehat{C}_U)[-1] & \longrightarrow & (\mathbb{G}_{m,U}\oplus\mathbb{G}_{m,U})[1],
\end{array}$$

where $\Delta:\widehat{C}_U\to\widehat{C}_U\oplus\widehat{C}_U$ denotes the diagonal map and where $\nabla:\mathbb{G}_{m,U}\oplus\mathbb{G}_{m,U}\to\mathbb{G}_{m,U}$ is the addition map. Now, the exact sequences of the shape (3.4) associated with the rows of the above diagram are given by

$$\begin{array}{ccccc}
\mathrm{Br}(U) & \xrightarrow{(f^\sigma)^*} & \mathrm{Br}_1(V^\sigma/U) & \xrightarrow{\varphi^\sigma} & \mathrm{H}^1(k,\widehat{C}) \\
\| & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \Delta} \\
\mathrm{Br}(U) & \xrightarrow{h^*} & \mathrm{Br}_1((V\times_k Z)/U) & \longrightarrow & \mathrm{H}^1(k,\widehat{C})\oplus\mathrm{H}^1(k,\widehat{C}) \\
\uparrow{\scriptstyle \mathrm{id}_{\mathrm{Br}(U)}+p^*} & & \uparrow{\scriptstyle p_1^*+p_2^*} & & \| \\
\mathrm{Br}(U)\oplus\mathrm{Br}(k) & \xrightarrow{f^*\oplus q^*} & \mathrm{Br}_1(V/U)\oplus\mathrm{Br}_1(Z) & \xrightarrow{\varphi^1\oplus\varphi} & \mathrm{H}^1(k,\widehat{C})\oplus\mathrm{H}^1(k,\widehat{C}).
\end{array}$$

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Since $\Delta(\varphi^\sigma(\alpha^\sigma)) = (\varphi^\sigma(\alpha^\sigma), \varphi^\sigma(\alpha^\sigma)) = (\varphi^1(\alpha^1), \varphi(\alpha))$, the elements $p_1^*\alpha^1 + p_2^*\alpha$ and $\pi^*\alpha^\sigma$ have the same image in $\mathrm{H}^1(k, \widehat{C}) \oplus \mathrm{H}^1(k, \widehat{C})$. By exactness of the middle row, there exists $\beta \in \mathrm{Br}(U)$ such that $p_1^*\alpha^1 + p_2^*\alpha = \pi^*\alpha^\sigma + h^*\beta$, or

$$p_1^*(\alpha^1 - f^*\beta) + p_2^*\alpha = \pi^*\alpha^\sigma.$$

On the other hand, $\varphi^1(\alpha^1 - f^*\beta) = \varphi^1(\alpha^1)$ since $\varphi^1 \circ f^* = 0$. Replacing α^1 by $\alpha^1 - f^*\beta$, we may assume that $\varphi^\sigma(\alpha^\sigma) + \varphi^1(\alpha^1) = \varphi(\alpha)$ and $p_1^*\alpha^1 + p_2^*\alpha = \pi^*\alpha^\sigma$. It remains to check that this implies $\alpha^1 \in \mathrm{Br}_{\mathrm{nr}}(Y)$. Indeed, since $\alpha^\sigma \in \mathrm{Br}_{\mathrm{nr}}(V^\sigma) = \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$, we have $p_1^*\alpha^1 + p_2^*\alpha \in \mathrm{Br}_{\mathrm{nr}}(Y \times_k Z)$. Let ξ be any codimension 1 point on a smooth compactification of Y, and let η denote the generic point $\xi \times_k Z$ (which is a $k(\xi)$-torsor under $G_{k(\xi)}$). Then η is a codimension 1 point on a smooth compactification of $Y \times_k Z$, hence $\partial_\eta(p_1^*\alpha^1 + p_2^*\alpha) = 0$. It follows that $\partial_\eta(p_1^*\alpha^1) = 0 \in \mathrm{H}^1(k(\eta), \mathbb{Q}/\mathbb{Z})$ because $p_2^*\alpha \in \mathrm{Br}(Y \times_k Z)$. Since $\xi \times_k Z$ is geometrically integral, the restriction map $\mathrm{H}^1(k(\xi), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k(\eta), \mathbb{Q}/\mathbb{Z})$ is injective. By functoriality of the residue map [CTS21, Theorem 3.7.5], we have $\partial_\xi(\alpha^1) = 0 \in \mathrm{H}^1(k(\xi), \mathbb{Q}/\mathbb{Z})$. Since ξ is arbitrary, we conclude that $\alpha^1 \in \mathrm{Br}_{\mathrm{nr}}(Y)$.

We have shown that $\varphi^\sigma(\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)) \subseteq \varphi^1(\mathrm{Br}_{\mathrm{nr}}(Y))$ for any torsor $Y \to X$ under any reductive group G and any $\sigma \in \mathrm{Z}^1(k, G)$. If $\sigma, \tau \in \mathrm{Z}^1(k, G)$, then $\varphi^\sigma(\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)) \subseteq \varphi^\tau(\mathrm{Br}_{\mathrm{nr}}(Y^\tau))$ (resp. $\varphi^\tau(\mathrm{Br}_{\mathrm{nr}}(Y^\tau)) \subseteq \varphi^\sigma(\mathrm{Br}_{\mathrm{nr}}(Y^\sigma))$) thanks to a twisting argument by τ (resp. σ). It follows that $\varphi^\tau(\mathrm{Br}_{\mathrm{nr}}(Y^\tau)) = \varphi^\sigma(\mathrm{Br}_{\mathrm{nr}}(Y^\sigma))$.

(ii) It suffices to repeat the proof of [HW20, Proposition 2.3 (iii)]. □

We can now proceed to prove Theorem 3.4. Keep the notations and assumptions therein.

Proof of Theorem 3.4. For any Galois cocycle $\sigma \in \mathrm{Z}^1(k, G)$, the subgroup $A \subseteq \mathrm{Br}(X)$ is the preimage of $\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$ by $(f^\sigma)^* : \mathrm{Br}(X) \to \mathrm{Br}(Y^\sigma)$, thanks to Proposition 3.6 (ii). The inclusion

$$\bigcup_{[\sigma] \in \mathrm{H}^1(k, G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}) \subseteq X(\mathbb{A}_k)^A$$

is then obvious, by functoriality of the Brauer–Manin pairing ($\mathrm{BM}_{\mathbb{A}}$). Let us take a family $x_{\mathbb{A}} = (x_v)_{v \in \Omega_k} \in X(\mathbb{A}_k)^A$ and show that it can be lifted to $Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}$ for some $\sigma \in \mathrm{Z}^1(k, G)$. First, we try to lift $x_{\mathbb{A}}$ to $Y^\sigma(\mathbb{A}_k)$ by exploiting the orthogonality to $\mathrm{Ker}(f^*)$. This has already been done in [CDX19, Proof of Theorem 3.5]. We present here an alternative proof in the spirit of [HW20, Proposition 2.5] for the sake of uniformity.

Lemma 3.7. *There exists a Galois cocycle $\sigma \in \mathrm{Z}^1(k, G)$ such that $x_{\mathbb{A}} \in f^\sigma(Y^\sigma(\mathbb{A}_k))$.*

Proof. Since the torsor $Y \times_X Y \to Y$ under G is trivial, the map $f^* : \mathrm{H}^1(X, G) \to \mathrm{H}^1(Y, G)$ takes $[Y]$ to 1, hence $f^*(\mathrm{ab}^1([Y])) = 0$ in $\mathrm{H}^1(Y, C)$ by functoriality of ab^1. It follows that $f^*(\mathrm{ab}^1([Y]) \cup p^*\widehat{c}) = 0$ for any $\widehat{c} \in \mathrm{H}^0(k, \widehat{C})$ (the cup-product being induced by the pairing (2.3)), or $\mathrm{ab}^1([Y]) \cup p^*\widehat{c} \in \mathrm{Ker}(f^*) \subseteq A$. Since $x_{\mathbb{A}} \in X(\mathbb{A}_k)^A$, the family $\mathrm{ab}^1([Y](x_{\mathbb{A}})) \in \mathbb{P}^1(k, C)$ is orthogonal to $\mathrm{H}^0(k, \widehat{C})$. By the Poitou–Tate exact sequence for C (see [Dem11b,

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Théorème 6.1]), there exists $c \in \mathrm{H}^1(k,C)$ such that $\mathrm{ab}^1([Y](x_v)) = \mathrm{loc}_v(c) \in \mathrm{H}^1(k_v,C)$ for all $v \in \Omega_k$.

Let us now apply Proposition 2.4 (iii). First, let $c = \mathrm{ab}^1([\sigma])$ for some cocycle $\sigma \in \mathrm{Z}^1(k,G)$, so that $\mathrm{ab}^1([Y](x_v)) = \mathrm{ab}^1(\mathrm{loc}_v([\sigma]))$ in $\mathrm{H}^1(k_v,C)$ for all $v \in \Omega_k$. By Proposition 2.5 (iii), this is equivalent to $\mathrm{ab}^1([Y^\sigma](x_v)) = 0$. Twisting by σ, we may assume that $\mathrm{ab}^1([Y](x_v)) = 0$ for all $v \in \Omega_k$. If $v \in \Omega_k^{\mathrm{f}}$, this means $[Y](x_v) = 1 \in \mathrm{H}^1(k_v,G)$ by virtue of Proposition 2.4 (ii). Since $\mathrm{H}^1(k,G)$ is the fibre product of $\prod_{v\in\Omega_k^\infty} \mathrm{H}^1(k_v,G)$ and $\mathrm{H}^1(k,C)$ over $\prod_{v\in\Omega_k^\infty} \mathrm{H}^1(k_v,C)$, there exists $\sigma \in \mathrm{Z}^1(k,G)$ such that $[Y](x_v) = \mathrm{loc}_v([\sigma])$ for all $v \in \Omega_k^\infty$ and $\mathrm{ab}^1([\sigma]) = 0$ (in particular, $\mathrm{loc}_v([\sigma]) = 1$ for $v \in \Omega_k^{\mathrm{f}}$). It follows that $[Y^\sigma](x_v) = 1$ in $\mathrm{H}^1(k_v,G^\sigma)$ for all $v \in \Omega_k$, *i.e.*, $x_{\mathbb{A}} \in f^\sigma(Y^\sigma(\mathbb{A}_k))$. □

Twisting by the cocycle provided by Lemma 3.7, we may assume that $x_{\mathbb{A}} \in f(Y(\mathbb{A}_k))$. Then $x_{\mathbb{A}} \in f^\sigma(Y^\sigma(\mathbb{A}_k))$ for any cocycle σ such that $[\sigma] \in Ш(G)$. For such a cocycle, we define a map

$$\varepsilon_\sigma : Ш^1(k,\widehat{C}) \cap \varphi^1(\mathrm{Br}_{\mathrm{nr}}(Y)) \to \mathbb{Q}/\mathbb{Z}, \quad \alpha' \mapsto \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}}, \tag{3.7}$$

where $y_{\mathbb{A}} = (y_v)_{v\in\Omega_k} \in Y^\sigma(\mathbb{A}_k) = Y(\mathbb{A}_k)$ is any lifting of $x_{\mathbb{A}}$ and where $\alpha^\sigma \in \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$ is such that $\varphi^\sigma(\alpha^\sigma) = \alpha'$ (which exists by Proposition 3.6 (i)). The map ε_σ is well-defined:

- It does not depend on the choice of α^σ. Indeed, upon replacing α^σ by $\alpha^\sigma + (f^\sigma)^*\beta \in \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$ (in view of the exact sequence (3.5)) for some $\beta \in A$, one has
$$\begin{aligned}\langle y_{\mathbb{A}}, \alpha^\sigma + (f^\sigma)^*\beta \rangle_{\mathrm{BM}} &= \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}} + \langle y_{\mathbb{A}}, (f^\sigma)^*\beta \rangle_{\mathrm{BM}} \\ &= \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}} + \langle f^\sigma(y_{\mathbb{A}}), \beta \rangle_{\mathrm{BM}} = \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}},\end{aligned}$$
where the last equality is due to the fact that $f^\sigma(y_{\mathbb{A}}) = x_{\mathbb{A}} \in X(\mathbb{A}_k)^A$.
- It does not depend on the choice of $y_{\mathbb{A}}$. Indeed, since $Y \to X$ is a torsor under G, upon replacing $y_{\mathbb{A}}$ by $y_{\mathbb{A}} \cdot g_{\mathbb{A}} := (y_v \cdot g_v)_{v\in\Omega_k}$ for some $g_{\mathbb{A}} = (g_v)_{v\in\Omega_k} \in \mathbb{G}(\mathbb{A}_k)$, one has
$$\langle y_{\mathbb{A}} \cdot g_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}} = \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}} + \sum_{v\in\Omega_k} \mathrm{ab}^0(g_v) \cup \mathrm{loc}_v(\alpha')$$
by Lemma 3.5 applied to each fibre $(f^\sigma)^{-1}(x_v) \to \mathrm{Spec}(k_v)$. But $\mathrm{loc}_v(\alpha') = 0$ for all $v \in \Omega_k$ since $\alpha' \in Ш^1(k,\widehat{C})$, hence $\langle y_{\mathbb{A}} \cdot g_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}} = \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}}$.

In what follows, we denote by $\langle -,- \rangle_{\mathrm{PT}} : Ш^1(k,C) \times Ш^1(k,\widehat{C}) \to \mathbb{Q}/\mathbb{Z}$ the global Poitou–Tate pairing induced by (2.3), which is a perfect duality of finite groups (see [Dem11b, Théorème 5.7]).

Lemma 3.8. *Let $[\sigma] \in Ш(G)$. Let Z be the k-torsor under G^σ in Proposition 3.6 (hence $Z(\mathbb{A}_k) \neq \varnothing$) and φ the map from (3.6). For any $\alpha \in Б(Z)$ (cf. (1.2)) and $z_\Omega \in Z(k_\Omega)$, one has*

$$\langle z_\Omega, \alpha \rangle_{\mathrm{BM}} = -\big\langle \mathrm{ab}^1([\sigma]), \varphi(\alpha) \big\rangle_{\mathrm{PT}}.$$

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Proof. Since $\alpha \in Б(Z)$, it suffices to show the identity in the statement for any family z_Ω, which can be chosen at our disposal. We deploy the same trick as in the proof of Lemma 3.5. If G is a torus, the result is [San81, Lemme 8.4]. When $G^{\mathrm{ss}} = G^{\mathrm{sc}}$, we have $C \simeq G^{\mathrm{tor}}$, and the map ab^1 is just the natural pushforward $\mathrm{H}^1(k,G) \to \mathrm{H}^1(k,G^{\mathrm{tor}})$. Let $Z' \to \mathrm{Spec}(k)$ denote the k-torsor $Z \times_k^G G^{\mathrm{tor}}$ under G_{tor}. Note that the map φ induces an isomorphism $Б(Z)/\mathrm{Br}(k) \xrightarrow{\cong} Ш^1(k,\widehat{C})$, and similarly, $Б(Z')/\mathrm{Br}(k) \cong Ш^1(k,\widehat{C}) \cong Б(Z)/\mathrm{Br}(k)$. By functoriality and the toric case, the result also holds for the case $G^{\mathrm{ss}} = G^{\mathrm{sc}}$. In the general case, it suffices to show that there exist a surjection $\varpi : H \to G$, where H is a connected reductive k-group with $H^{\mathrm{ss}} = H^{\mathrm{sc}}$ such that $[\sigma]$ lifts to $Ш(H)$. Indeed, as in [Kot86, p. 369], there is a *z-extension* $1 \to T \to H \to G \to 1$, *i.e.*, a t-resolution with T quasi-trivial, such that $[\sigma] \in \mathrm{H}^1(k,G)$ lifts to a class $[\tau] \in \mathrm{H}^1(k,H)$. Since $[\sigma] \in Ш(G)$ and $\mathrm{H}^1(k_v,T) = 0$ for all $v \in \Omega_k$, one has $[\tau] \in Ш(H)$. This concludes the proof of the lemma. □

Lemma 3.8 allows us to relate the maps ε_1 and ε_σ from (3.7) as follows. Let Z be as above, then $Y^\sigma = Y \times_k^G Z$. Let $\pi : Y \times_k Z \to Y^\sigma$, $p_1 : Y \times_k Z \to Y$ and $p_2 : Y \times_k Z \to Z$ be the canonical projections. For any $\alpha' \in Ш^1(k,\widehat{C}) \cap \varphi^1(\mathrm{Br}_{\mathrm{nr}}(Y))$, by Proposition 3.6 (i), there are $\alpha \in \mathrm{Br}_1(Z)$, $\alpha^1 \in \mathrm{Br}_{\mathrm{nr}}(Y)$ and $\alpha^\sigma \in \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$ such that $\varphi(\alpha) = \varphi^1(\alpha^1) = \varphi^\sigma(\alpha^\sigma) = \alpha'$ and $p_1^*\alpha^1 + p_2^*\alpha = \pi^*\alpha^\sigma$. Take any $z_{\mathbb{A}} \in Z(\mathbb{A}_k)$, then

$$\begin{aligned}
\varepsilon_\sigma(\alpha') &= \langle y_{\mathbb{A}}, \alpha^\sigma \rangle_{\mathrm{BM}}, && \text{by definition of } \varepsilon_\sigma,\\
&= \langle (y_{\mathbb{A}}, z_{\mathbb{A}}), \pi^*\alpha^\sigma \rangle_{\mathrm{BM}}, && \text{by functoriality of the pairing } (\mathrm{BM}_{\mathbb{A}}),\\
&= \left\langle (y_{\mathbb{A}}, z_{\mathbb{A}}), p_1^*\alpha^1 \right\rangle_{\mathrm{BM}} + \langle (y_{\mathbb{A}}, z_{\mathbb{A}}), p_2^*\alpha \rangle_{\mathrm{BM}} &&\\
&= \left\langle y_{\mathbb{A}}, \alpha^1 \right\rangle_{\mathrm{BM}} + \langle z_{\mathbb{A}}, \alpha \rangle_{\mathrm{BM}}, && \text{by functoriality of the pairing } (\mathrm{BM}_{\mathbb{A}}),\\
&= \varepsilon_1(\alpha') - \left\langle \mathrm{ab}^1([\sigma]), \alpha' \right\rangle_{\mathrm{BM}}, && \text{by definition of } \varepsilon_1 \text{ and Lemma 3.8.}
\end{aligned}$$

Since the group $\mathbb{Q}/\mathbb{Z}$ is divisible, the homomorphism ε_1 can be extended to the whole group $Ш^1(k,\widehat{C})$. Since the Poitou–Tate pairing $\langle -,- \rangle_{\mathrm{PT}} : Ш^1(k,C) \times Ш^1(k,\widehat{C}) \to \mathbb{Q}/\mathbb{Z}$ is a perfect duality and since the map $\mathrm{ab}^1 : \mathrm{H}^1(k,G) \to \mathrm{H}^1(k,C)$ induces a bijection $Ш(G) \xrightarrow{\cong} Ш^1(k,C)$ by Proposition 2.4 (iii), there exists $[\sigma] \in Ш(G)$ such that $\varepsilon_1(\alpha') = \left\langle \mathrm{ab}^1([\sigma]), \alpha' \right\rangle_{\mathrm{BM}}$ for all $\alpha' \in Ш^1(k,\widehat{C}) \cap \varphi^1(\mathrm{Br}_{\mathrm{nr}}(Y))$. By the above calculation, we have $\varepsilon_\sigma = 0$. Twisting by the cocycle σ as above, we may assume that $\varepsilon_1 = 0$. The rest of the proof is to show that this implies $x_{\mathbb{A}} \in f(Y(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y)})$. As always, we fix an adelic point $y_{\mathbb{A}} \in Y(\mathbb{A}_k)$ lifting $x_{\mathbb{A}}$, and consider the homomorphism $\varepsilon : \mathrm{Br}_{\mathrm{nr}}(Y) \to \mathbb{Q}/\mathbb{Z}$ given by $\varepsilon(\alpha^1) := \left\langle y_{\mathbb{A}}, \alpha^1 \right\rangle_{\mathrm{BM}}$. Then, one has

$$\varepsilon|_{\mathrm{Br}_{\mathrm{nr}}(Y) \cap (\varphi^1)^{-1}(Ш^1(k,\widehat{C}))} = \varepsilon_1 \circ \varphi^1|_{\mathrm{Br}_{\mathrm{nr}}(Y) \cap (\varphi^1)^{-1}(Ш^1(k,\widehat{C}))} = 0.$$

Hence, ε factors through a homomorphism $\bar{\varepsilon} : \frac{\mathrm{Br}_{\mathrm{nr}}(Y)}{\mathrm{Br}_{\mathrm{nr}}(Y) \cap (\varphi^1)^{-1}(Ш^1(k,\widehat{C}))} \to \mathbb{Q}/\mathbb{Z}$. On the other hand, the map φ^1 induces an injection $\bar{\varphi}^1 : \frac{\mathrm{Br}_{\mathrm{nr}}(Y)}{\mathrm{Br}_{\mathrm{nr}}(Y) \cap (\varphi^1)^{-1}(Ш^1(k,\widehat{C}))} \hookrightarrow \frac{\mathrm{H}^1(k,\widehat{C})}{Ш^1(k,\widehat{C})}$. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible group, there exists a map $\bar{\varepsilon}' : \frac{\mathrm{H}^1(k,\widehat{C})}{Ш^1(k,\widehat{C})} \to \mathbb{Q}/\mathbb{Z}$ such that $\bar{\varepsilon}' \circ \bar{\varphi}^1 = \bar{\varepsilon}$, or

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equivalently, a map $\varepsilon' : \mathrm{H}^1(k,\widehat{C}) \to \mathbb{Q}/\mathbb{Z}$ such that $\varepsilon'|_{Ш^1(k,\widehat{C})} = 0$ and $\varepsilon' \circ \varphi^1 = \varepsilon$. By Proposition 2.4 (i), one has a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & Ш(G) & \xrightarrow{\cong} & Ш^1(k,C) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \mathrm{H}^0(k,C) & \longrightarrow & \mathrm{H}^1(k,G^{\mathrm{sc}}) & \longrightarrow & \mathrm{H}^1(k,G) & \xrightarrow{\mathrm{ab}^1} & \mathrm{H}^1(k,C) \\
 & & \downarrow & & \downarrow\cong & & \downarrow & & \downarrow \\
G(\mathbb{A}_k) & \xrightarrow{\mathrm{ab}^0} & \mathbb{P}^0(k,C) & \longrightarrow & \mathbb{P}^1(k,G^{\mathrm{sc}}) & \longrightarrow & \mathbb{P}^1(k,G) & \xrightarrow{\mathrm{ab}^1} & \mathbb{P}^1(k,C) \\
 & & \downarrow & & & & & & \\
 & & \mathrm{H}^1(k,\widehat{C})^D & & & & & & \\
 & & \downarrow & & & & & & \\
 & & Ш^1(k,\widehat{C})^D. & & & & & &
\end{array}$$

Actually, the first column from the left is a complex, which is exact at the term $\mathrm{H}^1(k,\widehat{C})^D$ [Dem11a, Théorème 2.9]. The arrow on the second column is bijective (Kneser–Harder–Chernousov; see [PR94, Theorems 6.4 and 6.6]). The other columns are obviously exact. Since $\varepsilon' \in \mathrm{H}^1(k,\widehat{C})^D$ is mapped to $0 \in Ш^1(k,\widehat{C})^D$, it comes from an element $c_{\mathbb{A}} = (c_v)_{v\in\Omega_k} \in \mathbb{P}^0(k,C)$. Its image $\gamma_{\mathbb{A}}$ in $\mathbb{P}^1(k,G^{\mathrm{sc}})$ comes from a unique element $\gamma \in \mathrm{H}^1(k,G^{\mathrm{sc}})$, whose image δ in $\mathrm{H}^1(k,G)$ lies in $Ш(G)$. Since $\mathrm{ab}^1(\delta) = 0$, we have $\delta = 1$ by Proposition 2.4 (iii). It follows that γ comes from an element $c \in \mathrm{H}^0(k,C)$. Its localization $\mathrm{loc}_\Omega(c) := (\mathrm{loc}_v(c))_{v\in\Omega_k} \in \mathbb{P}^0(k,C)$ has the same image in $\mathbb{P}^1(k,G^{\mathrm{sc}})$ as that of $c_{\mathbb{A}}$, hence there exists $g_{\mathbb{A}} = (g_v)_{v\in\Omega_k} \in G(\mathbb{A}_k)$ such that $\mathrm{ab}^0(g_v) = \mathrm{loc}_v(c) - c_v$ for all $v \in \Omega_k$. Since the first column is a complex, the image of $(\mathrm{ab}^0(g_v))_{v\in\Omega_k}$ in $\mathrm{H}^1(k,\widehat{C})^D$ is the same as that of $-c_{\mathbb{A}}$, *i.e.*, $-\varepsilon'$. Let $y_{\mathbb{A}} \cdot g_{\mathbb{A}} := (y_v \cdot g_v)_{v\in\Omega_k}$. By virtue Lemma 3.5, we have, for all $\alpha^1 \in \mathrm{Br}_{\mathrm{nr}}(Y)$ (noting that $\varepsilon = \varepsilon' \circ \varphi^1$),

$$\langle y_{\mathbb{A}} \cdot g_{\mathbb{A}}, \alpha^1 \rangle_{\mathrm{BM}} = \langle y_{\mathbb{A}}, \alpha^1 \rangle_{\mathrm{BM}} + \sum_{v\in\Omega_k} \mathrm{ab}^0(g_v) \cup \varphi^1(\alpha^1) = \varepsilon(\alpha^1) - \varepsilon'(\varphi^1(\alpha^1)) = 0.$$

We have found a lifting $y_{\mathbb{A}} \cdot g_{\mathbb{A}} \in Y(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y)}$ of $x_{\mathbb{A}} \in X(\mathbb{A}_k)^A$, and this concludes the proof of Theorem 3.4. □

3.3. End of proof of the main theorem

Let us now finish the proof of Theorem A, using Theorem 3.4 from the previous subsection. The first step is to drop the reductive assumption.

Theorem 3.9. *Let G be a connected (not necessarily reductive) linear algebraic group over a number field k. Let X be a smooth geometrically integral k-variety and $f : Y \to X$ a*

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torsor under G. Let $A \subseteq \mathrm{Br}(X)$ denote the preimage of $\mathrm{Br}_{\mathrm{nr}}(Y)$ by the map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$. Then

$$X(\mathbb{A}_k)^A = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}).$$

Proof. Let G^{u} be the unipotent radical of G and $G^{\mathrm{red}} := G/G^{\mathrm{u}}$ (which is connected reductive). Letting $Z := Y/G^{\mathrm{u}}$, we obtain a torsor $g : Z \to X$ under G^{red} and a torsor $h : Y \to Z$ under G^{u}. Now, [San81, Lemme 1.13] tells us that $\mathrm{H}^1(K,(G^{\mathrm{u}})^\sigma) = 1$ for any field extension K/k and any cocycle $\sigma \in \mathrm{Z}^1(k,G)$ and that the map $\mathrm{H}^1(k,G) \to \mathrm{H}^1(k,G^{\mathrm{red}})$ is bijective. For every $[\tau] \in \mathrm{H}^1(k,G^{\mathrm{red}})$, there is a unique class $[\sigma] \in \mathrm{H}^1(k,G)$ lifting $[\tau]$, hence the torsor $f^\sigma : Y^\sigma \to X$ has a factorization $Y^\sigma \xrightarrow{h^\sigma} Z^\tau \xrightarrow{g^\tau} X$. Since h^σ is a torsor under $(G^{\mathrm{u}})^\sigma$ and $\mathrm{H}^1(k(Z^\tau),(G^{\mathrm{u}})^\sigma) = 1$, it is a stable birational equivalence, hence induces an isomorphism $(h^\sigma)^* : \mathrm{Br}_{\mathrm{nr}}(Z^\tau) \xrightarrow{\cong} \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$. In particular, A is equal to the preimage of $\mathrm{Br}_{\mathrm{nr}}(Z)$ by the map $g^* : \mathrm{Br}(X) \to \mathrm{Br}(Z)$. In view of Theorem 3.4, it remains to show that $h^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}) = Z^\tau(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Z^\tau)}$. The inclusion '$\subseteq$' is obvious. Conversely, let $z_{\mathbb{A}} \in Z^\tau(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Z^\tau)}$. Viewing that $\mathrm{H}^1(k_v,(G^{\mathrm{u}})^\sigma) = 1$ for all $v \in \Omega_k$, we may lift $z_{\mathbb{A}}$ to a family $y_{\mathbb{A}} \in Y^\sigma(\mathbb{A}_k)$. Since $\mathrm{Br}_{\mathrm{nr}}(Z^\tau) \cong \mathrm{Br}_{\mathrm{nr}}(Y^\sigma)$, we have $y_{\mathbb{A}} \in Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}$ by functoriality of the pairing ($\mathrm{BM}_{\mathbb{A}}$). The theorem is proved. □

If X is proper, then $\mathrm{Br}_{\mathrm{nr}}(X) = A = \mathrm{Br}(X)$, and the statement of Theorem 3.9 becomes

$$X(k_\Omega)^{\mathrm{Br}(X)} = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}) = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}),$$

which implies Conjecture B for the torsor $f : Y \to X$. Under the rationally connected assumption, one can establish Theorem 3.11 below (*i.e.*, Theorem A) without properness. To this end, it will be sufficient to deploy *Harari's formal lemma* [Har94, Théorème 2.1.1, Corollaire 2.6.1].

Lemma 3.10. *Let X be a smooth geometrically integral variety over a field k of characteristic 0.*

(i) *If X is rationally connected, then $\mathrm{Br}_{\mathrm{nr}}(X)/\mathrm{Br}(k)$ is finite.*

(ii) *Suppose that k is a number field and $A \subseteq \mathrm{Br}(X)$ is a subgroup containing $\mathrm{Br}_{\mathrm{nr}}(X)$ such that $A/\mathrm{Br}(k)$ is finite. Then, one has $\overline{X(\mathbb{A}_k)^A} = X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ in the topological space $X(k_\Omega)$ (both sides can be empty).*

Proof.

(i) Combine [CTS13, Lemma 1.3] and [Deb01, Corollary 4.18].

(ii) Let X^c be a smooth compactification of X (so that $\mathrm{Br}_{\mathrm{nr}}(X) = \mathrm{Br}(X^c)$) and apply [CTS00, Proposition 1.1] (which is a consequence of Harari's formal lemma) to the inclusion $X \subseteq X^c$. This result says that $\overline{X(\mathbb{A}_k)^A} = X^c(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ in $X^c(k_\Omega)$, *a fortiori* $\overline{X(\mathbb{A}_k)^A} = X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ in $X(k_\Omega)$. □

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Theorem 3.11 (Theorem A). *Let k be a number field, X a smooth rationally connected k-variety, and G a connected linear algebraic k-group. If $f : Y \to X$ is a torsor under G, then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)})}$$

in the topological space $X(k_\Omega)$. In particular, if the varieties Y^σ satisfy (**BMHP**) *(resp.* (**BMWA**)*) for all $[\sigma] \in \mathrm{H}^1(k,G)$, then it is also the case for X.*

Proof. The inclusion '$\supseteq$' is obvious by functoriality and continuity of the pairing ($\mathrm{BM}_{\mathrm{nr}}$), so let us show the inclusion '$\subseteq$'. Let $A \subseteq \mathrm{Br}(X)$ denote the preimage of $\mathrm{Br}_{\mathrm{nr}}(Y)$ by the map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$. By Theorem 3.9, we have

$$X(\mathbb{A}_k)^A = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}).$$

Thus, it remains to show that $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{X(k_\Omega)^A}$. In view of Lemma 3.10 (ii), we are reduced to show that the group $A/\mathrm{Br}(k)$ is finite. Since X is rationally connected, it follows from the Graber–Harris–Starr theorem [GHS03] that Y is also rationally connected (see [Deb03, Corollaire 3.1]). By Lemma 3.10 (i), the group $\mathrm{Br}_{\mathrm{nr}}(Y)/\mathrm{Br}(k)$ is finite. Thus, to establish the finiteness of $A/\mathrm{Br}(k)$, it is enough to show that $\mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(Y))$ is finite. Using Sansuc's units-Picard-Brauer exact sequence [San81, (6.10.1)], we are reduced to prove the finiteness of $\mathrm{Pic}(G)$. The same sequence tells us that this property is stable under extensions of groups. Hence, we may deal separately with the following cases.

- If G is unipotent, then $\mathrm{Pic}(G) = 0$ since the k-variety G is isomorphic to an affine space.
- If G is a torus, then $\mathrm{Pic}(G) = \mathrm{H}^1(k,\widehat{G})$ by [San81, Lemme 6.9 (ii)]. Let K/k be a finite Galois extension splitting G, then $\mathrm{H}^1(K,\widehat{G}) = 0$, so $\mathrm{H}^1(k,\widehat{G}) = \mathrm{H}^1(\mathrm{Gal}(K/k),\widehat{G})$ by the inflation-restriction sequence. This latter is torsion and finitely generated, hence finite.
- If G is semisimple, by [San81, Lemme 6.9 (iii)], $\mathrm{Pic}(G)$ injects into the character module of the algebraic fundamental group $\pi_1(\bar{G}) = \mathrm{Ker}(\bar{G}^{\mathrm{sc}} \to \bar{G})$, which is finite. □

Having established Theorem 3.11 for torsors, we can now generalize it to the situation where no torsors are given *a priori*. Let us bring the notion of nonabelian descent types from Subsection 2.1 into play.

Theorem 3.12. *Let X be a smooth quasi-projective geometrically integral variety over a number field k. Let $L = (\bar{G},\kappa)$ be a k-kernel, where $\bar{G}$ is a connected linear algebraic $\bar{k}$-group, and $\lambda = [\bar{Y},E]$ a descent type on X bound by L. If X is rationally connected, then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{\substack{f:Y\to X\\ \mathbf{dtype}(Y)=\lambda}} f(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)})}$$

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in the topological space $X(k_\Omega)$. In particular, if the torsors $f : Y \to X$ of descent type λ satisfy (**BMHP**) *(resp.* (**BMWA**)*), then it is also the case for X. By convention, this hypothesis holds vacuously if there are no such torsors at all.*

Proof. By Lemma 2.1 (iii), two X-torsors of descent type λ differ by twisting by a Galois cycle. Hence, deducing the present theorem from Theorem 3.11 amounts to showing that torsors of descent type λ *exist* whenever $X(k_\Omega)^{\mathrm{Br_{nr}}(X)} \neq \varnothing$. To this end, the condition $X(k_\Omega)^{\text{Б}(X)} \neq \varnothing$ is in fact sufficient, where the subgroup $\text{Б}(X) \subseteq \mathrm{Br_{nr}}(X)$ was defined in (1.2). Indeed, this has been proved in [Lin25, Proposition 3.10]. □

As an application of Theorem 3.12, we give a direct proof of Borovoi's theorem on the Brauer–Manin obstruction for homogeneous spaces with connected stabilizers [Bor96, Theorems 2.2 and 2.3], using Sansuc's result for connected linear algebraic groups.

Corollary 3.13. *Let k be a number field. Then, the property* (**BMHP**) *and* (**BMWA**) *hold for homogeneous spaces of connected linear algebraic groups over k with connected geometric stabilizers.*

Proof. Let G be a connected linear algebraic group over k and let X be a homogeneous space of G. Fix a $\bar{k}$-point of X and let $\bar{H} \subseteq \bar{G}$ be its stabilizer, which we assume to be connected. As in [HS02, Proposition 3.3], the subgroup $E_X \subseteq \mathrm{SAut}(\bar{G}/X)$ consisting of semilinear s-automorphisms (for some $s \in \Gamma_k$) for which $\alpha(x \cdot g) = x \cdot {}^s g$ for all $x \in X(\bar{k})$ and $g \in G(\bar{k})$ fits in an exact sequence

$$1 \to \bar{H}(\bar{k}) \to E_X \to \Gamma_k \to 1$$

of topological groups, which defines a k-kernel L_X as well as a descent type $\lambda = [\bar{G}, E_X]$ on X bound by L_X. The torsors $Y \to X$ of descent type λ (if they exist) are left torsors under G (they are k-forms of the variety G twisted by cocycles with values in $\bar{H}(\bar{k})$). According to [San81, Corollaires 8.7 and 8.13], these torsors satisfy (**BMHP**) and (**BMWA**). In fact, Sansuc's original result concerning the property (**BMHP**) requires the absence of E_8 factors, since his proof used the result of Kneser–Harder (see Théorème 4.2 in *loc. cit.*) which excluded the E_8 case. This problem, however, was later resolved by Chernousov [Che89] (see also [PR94, Theorems 6.4 and 6.6]). It follows from Theorem 3.12 that the properties (**BMHP**) and (**BMWA**) hold for X. □

3.4. Reduction to the finite case

Next, let us show that Conjecture B can be reduced to the case of torsors under finite group schemes. Actually, instead of working directly with torsors, it is more convenient to replace them with descent types.

In what follows, when G is an algebraic group over k, we denote by G° its identity component, and $G^{\mathrm{f}} := \pi_0(G) = G/G^\circ$. When $L = (\bar{G}, \kappa)$ is a k-kernel, the outer Galois action κ induces a canonical outer action on $\bar{G}^{\mathrm{f}}$ (since $\bar{G}^\circ$ is characteristic in $\bar{G}$). This gives rise to a k-kernel $L^{\mathrm{f}} = (\bar{G}, \kappa^{\mathrm{f}})$ and a surjective morphism $\pi : L \to L^{\mathrm{f}}$ of k-kernels. If X is a nonempty variety and $\lambda \in \mathrm{DType}(X, L)$, we shall write $\lambda^{\mathrm{f}} := \pi_* \lambda \in \mathrm{DType}(X, L^{\mathrm{f}})$ (*cf.* Lemma 2.2 (i)), which can then be thought of as a finite descent type on X in the sense of

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Harpaz–Wittenberg, *i.e.*, λ^{f} is completely determined by the $\bar{X}$-torsor $\bar{Z} := \bar{Y}/\bar{G}^{\circ}$ under $\bar{G}^{\mathrm{f}}$ (see p. 11).

Theorem 3.14. *Let k be a number field, $L = (\bar{G}, \kappa)$ a k-kernel with $\bar{G}$ linear (not necessarily connected), X a smooth quasi-projective k-variety, and $\lambda = [\bar{Y}, E] \in \mathrm{DType}(X, L)$ a descent type with $\bar{Y}$ rationally connected (in particular, X is rationally connected). Then, one has*

$$\overline{\bigcup_{\substack{f: Y \to X \\ \mathbf{dtype}(Y) = \lambda}} f(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)})} = \overline{\bigcup_{\substack{g: Z \to X \\ \mathbf{dtype}(Z) = \lambda^{\mathrm{f}}}} g(Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)})} \tag{3.8}$$

in the topological space $X(k_\Omega)$. Here, we use the convention that both sides can be empty (for example, when there are no torsors of mentioned descent types at all). Furthermore, consider the following statements.

(i) *If the X-torsors of descent type λ^{f} satisfy* (**BMHP**)*, then X itself satisfies* (**BMHP**).

(ii) *If the X-torsors of descent type λ^{f} satisfy* (**BMWA**)*, then X itself satisfies* (**BMWA**).

(iii) *If the X-torsors of descent type λ satisfy* (**BMHP**)*, then X itself satisfies* (**BMHP**).

(iv) *If the X-torsors of descent type λ satisfy* (**BMWA**)*, then X itself satisfies* (**BMWA**).

Then, (i) implies (iii) and (ii) implies (iv).

Proof. The inclusion '$\subseteq$' of (3.8) follows from the following fact. If $Y \xrightarrow{f} X$ is a torsor under a k-form G of L, of descent type λ, then $Z := Y/G^{\circ} \xrightarrow{g} X$ is a torsor under the k-form G^{f} of L^{f}, of descent type λ^{f} (see *e.g.* [Lin25, Construction 2.6]). Conversely, let $x_\Omega \in X(k_\Omega)$ be a family lying in the right hand side of (3.8). Then, for any neighborhood $\mathscr{U} \subseteq X(k_\Omega)$ of x_Ω, there exists a torsor $g: Z \to X$ of descent type λ^{f} such that $g(Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)}) \cap \mathscr{U} \neq \varnothing$, or $Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)} \cap g^{-1}(\mathscr{U}) \neq \varnothing$. By Lemma 2.2 (ii), we obtain a k-kernel $L_Z = (\bar{G}^{\circ}, \kappa_Z)$ and a descent type $\lambda_Z = [\bar{Y}, E_Z] \in \mathrm{DType}(Z, L_Z)$. By Theorem 3.12, there exists a torsor $h: Y \to Z$ of descent type λ_Z such that $h(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}) \cap g^{-1}(\mathscr{U}) \neq \varnothing$, or $g(h(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)})) \cap \mathscr{U} \neq \varnothing$. By Lemma 2.2 (iii), the composite $g \circ h: Y \to X$ is a torsor of descent type λ. Hence, x_Ω lies in the left hand side of (3.8). The identity (3.8) is thus proved.

Now, we suppose that (i) holds true and proceed to show the contraposition of (iii). Assume that the property (**BMHP**) fails for X. Under the assumption that (i) holds true, there exists a torsor $Z \to X$ of descent type λ^{f} that fails the property (**BMHP**), that is, $Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)} \neq \varnothing$ but $Z(k) = \varnothing$. In particular, the right hand side of (3.8) is nonempty. Hence, the left hand side is also nonempty, that is, there exists a torsor $Y \to X$ of descent type λ such that $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)} \neq \varnothing$. But, since $X(k) = \varnothing$, we have necessarily that $Y(k) = \varnothing$, *i.e.*, Y fails the property (**BMHP**). This means (iii) holds true.

Finally, we suppose that (ii) holds true and proceed to show (iv) directly. Assume that every torsor $Y \to X$ of descent type λ satisfies (**BMWA**). Let $g: Z \to X$ be any torsor of

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descent type λ^{f}. Assume that there is a family $z_\Omega \in Z(k_\Omega)^{\mathrm{Br_{nr}}(Z)}$, and let $\mathscr{U} \subseteq Z(k_\Omega)$ be any neighborhood of it. Again, by Lemma 2.2 (ii) and Theorem 3.12, we obtain a torsor $h : Y \to Z$ under a k-form of $\bar{G}^\circ$ such that $h(Y(k_\Omega)^{\mathrm{Br_{nr}}(Y)}) \cap \mathscr{U} \neq \varnothing$, and the composite $g \circ h : Y \to X$ is a torsor of descent type λ. Under our assumption, one has $Y(k) \cap h^{-1}(\mathscr{U}) \neq \varnothing$, *a fortiori* $Z(k) \cap \mathscr{U} \neq \varnothing$. This means Z satisfies (**BMWA**). Under the assumption that (ii) holds true, the variety X also satisfies (**BMWA**), which proves (iv). □

In Theorem 3.14, the quasi-projective assumption is required since we were working with descent types instead of torsors. If a torsor $Y \to X$ is given, it is possible to drop this condition, allowing one to recover the following.

Theorem 3.15 (Theorem B). *Let k be a number field, G a linear (not necessarily connected) algebraic k-group, X be a smooth k-variety and $f : Y \to X$ a torsor under G, with Y rationally connected (in particular, X is rationally connected). Put $Z := Y/G^\circ$, which gives a torsor $g : Z \to X$ under G^{f} and a torsor $h : Y \to Z$ under G°. Then, one has*

$$\overline{\bigcup_{[\sigma] \in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(k_\Omega)^{\mathrm{Br_{nr}}(Y^\sigma)})} = \overline{\bigcup_{[\tau] \in \mathrm{H}^1(k,G^{\mathrm{f}})} g^\tau(Z^\tau(k_\Omega)^{\mathrm{Br_{nr}}(Z^\tau)})} \tag{3.9}$$

in the topological space $X(k_\Omega)$. Furthermore, consider the following statements.

(i) *If Z^τ satisfies* (**BMHP**) *for every $\tau \in \mathrm{Z}^1(k,G^{\mathrm{f}})$, then X itself satisfies* (**BMHP**).
(ii) *If Z^τ satisfies* (**BMWA**) *for every $\tau \in \mathrm{Z}^1(k,G^{\mathrm{f}})$, then X itself satisfies* (**BMWA**).
(iii) *If Y^σ satisfies* (**BMHP**) *for every $\sigma \in \mathrm{Z}^1(k,G)$, then X itself satisfies* (**BMHP**).
(iv) *If Y^σ satisfies* (**BMWA**) *for every $\sigma \in \mathrm{Z}^1(k,G)$, then X itself satisfies* (**BMWA**).

Then, (i) implies (iii) and (ii) implies (iv).

Proof. If X is quasi-projective, then the theorem follows from Theorem 3.14 and Lemma 2.1 (iii). In the general case, take any dense affine open subset $U \subseteq X$, and let $V := f^{-1}(U)$ and $W := g^{-1}(U)$. Then, the identity (3.8) from Theorem 3.14 becomes

$$\overline{\bigcup_{[\sigma] \in \mathrm{H}^1(k,G)} f(V^\sigma(k_\Omega)^{\mathrm{Br_{nr}}(V^\sigma)})} = \overline{\bigcup_{[\tau] \in \mathrm{H}^1(k,G^{\mathrm{f}})} g(W^\tau(k_\Omega)^{\mathrm{Br_{nr}}(W^\tau)})},$$

which implies (3.9), because $V^\sigma(k_\Omega)^{\mathrm{Br_{nr}}(V^\sigma)}$ (resp. $W^\tau(k_\Omega)^{\mathrm{Br_{nr}}(W^\tau)}$) is dense in $Y^\sigma(k_\Omega)^{\mathrm{Br_{nr}}(Y^\sigma)}$ (resp. $Z^\tau(k_\Omega)^{\mathrm{Br_{nr}}(Z^\tau)}$) for all $\sigma \in \mathrm{Z}^1(k,G)$ (resp. $\tau \in \mathrm{Z}^1(k,G^{\mathrm{f}})$), thanks to Lemma 3.10. The implications '(i) ⇒ (iii)' and '(ii) ⇒ (iv)' follow from the respective implications in Theorem 3.14 (applied to U) and the birational invariance of the properties (**BMHP**) and (**BMWA**). □

The 'finite case' of Conjecture B is hence the most difficult one. The inductive argument used in the proofs of Theorems 3.14 and 3.15 was used in a similar way by Harpaz and Wittenberg in their work [HW24] on supersolvable descent. The base case (*i.e.*, 'cyclic descent') relies on the Graber–Harris–Starr theorem [GHS03]. If we were able to prove Conjecture B (for torsors) in the finite abelian case, then we would obtain the same

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result for descent types thanks to [HW24, Lemma 3.7]. Using the same sort of dévissage argument, the descent conjecture for the descent types whose underlying $\bar{k}$-group are finite solvable (as an abstract group!) would then be proved. In particular, this would imply that homogeneous spaces of connected linear algebraic groups with finite solvable geometric stabilizers have the properties (**BMHP**) and (**BMWA**) (it suffices to repeat the proof of Corollary 3.13). A direct consequence of this would be Shafarevich's theorem on solvable Galois groups. Unfortunately, at least to the author's knowledge, a proof of Conjecture B in the finite abelian case is currently out of reach. It should be noted that if G is a connected linear algebraic group over a number field k such that $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ and X is a homogeneous space of G with finite abelian geometric stabilizers, then X satisfies (**BMHP**) and (**BMWA**) (see [Bor96, Theorems 2.2 and 2.3]).

Just like in the cyclic case, where Harpaz and Wittenberg applied the sophisticated theorem of Graber–Harris–Starr, a proof of the descent conjecture for G finite abelian should use the rational connected assumption in an essential way. As a matter of fact, this assumption is *absolutely indispensable*, even when $G = \mathbb{Z}/2$. Indeed, in [HS05, Theorem 2.10, Proposition 2.11], Harari and Skorobogatov constructed a double cover $f : Y \to X$, where X is an Enriques surface X over $k = \mathbb{Q}$, and Y is a Kummer surface, such that $X(k_\Omega)^{\mathrm{Br}(X)} \not\subseteq \bigcup_{[\sigma]\in \mathrm{H}^1(k,\mathbb{Z}/2)} f^\sigma(Y^\sigma(k_\Omega))$ (by properness, this latter is a closed subset of $X(k_\Omega)$). Worse, the work of Balestrieri–Berg–Manes–Park–Viray [BBM+16, Theorem 1.2] gave an Enriques surface X over $k = \mathbb{Q}$, and a double cover $f : Y \to X$, where Y is a $K3$ surface, such that $X(k) = Y^\sigma(k_\Omega)^{\mathrm{Br}(Y^\sigma)} = \varnothing$ for all $\sigma \in \mathrm{H}^1(k,\mathbb{Z}/2)$, but $X(k_\Omega)^{\mathrm{Br}(X)} \neq \varnothing$ (that is, the descent varieties Y^σ vacuously satisfy (**BMHP**) and (**BMWA**), but X does not).

4. The descent conjecture for zero-cycles

In this section, we prove Theorem C. For convenience, all of the subsequent results shall be formulated in the language of descent types. They can be easily converted into ones for torsors (dropping the quasi-projective assumption) using the same trick deployed in the proof of Theorem 3.15.

4.1. Brauer–Manin obstruction for zero-cycles

We recall some facts on the Chow groups of zero-cycles.

Zero-cycles. The standard reference for the Chow groups is [Ful84]. When X is a *smooth* variety over a field k, denote by $\mathrm{Z}_0(X)$ the group of zero-cycles on X, which is functorial contravariantly with respect to flat morphisms of relative dimension 0, and covariantly with respect to any morphism. Its quotient $\mathrm{CH}_0(X)$ by rational equivalence is the *zero-dimensional Chow group* of X, which is functorial covariantly with respect to proper morphisms, and contravariantly with respect to generically finite morphisms (between smooth varieties). For any field extension (not necessarily finite) K/k, we have a restriction homomorphism $\mathrm{res}_{K/k} : \mathrm{Z}_0(X) \to \mathrm{Z}_0(X_K)$, which preserves degree. If $[K : k]$ is finite, we also have a corestriction homomorphism $\mathrm{N}_{K/k} : \mathrm{Z}_0(X_K) \to \mathrm{Z}_0(X)$ (also called the (K/k)-*norm*) which multiplies the degree by $[K : k]$. Both of these maps induce

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homomorphisms on Chow groups, and the composite $\mathrm{N}_{K/k} \circ \mathrm{res}_{K/k} : \mathrm{Z}_0(X) \to \mathrm{Z}_0(X)$ is the multiplication by $[K : k]$.

Local (co-)restrictions over number fields. If k is a number field, put $\mathrm{Z}_{0,\Omega}(X) := \prod_{v \in \Omega_k} \mathrm{Z}_0(X_v)$. For $v \in \Omega_k$, denote by $\mathrm{loc}_v : \mathrm{Z}_0(X) \to \mathrm{Z}_0(X_v)$ the restriction map $\mathrm{res}_{k_v/k}$, and let $\mathrm{loc}_\Omega = ((\mathrm{loc}_v)_{v \in \Omega_k}) : \mathrm{Z}_0(X) \to \mathrm{Z}_{0,\Omega}(X)$. The *local norm*

$$\mathrm{N}^{\Omega}_{K/k} : \mathrm{Z}_{0,\Omega}(X_K) \to \mathrm{Z}_{0,\Omega}(X) \tag{4.1}$$

is defined as follows. For each family $x_\Omega = (x_w)_{w \in \Omega_K} \in \mathrm{Z}_{0,\Omega}(X_K)$, we define $\mathrm{N}_{K/k}(x_\Omega) \in \mathrm{Z}_{0,\Omega}(X)$ to be the family whose v-component, for each $v \in \Omega_k$, is $(\mathrm{N}^{\Omega}_{K/k}(x_\Omega))_v := \sum_{w \in \Omega_K,\, w|v} \mathrm{N}_{K_w/k_v}(x_w)$. One checks with ease that the construction (4.1) is functorial and compatible with the global norm $\mathrm{N}_{K/k}$. Similarly, we define the *local restriction*

$$\mathrm{res}^{\Omega}_{K/k} : \mathrm{Z}_{0,\Omega}(X) \to \mathrm{Z}_{0,\Omega}(X_K) \tag{4.2}$$

as follows. For each family $x_\Omega = (x_v)_{v \in \Omega_k} \in Z_{0,\Omega}(X)$ and each place $w \in \Omega_K$ lying over a place $v \in \Omega_k$, let $(\mathrm{res}^{\Omega}_{K/k}(x_\Omega))_w := \mathrm{res}_{K_w/k_v}(x_v)$. Then, (4.2) is functorial and compatible with the global restriction $\mathrm{res}_{K/k}$. Furthermore, $\mathrm{N}^{\Omega}_{K/k} \circ \mathrm{res}^{\Omega}_{K/k}$ is the multiplication by $[K : k]$ on $\mathrm{Z}_{0,\Omega}(X)$.

Brauer–Manin obstruction. A complete exposition on the Brauer–Manin obstruction in the context of zero-cycles can be found in [CTS21, §15.1]. Let X be a smooth geometrically integral variety over a number field k. We have the local pairings

$$\langle -, - \rangle_v : \mathrm{Z}_0(X_v) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

defined for all $v \in \Omega_k$, which factors through a pairing $\mathrm{CH}_0(X_v) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ whenever X is proper. If $v \in \Omega^{\infty}_k$, this pairing vanishes on the image of the norm $\mathrm{N}_{\bar{k}_v/k_v} : \mathrm{CH}_0(\bar{X}_v) \to \mathrm{CH}_0(X_v)$. By restricting to the unramified elements of $\mathrm{Br}(X)$, we obtain from (4.1) a pairing

$$\langle -, - \rangle_{\mathrm{BM}} : \mathrm{Z}_{0,\Omega}(X) \times \mathrm{Br}_{\mathrm{nr}}(X) \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_{v \in \Omega_k}, \alpha) \mapsto \sum_{v \in \Omega_k} \langle x_v, \alpha \rangle_v . \tag{$\mathrm{BM}_{\mathrm{cyc}}$}$$

The pairing ($\mathrm{BM}_{\mathrm{cyc}}$) vanishes on the image of loc_Ω and on $\mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$, thanks to the global reciprocity law. When X is proper, we let

$$\mathrm{CH}'_0(X_v) := \begin{cases} \mathrm{CH}_0(X_v) & \text{if } v \in \Omega^{\mathrm{f}}_k, \\ \mathrm{Coker}(\mathrm{N}_{\bar{k}_v/k_v}) & \text{if } v \in \Omega^{\infty}_k. \end{cases}$$

Put $\mathrm{CH}_{0,\Omega} := \prod_{v \in \Omega_k} \mathrm{CH}'_0(X_v)$. Then, the pairing ($\mathrm{BM}_{\mathrm{cyc}}$) induces a pairing

$$\mathrm{CH}_{0,\Omega}(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

which vanishes on the image of loc_Ω, hence a map $\mathrm{CH}_{0,\Omega}(X) \to \mathrm{Br}(X)^D$ fitting in a complex

$$\mathrm{CH}_0(X) \xrightarrow{\mathrm{loc}_\Omega} \mathrm{CH}_{0,\Omega}(X) \to \mathrm{Br}(X)^D. \tag{E$'$}$$

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Applying the functor $A \mapsto \varprojlim_n (A/n)$ to (E′) (noting that the group $\mathrm{Br}(X)$ is torsion) yields the complex (E). Recall that the conjecture (E) (*i.e.*, Conjecture C) predicts that (E) is exact for any smooth proper geometrically integral variety X. Note that the exactness of (E) is a stable birational invariant of smooth proper geometrically integral varieties, because the groups $\mathrm{CH}_0(X)$ and $\mathrm{Br}(X)$ are so (see [Wit12, Remarques 1.1 (vi)], [Ful84, Example 16.1.11], and [CTC79, Proposition 6.3]).

Weak topology on $\mathbf{Z_{0,\Omega}(X)}$. Following Liang [Lia13, §1.2], we consider the problem of weak approximation for zero-cycles. Let X be a (not necessarily proper) smooth geometrically integral variety defined over a number field k. Let X^c be a smooth compactification of X. For each $v \in \Omega_k$, we equip the group $\mathrm{Z}_0(X_v)$ with the initial topology defined by the composites

$$\mathrm{Z}_0(X_v) \subseteq \mathrm{Z}_0(X_v^c) \to \mathrm{CH}_0'(X_v^c) \to \mathrm{CH}_0'(X_v^c)/n,$$

for $n \geq 1$, where $\mathrm{CH}_0'(X_v^c)/n$ is equipped with the discrete topology. By birational invariance of the group CH_0, this topology is independent of the choice of the smooth compactification X^c. If K/k_v is any finite extension and $P, P' \in X(K)$ are two points which are close to each other, then $[P]$ and $[P']$ are close in $\mathrm{Z}_0(X_v)$ [Wit12, Lemme 1.8]. Finally, we equip the group $\mathrm{Z}_{0,\Omega}(X) = \prod_{v\in\Omega_k} \mathrm{Z}_0(X_v)$ with the product topology. Hence, two families $x_\Omega = (x_v)_{v\in\Omega_k}$ and $x'_\Omega = (x'_v)_{v\in\Omega_k}$ are close to each other if for a large (but finite) subset $S \subseteq \Omega_k$ and a large integer $n \geq 1$, the local zero-cycles x_v and x'_v have the same image in $\mathrm{CH}_0'(X_{k_v}^c)/n$ for every $v \in S$.

Descent. In [BB24], Balestrieri and Berg developed a formalism allowing one to study descent theory in the context of zero-cycles. In short, given a torsor $f : Y \to X$ under an algebraic group G over k, we consider all finite extensions K/k and all the twisted torsors $Y_K^\sigma \to K$, where $[\sigma] \in \mathrm{H}^1(K,G)$. Since the notion of descent type (introduced in Subsection 2.1) classifies torsors up to twisting by a Galois cocycle, it is more convenient to translate the formalism of Balestrieri–Berg into one in this new language. Let λ be a descent type on X bound by a k-kernel L. We fix an algebraic closure $\bar{k}$ of k, and let $\mathscr{T}_\lambda$ be the set of pairs $(K,[Y])$, where $K \subseteq \bar{k}$ is a finite extension of k, and $[Y]$ is the class of a torsor $Y \to X_K$ of descent type $\mathbf{dtype}(Y) = \lambda_K := \mathrm{res}_{K/k}(\lambda) \in \mathrm{DType}(X_K, L_K)$. Following [BB24, Definition 2.2], we define the 'recombining' map associated with λ as

$$\mathrm{rec}_\lambda = \sum_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}_{K/k} \circ f_* : \bigoplus_{(K,[Y])\in\mathscr{T}_\lambda} \mathrm{Z}_0(Y) \to \mathrm{Z}_0(X). \tag{4.3}$$

On the local level, one defines

$$\mathrm{rec}_{\lambda,\Omega} = \sum_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}^\Omega_{K/k} \circ f_* : \bigoplus_{(K,[Y])\in\mathscr{T}_\lambda} \mathrm{Z}_{0,\Omega}(Y) \to \mathrm{Z}_{0,\Omega}(X), \tag{4.4}$$

where $\mathrm{N}^\Omega_{K/k} : \mathrm{Z}_{0,\Omega}(X_K) \to \mathrm{Z}_{0,\Omega}(X)$ is the local norm map (4.1). Denote by $\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ the subgroup of $\mathrm{Z}_{0,\Omega}(X)$ consisting of the families orthogonal to $\mathrm{Br}_{\mathrm{nr}}(X)$ relative to the pairing ($\mathrm{BM}_{\mathrm{cyc}}$). By functoriality of this pairing, (4.4) restricts to a homomorphism

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$$\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega}: \bigoplus_{(K,[Y])\in\mathscr{T}_\lambda} \mathrm{Z}_{0,\Omega}(Y)^{\mathrm{Br}_{\mathrm{nr}}(Y)} \to \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)}. \tag{4.5}$$

Here is the main result of this section.

Theorem 4.1 (Theorem C). *Let X be a smooth quasi-projective variety over a number field k. Let $\lambda=[\bar{Y},E]$ be a descent type on X bound by a k-kernel $L=(\bar{G},\kappa)$, with $\bar{Y}$ rationally connected (the existence of such a descent type implies that X is rationally connected) and $\bar{G}$ linear. For any smooth compactification X^c of X, one has*

$$\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)}=\overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c))+\mathrm{Im}(\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega})}$$

in the set $\mathrm{Z}_{0,\Omega}(X)$ equipped with the topology defined in p. 34. In particular, if for every finite extension K/k and every torsor $Y\to X_K$ of descent type $\mathbf{dtype}(Y)=\lambda_K$, Conjecture C holds for the smooth compactifications of Y, then it is also the case for the smooth compactifications of X.

The strategy for the proof of Theorem 4.1 is as follows. In Subsection 4.2, we combine Liang's method (passing from rational points to zero-cycles) in [Lia13, §3.2] with Theorem 3.12 to deal with the case where $\bar{G}$ is connected. In particular, this can be applied to the case where $\bar{G}=\bar{T}$ is a torus. Next, using the fibration argument of Harpaz–Wittenberg in [HW24, §3.1 and 3.2] for rational points to deal with the case where $\bar{G}$ is a group of multiplicative type in Subsection 4.3 (this requires the previous case of torsors under a torus). In particular, this implies the result in the case where $\bar{G}$ is finite abelian. An inductive argument then yields the result in the case where $\bar{G}$ is finite solvable. The case where $\bar{G}$ is finite is treated in Subsection 4.4, by passing to its Sylow subgroups (this uses an argument similar to one in [HW20, §7.2]). Finally, we deploy (again) a dévissage argument to combine the results in the finite and connected cases.

4.2. The connected case

In this subsection, we prove the following result.

Proposition 4.2. *Let X be a smooth quasi-projective rationally connected variety over a number field k. Let λ be a descent type on X bound by a k-kernel $L=(\bar{G},\kappa)$, with $\bar{G}$ linear and connected. Then, one has*

$$\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)}=\overline{\bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}^{\Omega}_{K/k}(f_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y_K)}))},$$

where $\mathrm{Z}_{0,\Omega}(X)$ is equipped with the topology defined in p. 34.

First, we show the following.

Lemma 4.3. *Let X be a smooth proper geometrically integral variety over a number field k such that the Néron–Severi group of $\bar{X}$ is torsion-free and $\mathrm{H}^1(\bar{X},\mathcal{O}_{\bar{X}})=\mathrm{H}^2(\bar{X},\mathcal{O}_{\bar{X}})=0$ (this is the case if X is rationally connected). For any $x_\Omega=(x_v)_{v\in\Omega_k}\in\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X)}$, any finite set $S\subseteq\Omega_k$ of places and any integer $n\geq 1$, there exist a finite extension K/k and a*

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family $P_\Omega = (P_w)_{w\in\Omega_K} \in X_K(K_\Omega)^{\mathrm{Br}(X_K)}$ *such that* $(\mathrm{N}^\Omega_{K/k}([P_\Omega]))_v$ *and* x_v *have the same image in* $\mathrm{CH}'_0(X)/n$ *for all* $v \in S$.

Proof. Note that, for (smooth proper) rationally connected varieties, the complex (E′) is exact if and only if (E) is so (see [HW20, Rappel 1.3]). For $X = \mathrm{Spec}(k)$, the complex (E) is exact because it is the dual of the Albert–Brauer–Noether–Hasse sequence from global class field theory:

$$0 \to \mathrm{Br}(k) \to \bigoplus_{v\in\Omega_k} \mathrm{Br}(k_v) \xrightarrow{\sum} \mathbb{Q}/\mathbb{Z} \to 0.$$

It follows that (E′) is also exact for $X = \mathrm{Spec}(k)$. Since $\mathrm{CH}'_0(\mathrm{Spec}(k_v)) = \mathbb{Z}$ for $v \in \Omega^{\mathrm{f}}_k$ and $\mathrm{CH}'_0(\mathrm{Spec}(\mathbb{R})) = \mathbb{Z}/2$, any family $(x_v)_{v\in\Omega_k} \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X)}$ has 'the same' local degree in the following sense. There exists an integer $d \in \mathbb{Z}$ such that $\deg_{k_v}(x_v) = d$ for all $v \in \Omega^{\mathrm{f}}_k$ and $\deg_{k_v}(x_v) \equiv d (\mathrm{mod}\ 2)$ for all real places $v \in \Omega^\infty_k$. For $v \in \Omega^\infty_k$, we take any point $\bar{P}_v \in X(\bar{k}_v)$ and add a suitable multiple of the 0-cycle $\mathrm{N}_{\bar{k}_v/k_v}(\bar{P}_v)$ to x_v (which does not affect the class of x_v in $\mathrm{CH}'_0(X_{k_v})$). Hence, we may assume that $\deg_{k_v}(x_v) = d$ for all $v \in \Omega_k$.

The rest of the proof is essentially done in [Lia13, Proof of Theorem 3.2.1] if $d = 1$, and in *loc. cit.*, Proposition 3.4.1 if $d \neq 1$. At the end of these proofs, we obtain a finite extension K/k and a family $P_\Omega = (P_w)_{w\in\Omega_K} \in X_K(K_\Omega)^{\mathrm{Br}(X_K)}$ satisfying the stated condition. □

Proof of Proposition 4.2. Let X^c be a smooth compactification of X and let $x_\Omega \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X^c)}$. By Lemma 4.3, we find a finite extension K/k and a family $P_\Omega \in X^c_K(K_\Omega)^{\mathrm{Br}(X^c)}$ such that $\mathrm{N}^\Omega_{K/k}([P_\Omega]) \in \mathrm{Z}_{0,\Omega}(X^c)^{\mathrm{Br}(X^c)}$ is sufficiently close to x_Ω. By Lemma 3.10, we may assume that $P_\Omega \in X_K(K_\Omega)^{\mathrm{Br}(X^c)}$. By Theorem 3.12, there exists a torsor $f : Y \to X_K$ of descent type λ_K and a family $Q_\Omega \in Y(K_\Omega)^{\mathrm{Br_{nr}}(Y)}$ such that $f(Q_\Omega)$ is sufficiently close to P_Ω, hence $\mathrm{N}^\Omega_{K/k}(f_*[Q_\Omega])$ is sufficiently close to x_Ω. □

4.3. The multiplicative type case

In this subsection, we adapt the argument from [HW24, §3.2] to show the following.

Proposition 4.4. *Let* X *be a smooth quasi-projective variety over a number field* k. *Let* M *be a* k*-group of multiplicative type and let* $\lambda = [\bar{Y}, E] \in \mathrm{DType}(X, M)$, *with* $\bar{Y}$ *rationally connected (in particular,* X *is rationally connected). Then, for any smooth compactification* X^c *of* X, *one has*

$$\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br_{nr}}(X)} = \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}^\Omega_{K/k}(f_*(Y(K_\Omega)^{\mathrm{Br_{nr}}(Y_K)}))}$$

in the set $\mathrm{Z}_{0,\Omega}(X)$ *equipped with the topology defined in p. 34.*

We require the following analogue of [HW24, Theorem 3.6] in the context of zero-cycles.

Lemma 4.5. *Let* Z *be a smooth geometrically integral variety equipped with a dominant morphism* $\pi : Z \to \mathbb{A}^n_k$, *for some* $n \geq 1$. *Let* $U \subseteq \mathbb{A}^n_k$ *be a dense open subset over which* π *is smooth, and suppose that* $Z_q = \pi^{-1}(q)$ *is rationally connected for all* $q \in U$. *Then, for any*

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smooth compactification Z^c of Z and any family $z_\Omega \in \mathrm{Z}_{0,\Omega}(Z)^{\mathrm{Br}(Z^c)}$, there exist a global zero-cycle $z \in \mathrm{Z}_0(Z^c)$, a closed point $q \in U$ and a family $z'_\Omega \in \mathrm{Z}_{0,\Omega}(Z_q)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)}$ such that $\mathrm{loc}_\Omega(z) + \mathrm{N}_{k(q)/k}(z'_\Omega)$ is sufficiently close to z_Ω in $\mathrm{Z}_{0,\Omega}(Z^c)$ (equipped with the topology defined in p. 34).

Proof. Since the group CH_0 is a birational invariant of smooth proper geometrically integral variety, it is enough to prove the lemma for an *arbitrarily chosen* smooth compactification Z^c. Hence, by Nagata's theorem, we may assume that π extends into a proper morphism $\pi^c : Z^c \to \mathbb{P}^n_k$, which is smooth over U (after shrinking U if necessary).

We proceed by induction on n. For $n = 1$, this follows from [HW16, Proposition 8.7] (note that $\mathrm{Z}_{0,\Omega}(Z_q)^{\mathrm{Br}(Z_q^c)}$ is dense in $\mathrm{Z}_{0,\Omega}(Z_q^c)^{\mathrm{Br}(Z_q^c)}$ by continuity of ($\mathrm{BM}_{\mathrm{cyc}}$) and Lemma 3.10 (i)).

Let $n \geq 2$ and let $p : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be the first projection. Let $V \subseteq \mathbb{A}^1_k$ be a dense open subset over which $p \circ \pi$ is smooth, such that $U \cap p^{-1}(r) \neq \varnothing$ for every $r \in V$. For such r, the fibre $Z_r = (p \circ \pi)^{-1}(r)$ is equipped with a dominant morphism to $p^{-1}(r) \cong \mathbb{A}^{n-1}_k$ with rationally connected generic fibre, hence is rationally connected by the theorem of Graber–Harris–Starr (see [Deb03, Corollaire 3.1]). Given a family $z_\Omega \in \mathrm{Z}_{0,\Omega}(Z)^{\mathrm{Br}(Z^c)}$, by the case $n = 1$ applied to $p \circ \pi$, we find a global zero-cycle $w \in \mathrm{Z}_0(Z^c)$, a closed point $r \in V$ and a family $w'_\Omega \in \mathrm{Z}_{0,\Omega}(Z_r)^{\mathrm{Br}(Z_r^c)}$ such that $\mathrm{loc}_\Omega(w) + \mathrm{N}_{k(r)/k}(w'_\Omega)$ is sufficiently close to z_Ω. By the induction hypothesis applied to the restriction $\pi|_{Z_r} : Z_r \to p^{-1}(r)$, we obtain a global zero-cycle $w^1 \in \mathrm{Z}_0(Z_r^c)$, a closed point $q \in U \cap \pi^{-1}(r)$, and a family $z'_\Omega \in \mathrm{Z}_{0,\Omega}(Z_q)^{\mathrm{Br}(Z_q^c)}$ such that $\mathrm{loc}_\Omega(w^1) + \mathrm{N}_{k(q)/k(r)}(z'_\Omega)$ is sufficiently close to w'_Ω. Let $z := w + \mathrm{N}_{k(r)/k}(w^1)$, then $\mathrm{loc}_\Omega(z) + \mathrm{N}_{k(q)/k}(z'_\Omega)$ is sufficiently close to z_Ω. □

Proof of Proposition 4.4. Let $x_\Omega = (x_v)_{v \in \Omega_k} \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X^c)}$. There is an exact sequence

$$1 \to M \xrightarrow{\iota} T \to Q \to 1, \tag{4.6}$$

where T is a k-torus and Q is a quasi-trivial k-torus (see *e.g.* [HW24, Lemma 3.8]). Let us apply Proposition 4.2 to the pushforward descent type $\iota_*\lambda \in \mathrm{DType}(X,T)$ (*cf.* Lemma 2.3). We then obtain a finite extension k'/k, a torsor $g : Z \to X_{k'}$ under $T_{k'}$ with $\mathbf{dtype}(Z) = \iota_*\lambda_{k'}$, and a family $P'_\Omega \in Z(k'_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)}$ such that $\mathrm{N}^\Omega_{k'/k}(g_*[P'_\Omega])$ is sufficiently close to x_Ω. Furthermore, by the exact sequence (2.2) (also from Lemma 2.3)), we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^1(k',M) & \xrightarrow{p^*} & \mathrm{H}^1(X_{k'},M) & \xrightarrow{\mathbf{dtype}} & \mathrm{DType}(X_{k'},M_{k'}) & \xrightarrow{\partial} & \mathrm{H}^2(k',M) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\mathrm{H}^1(k',T) & \xrightarrow{p^*} & \mathrm{H}^1(X_{k'},T) & \xrightarrow{\mathbf{dtype}} & \mathrm{DType}(X_{k'},T_{k'}) & \xrightarrow{\partial} & \mathrm{H}^2(k',T)
\end{array} \tag{4.7}$$

with exact rows, where $p : X \to \mathrm{Spec}(k)$ denotes the structure morphism. Since $\mathrm{H}^1(k',Q) = 0$, the left arrow from (4.7) is surjective and the right arrow there is injective, by virtue of the long exact sequence associated with (4.6). Since $\iota_*\lambda_{k'} = \mathbf{dtype}(Z)$, one has $\iota_*\partial(\lambda_{k'}) = \partial(\iota_*\lambda_{k'}) = 0$. Thus, $\partial(\lambda_{k'}) = 0$ by injectivity of $\iota_* : \mathrm{H}^2(k',M) \to \mathrm{H}^2(k',T)$.

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It follows that there exists a torsors $f : Y \to X_{k'}$ with $\mathbf{dtype}(Y) = \lambda_{k'}$. By exactness of the bottom row, one has $\iota_*[Y] = [Z] - p^*[\tau]$ for some $[\tau] \in \mathrm{H}^1(k', T)$. By surjectivity of $\iota_* : \mathrm{H}^1(k', M) \to \mathrm{H}^1(k', T)$, we may write $[\tau] = \iota_*[\mu]$ for some $[\mu] \in \mathrm{H}^1(k', M)$. Twisting Y by a cocycle μ representing $[\mu]$ (which yields a torsor $Y^\mu \to X_{k'}$ of descent type $\mathbf{dtype}(Y^\mu) = \lambda_{k'}$), we may assume that $\iota_*[Y] = [Z]$. In other words, we have an isomorphism $Y \times_{k'}^{M_{k'}} T_{k'} \cong Z$ of $X_{k'}$-torsors under $T_{k'}$.

By Nagata's theorem [Nag63], we may extend g to a proper morphism $g^c : Z^c \to X^c_{k'}$, where Z^c is a smooth compactification of Z. Let us now follow the argument in [HW24, p. 797]. The second projection $Y \times_{k'} T_{k'} \to T_{k'}$ induces a morphism

$$\pi : Z \cong Y \times_{k'}^{M_{k'}} T_{k'} \to T_{k'}/M_{k'} = Q_{k'},$$

which has the following property. For every overfield K/k' and every K-point $q \in Q(K)$, if $\delta : Q(K) \to \mathrm{H}^1(K, M)$ is the connecting homomorphism associated with (4.6) and μ is a cocycle representing $[\mu] = \delta(q)$, then the fibre $Z_q = \pi^{-1}(q)$ is isomorphic to Y^μ_K as X_K-torsors under M_K (in particular, Z_q is rationally connected). Indeed, if $\varpi : T \to Q$ denotes the projection, then $[\mu] = [\varpi^{-1}(q)] \in \mathrm{H}^1(K, M)$ and hence $[Z_q] = [\pi^{-1}(q)] = [Y_K \times_K^{M_K} \varpi^{-1}(q)] = [Y^\mu_K] \in \mathrm{H}^1(X_K, M)$ by the very definition of twisting.

Since Q is a quasi-trivial torus, it is a dense open subset of an affine space. Hence, we may apply Lemma 4.5 to obtain a global zero-cycle $z \in \mathrm{Z}_0(Z^c)$, a closed point $q \in Q_{k'}$, and a family $z'_\Omega \in \mathrm{Z}_{0,\Omega}(Z_q)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)}$ such that $\mathrm{loc}_\Omega(z) + \mathrm{N}_{k'(q)/k'}(z'_\Omega)$ is sufficiently close to $[P'_\Omega]$. Twisting $Y_{k'(q)}$ by a cocycle μ representing $\delta(q) \in \mathrm{H}^1(k'(q), M)$, we may assume that $\delta(q) = 0$. As mentioned above, there is an isomorphism $Z_q \cong Y_{k'(q)}$ of $X_{k'(q)}$-torsors under $M_{k'(q)}$. Thus, we obtain a family $y_\Omega \in \mathrm{Z}_{0,\Omega}(Y_{k'(q)})^{\mathrm{Br}_{\mathrm{nr}}(Y_{k'(q)})}$ such that $f_* y_\Omega = g_* z'_\Omega$ in $\mathrm{Z}_{0,\Omega}(X_{k'(q)})$. Let $x := \mathrm{N}_{k'/k}(g^c_* z) \in \mathrm{Z}_0(X^c)$. Then $\mathrm{loc}_\Omega(x) + \mathrm{N}^\Omega_{k'(q)/k}(f_* y_\Omega)$ is sufficiently close to $\mathrm{N}^\Omega_{k'/k}(g_*[P'_\Omega])$, hence also to x_Ω. □

4.4. The general case

We prove Theorem 4.1 in this subsection. First, we start with the finite solvable case. Recall that a finite descent type on a smooth geometrically integral variety X is an isomorphism class of a (connected) finite étale cover $\bar{Y} \to \bar{X}$ such that $\bar{Y}$ is Galois over X (*cf.* p. 11).

Lemma 4.6. *Let X be a smooth quasi-projective variety over a number field k. Let $\lambda = [\bar{Y}]$ be a rationally connected finite descent type on X (in particular, X is rationally connected) such that the finite group $\bar{G} = \mathrm{Aut}(\bar{Y}/\bar{X})$ is solvable. For any smooth compactification X^c of X, one has*

$$\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}^\Omega_{K/k}(f_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y_K)}))}$$

in the set $\mathrm{Z}_{0,\Omega}(X)$ equipped with the topology defined in p. 34.

Proof. We proceed by induction on the derived length n of $\bar{G}$. If $n = 0$, there is nothing to prove.

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For $n \geq 1$, let $x_\Omega \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X^c)}$. The derived subgroup $[\bar{G},\bar{G}]$ of $\bar{G}$ is solvable of derived length $n-1$. Let $\bar{Z} := \bar{Y}/[\bar{G},\bar{G}]$, which defines a finite descent type λ^{ab} on X, and $\mathrm{Aut}(\bar{Z}/\bar{X}) = \bar{G}^{\mathrm{ab}}$ is finite abelian (hence of multiplicative type). By Proposition 4.4, we find a global zero-cycle $x \in \mathrm{Z}_0(X^c)$, a finite extension k'/k, a torsor $g : Z \to X_{k'}$ of descent type $\mathbf{dtype}(Z) = \lambda^{\mathrm{ab}}_{k'}$, and a family $P'_\Omega \in Z(k'_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)}$ such that $\mathrm{loc}_\Omega(x) + \mathrm{N}^\Omega_{k'/k}(g_*[P'_\Omega])$ is sufficiently close to x_Ω.

By Nagata's theorem [Nag63], g extends into a proper morphism $g^c : Z^c \to X^c_{k'}$, where Z^c is a smooth compactification of Z. Now, $\bar{Y}$ is Galois over Z since it is Galois over X, hence $\lambda_Z = [\bar{Y}]$ is a finite descent type on Z, with $\mathrm{Aut}(\bar{Y}/\bar{Z}) = [\bar{G},\bar{G}]$. By the induction hypothesis, there is a global zero-cycle $z \in \mathrm{Z}_0(Z^c)$, a finite extension K/k', a torsor $h : Y \to Z_K$ of descent type $\mathrm{res}_{K/k'}(\lambda_Z)$, and a family $P_\Omega \in \mathrm{Z}_{0,\Omega}(Y)^{\mathrm{Br}_{\mathrm{nr}}(Y)}$ such that $\mathrm{loc}_\Omega(z) + \mathrm{N}^\Omega_{K/k'}(h_*[P_\Omega])$ is sufficiently close to $[P'_\Omega]$. The composite $f = g_K \circ h : Y \to X_K$ is then a torsor of descent type $\mathbf{dtype}(Y) = \lambda_K$. We have $x + \mathrm{N}_{k'/k}(g^c_* z) \in \mathrm{Z}_{0,\Omega}(X^c)$ and $\mathrm{loc}_\Omega(x + \mathrm{N}_{k'/k}(g^c_* z)) + \mathrm{N}^\Omega_{K/k}(f_*[P_\Omega])$ is sufficiently close to x_Ω. □

We can now deal with the finite case using the strategy in [HW20, §5].

Proposition 4.7. *Let X be a smooth quasi-projective variety over a number field k. Let $\lambda = [\bar{Y}]$ be a rationally connected finite descent type on X (in particular, X is rationally connected). Then*

$$\mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \mathrm{Im}(\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega})}$$

in the set $\mathrm{Z}_{0,\Omega}(X)$ equipped with the topology defined in p. 34, for any smooth compactification X^c of X. Here, the 'recombining' map $\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega}$ was defined in (4.5). More precisely, for any $x_\Omega \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X^c)}$ and any prime number p, there is an integer n_p prime to p such that

$$n_p x_\Omega \in \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_\lambda} \mathrm{N}^\Omega_{K/k}(f_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}))}.$$

Proof. Let $\bar{G} := \mathrm{Aut}(\bar{Y}/\bar{X})$. Let $x_\Omega \in \mathrm{Z}_{0,\Omega}(X)^{\mathrm{Br}(X^c)}$ and let p be any prime number. By [HW20, Lemme 5.6], we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \bar{H} & \longrightarrow & E & \longrightarrow & \Gamma_{k'} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \bar{G} & \longrightarrow & \mathrm{Aut}(\bar{Y}/X_{k'}) & \longrightarrow & \Gamma_{k'} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \bar{G} & \longrightarrow & \mathrm{Aut}(\bar{Y}/X) & \longrightarrow & \Gamma_k & \longrightarrow & 1,
\end{array}$$

where $\bar{H}$ (resp. E, resp. $\Gamma_{k'}$) is a p-Sylow subgroup of $\bar{G}$ (resp. $\mathrm{Aut}(\bar{Y}/X)$, resp. Γ_k) in the profinite sense (see [Ser94, Chapitre I, §1.4]). The subgroup E is of index $n'_p := [\bar{G} : \bar{H}]$ in $\mathrm{Aut}(\bar{Y}/X_{k'})$, hence corresponds to an étale cover $Z' \to X_{k'}$ of degree n'_p, over which $\bar{Y}$ is Galois. The cover $Z' \to X_{k'}$ is the base change to k' of an étale cover $Z \to X_\ell$, for

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some finite extension ℓ of k contained in k' (hence of degree $n_p'' := [\ell : k]$ prime to p). Recall from (4.2) that we have a local restriction map $\mathrm{res}^{\Omega}_{\ell/k} : \mathrm{Z}_{0,\Omega}(X) \to \mathrm{Z}_{0,\Omega}(X_\ell)$. By functoriality of the pairing ($\mathrm{BM}_{\mathrm{cyc}}$), one has $\mathrm{res}^{\Omega}_{\ell/k}(x_\Omega) \in \mathrm{Z}_{0,\Omega}(X_\ell)^{\mathrm{Br}(X_\ell^c)}$.

By Nagata's theorem [Nag63], we may extend the $Z \to X_\ell$ to a morphism $\pi : Z^c \to X_\ell^c$, which is generically finite of degree n_p', where Z^c is a smooth compactification of Z. Applying the refined Gysin pullback $\pi^!$ [Ful84, Definition 8.1.2] to the rational equivalence class of $\mathrm{res}^{\Omega}_{\ell/k}(x_\Omega)$ yields a family $z_\Omega = (z_w)_{w\in\Omega_\ell} \in \mathrm{Z}_{0,\Omega}(Z^c)$ such that $\pi_* z_w$ is rationally equivalent to $n_p'(\mathrm{res}^{\Omega}_{\ell/k}(x_\Omega))_w$ for all $w \in \Omega_\ell$. Since $\mathrm{res}^{\Omega}_{\ell/k}(x_\Omega) \in \mathrm{Z}_{0,\Omega}(X_\ell)^{\mathrm{Br}(X_\ell^c)}$, we have $z_\Omega \in \mathrm{Z}_{0,\Omega}(Z^c)^{\mathrm{Br}(Z^c)}$ by (covariant) functoriality of the pairing ($\mathrm{BM}_{\mathrm{cyc}}$). We may assume that $z_\Omega \in \mathrm{Z}_{0,\Omega}(Z)^{\mathrm{Br}(Z^c)}$ by continuity of ($\mathrm{BM}_{\mathrm{cyc}}$) and Lemma 3.10 (i). Now, since $\bar{Y}$ is Galois over Z, it defines a finite descent type λ_Z on Z, with $\mathrm{Aut}(\bar{Y}/\bar{Z}) = \bar{H}$. Furthermore, any Z-torsor of descent type λ_Z is an X_ℓ-torsor of descent type λ_ℓ and an X-torsor of descent type λ. Since $\bar{H}$ is a p-group (*a fortiori* solvable), we have

$$z_\Omega \in \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(Z^c)) + \bigcup_{(K,[g:Y\to Z_K])\in\mathscr{T}_{\lambda_Z}} \mathrm{N}^{\Omega}_{K/\ell}(g_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}))}$$

by Lemma 4.6. Applying π_* to both side, we obtain

$$n_p' \mathrm{res}^{\Omega}_{\ell/k}(x_\Omega) \in \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X_\ell^c)) + \bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_{\lambda_\ell}} \mathrm{N}^{\Omega}_{K/\ell}(f_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}))}.$$

Finally, applying the local norm $\mathrm{N}^{\Omega}_{\ell/k}$ to both sides, one gets

$$n_p x_\Omega \in \overline{\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \bigcup_{(K,[f:Y\to X_K])\in\mathscr{T}_{\lambda}} \mathrm{N}^{\Omega}_{K/k}(f_*(Y(K_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}))},$$

where $n_p := n_p' n_p''$ is an integer prime to p.

Let us now show that x_Ω lies in the closure of $\mathrm{loc}_\Omega(\mathrm{Z}_0(X^c)) + \mathrm{Im}(\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega})$. Fix an integer $n \geq 1$ and a finite set $S \subseteq \Omega_k$ of places. For each prime $p|n$, we have seen that there are an integer n_p prime to p, a global zero-cycle $x^p \in \mathrm{Z}_0(X^c)$, and a family $z^p_\Omega = (z^p_v)_{v\in\Omega_k} \in \mathrm{Im}(\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega})$ such that $n_p x_v$ and $\mathrm{loc}_v(x^p) + z^p_v$ have the same image in $\mathrm{CH}_0'(X_v^c)/n$ for all $v \in S$. The integers n_p, together with n, are mutually coprime, hence there are integers d, d_p such that $1 = dn + \sum_{p|n} d_p n_p$. Let $x := \sum_{p|n} d_p x^p \in \mathrm{Z}_0(X^c)$ and $z_\Omega := (z_v)_{v\in\Omega_k} = \sum_{p|n} d_p z^p_\Omega \in \mathrm{Im}(\mathrm{rec}^{\mathrm{Br}}_{\lambda,\Omega})$. Then

$$x_v = dnx_v + \sum_{p|n} d_p n_p x_v \quad \text{and} \quad \mathrm{loc}_v(x) + z_v$$

have the same image in $\mathrm{CH}_0'(X_v^c)/n$ for all $v \in S$. This proves the proposition. □

We can now combine Propositions 4.2 and 4.7 to obtain Theorem 4.1 (*i.e.*, Theorem C).

Proof of Theorem 4.1. Let $L^{\mathrm{f}} := (\bar{G}^{\mathrm{f}}, \kappa^{\mathrm{f}})$ be the natural k-kernel (recall that $\bar{G}^{\mathrm{f}} = \pi_0(\bar{G}) = \bar{G}/\bar{G}^\circ$), and let $\lambda^{\mathrm{f}} := [\bar{Z}] \in \mathrm{DType}(X, L^{\mathrm{f}})$ be the pushforward of λ along the morphism $L \to L^{\mathrm{f}}$ induced by the projection $\bar{G} \to \bar{G}^{\mathrm{f}}$ (hence $\bar{Z} = \bar{Y}/\bar{G}^\circ$, *cf.* Lemma 2.2 (i)).

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For each $x_\Omega \in \mathrm{Z}_0(X)^{\mathrm{Br}_{\mathrm{nr}}(X)}$, applying Proposition 4.7 to λ^{f} yields a global zero-cycle $x \in \mathrm{Z}_0(X^c)$, a finite set $\{(k_i,[g_i : Z_i \to X_{k_i}])\}_{i\in\mathcal{I}} \subseteq \mathscr{T}_{\lambda^{\mathrm{f}}}$, and a family $z^i_\Omega \in \mathrm{Z}_0(Z_i)^{\mathrm{Br}_{\mathrm{nr}}(Z_i)}$ for each $i \in \mathcal{I}$, such that $\mathrm{loc}_\Omega(x) + \sum_{i\in\mathcal{I}} \mathrm{N}^\Omega_{k_i/k}((g_i)_* z^i_\Omega)$ is sufficiently close to x_Ω.

For $i \in \mathcal{I}$, by Lemma 2.2 (ii), the torsor $Z_i \to X_{k_i}$ gives rise to a k_i-kernel $L_i = (\bar{G}^\circ, \kappa_i)$ and a descent type $\lambda_i = [\bar{Y}, E_i] \in \mathrm{DType}(Z_i, L_i)$. By Proposition 4.2, there exist a finite extension K_i/k_i, a torsor $h_i : Y_i \to (Z_i)_{K_i}$ of descent type $\mathrm{res}_{K_i/k_i}(\lambda_i)$, and a family $P^i_\Omega \in Y_i((K_i)_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y_i)}$ such that $\mathrm{N}^\Omega_{K_i/k_i}((h_i)_*[P^i_\Omega])$ is sufficiently close to z^i_Ω. The composite $f_i = (g_i)_{K_i} \circ h_i : Y_i \to X_{K_i}$ is a torsor of descent type $\mathbf{dtype}(Y_i) = \mathrm{res}_{K_i/k}(\lambda)$ by Lemma 2.2 (iii), and $\mathrm{loc}_\Omega(x) + \sum_{i\in\mathcal{I}} \mathrm{N}^\Omega_{K_i/k}((f_i)_*[P^i_\Omega])$ is sufficiently close to x_Ω. The theorem is proved. □

Theorem 4.1 is a generalization of the following result, proved in [HW20, Théorème A].

Corollary 4.8. *Conjecture C holds for the smooth compactifications of homogeneous spaces of connected linear algebraic groups over number fields.*

Proof. Let X be a homogeneous space of a connected linear algebraic group G over a number field k. As in the proof of Corollary 3.13, we have a descent type $\lambda = [\bar{G}, E_X]$ on X. For any finite extension K/k, a torsor $Y \to X_K$ of type λ_K is a left K-torsor under G_K, hence satisfies (**BMHP**) and (**BMWA**) by [San81, Corollaires 8.7 and 8.13] and [Che89]. It follows from Liang's theorems [Lia13, Theorems A and B] that Conjecture C holds for their smooth compactifications. By virtue of Theorem 4.1, Conjecture C also holds for the smooth compactifications of X. □

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