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MEASURABLE E_0-SEMIGROUPS ARE CONTINUOUS

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Abstract

Let G be a second countable locally compact Hausdorff topological group and P be a closed subsemigroup of G containing the identity element $e \in G$. Assume that the interior of P is dense in P. Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of unital normal $*$-endomorphisms of a von Neumann algebra M with separable predual satisfying a natural measurability hypothesis. We show that α is an E_0-semigroup over P on M.

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1. Introduction

The theory of E_0-semigroups over $\mathbb{R}_+$ was initiated by Powers in his paper [16]. Arveson further developed the theory of E_0-semigroups in a series of seminal papers [3–6]. One-parameter E_0-semigroups are classified into three types. Type I E_0-semigroups were completely classified by Arveson and they are indeed canonical commutation relations (CCR) flows. It is known from the works of Powers [15, 17], Tsirelson [22] and Izumi and Srinivasan [10, 11] that there are uncountably many type II and type III E_0-semigroups. We refer the reader to the monograph [7] for a comprehensive bibliography.

Noncommutative dynamics and E_0-semigroups over several variables have been studied by several authors. The important papers in this direction are [19–21]. Recently the author in collaboration with Sundar [12, 13] has studied the connection between E_0-semigroups over closed convex cones and the associated product system. In [12, 13], it was established that Arveson's bijection between product systems and E_0-semigroups holds in the higher dimensional case as well.

The basic example in the theory of one-parameter E_0-semigroups is the CCR flow, which associates an E_0-semigroup to an isometric representation. This association can be carried out in the setting of general semigroups. The analysis of one-parameter CCR flows relies on the Wold decomposition, which states that every pure one-parameter isometric representation is a countable direct sum of the standard shift

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on $L^2([0,\infty))$. In the case of a general semigroup, we do not have an analogue of such a decomposition. In general, isometric representations of semigroups can be quite complicated. This makes understanding multiparameter CCR flows an interesting and a nontrivial question in its own right. This was demonstrated in [1, 2]. In particular, in [2] the authors made use of techniques from groupoid C^*-algebras, Schwartz theory of distributions and semigroup C^*-algebras to classify CCR flows that arise out of shifts. Because of this interaction, we believe that multiparameter E_0-semigroups are interesting mathematical objects worth pursuing.

The purpose of this paper is to prove that multiparameter CCR/CAR (canonical anti-commutation relations) flows are indeed genuine E_0-semigroups. We deal with this problem in the context of a closed subsemigroup of a locally compact group. In the definition of E_0-semigroups, it is natural to require the continuity condition. But, in practice, it is possible only to check the measurability hypothesis. Our main theorem, which is Theorem 3.6, states that this is indeed enough and there is no distinction between measurable and continuous E_0-semigroups. As an application, we also show that any measurable isometric representation is strongly continuous.

The results of this paper for the case of a closed convex cone were proved initially in [12]. The paper [12] is not intended for publication.

2. Preliminaries

Throughout this paper, the letter G stands for a fixed but an arbitrary second countable locally compact Hausdorff topological group. The letter P stands for a closed subsemigroup of G containing the identity element $e \in G$. We denote the interior of P by Ω. We assume that Ω is dense in P. Observe that Ω is actually an ideal in P in the sense that $P\Omega \subseteq \Omega$ and $\Omega P \subseteq \Omega$.

For the rest of this article, the letter M stands for a von Neumann algebra with separable predual. Denote the predual of M by M_*. For $T \in M_*$ and $A \in M$, we write $T(A)$ as $\langle T, A\rangle$.

Definition 2.1. By an E_0-semigroup over P on M, we mean a family $\alpha = \{\alpha_x\}_{x\in P}$ of normal $*$-endomorphisms of M such that the following conditions hold:

(1) for $x, y \in P$, $\alpha_x \circ \alpha_y = \alpha_{xy}$;
(2) for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is continuous;
(3) for $x \in P$, $\alpha_x(1) = 1$; and
(4) the endomorphism α_e is the identity.

An E_0-semigroup over Ω of M is a family $\{\alpha_x\}_{x\in\Omega}$ of normal $*$-endomorphisms of M which satisfies the first three conditions of Definition 2.1. Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P on M. We say that α is faithful if $\bigcap_{x\in\Omega} \ker(\alpha_x) = \{0\}$.

Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P on M. A σ-weakly continuous family $\{U_x\}_{x\in P}$ of unitaries in M is called an α-cocycle if $U_{xy} = U_x\alpha_x(U_y)$ for all x, $y \in P$. Suppose that $\{U_x\}_{x\in P}$ is an α-cocycle. For $x \in P$, define $\beta_x = Ad(U_x) \circ \alpha_x$.

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Then $\beta = \{\beta_x\}_{x\in P}$ is an E_0-semigroup over P on M. We call such an E_0-semigroup a cocycle perturbation of α.

The aim of this article is to prove the following two statements.

THEOREM A. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of unital normal $*$-endomorphisms of M such that for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is measurable. Assume that α is faithful. Then $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on M.*

THEOREM B. *Let $\alpha = \{\alpha_x\}_{x\in\Omega}$ be an E_0-semigroup over Ω on M. Assume that α is faithful. Then there exists a unique E_0-semigroup over P on M, say $\widetilde{\alpha} = \{\widetilde{\alpha}_x\}_{x\in P}$, such that $\widetilde{\alpha}_x = \alpha_x$ for all $x \in \Omega$.*

3. Measurability implies continuity

Suppose that α is a normal $*$-endomorphism of M. Then, by duality, there exists a linear map $\beta : M_* \to M_*$ such that $\langle\beta(T), A\rangle = \langle T, \alpha(A)\rangle$ for all $T \in M_*$ and $A \in M$. In this case, we say that α and β are dual to each other.

Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of normal $*$-endomorphisms of M. Fix $x \in P$. Let $\beta_x : M_* \to M_*$ be the linear map on M_* such that $\langle\beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $T \in M_*$ and $A \in M$. Since $\|\alpha_x\| = 1$, $\|\beta_x\| = 1$. Hence, the family $\beta = \{\beta_x\}_{x\in P}$ is norm bounded.

PROPOSITION 3.1. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of unital normal $*$-endomorphisms of M. Assume that α restricted to Ω is an E_0-semigroup over Ω on M. If α is faithful, then α is an E_0-semigroup over P on M.*

PROOF. It is enough to prove that:

(1) the endomorphism α_e is the identity; and
(2) for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A))\rangle \in \mathbb{C}$ is continuous.

Let $A \in M$ and $x \in \Omega$ be given. Now $\alpha_x(A - \alpha_e(A)) = \alpha_x(A) - \alpha_x(A) = 0$. Since α is faithful, $\alpha_e(A) = A$. This proves (1). Let $x \in P$ be given. Let $T \in M_*$ and $A \in M$ be given. Suppose that (x_k) is a sequence in P such that $x_k \to x$. Note that $\alpha_{x_k}(A) \in \{T \in M : \|T\| \le \|A\|\}$. Since $\{T \in M : \|T\| \le \|A\|\}$ is σ-weakly compact, it is enough to prove that any convergent subsequence $(\alpha_{x_{k_i}}(A))$ of $(\alpha_{x_k}(A))$ converges to $\alpha_x(A)$. Let $(\alpha_{x_{k_i}}(A))$ be a subsequence of $(\alpha_{x_k}(A))$ such that $\alpha_{x_{k_i}}(A) \to B$ for some $B \in M$. We claim that $B = \alpha_x(A)$. Let $s \in \Omega$ be given. Note that (sx_{k_i}) is a sequence in Ω and $sx_{k_i} \to sx \in \Omega$. Hence, $\alpha_{sx_{k_i}}(A) \to \alpha_{sx}(A)$. On the other hand, $\alpha_{sx_{k_i}}(A) = \alpha_s(\alpha_{x_{k_i}}(A)) \to \alpha_s(B)$. Thus, $\alpha_s(B) = \alpha_s(\alpha_x(A))$ for all $s \in \Omega$. Since α is faithful, we have $B = \alpha_x(A)$. This proves (2). Hence, α is an E_0-semigroup over P on M. □

Denote the space of all compactly supported continuous functions on P such that the support is contained in Ω by $C_c(\Omega)$. Let μ be a left-invariant Haar measure on G, which will be fixed for the rest of this paper. We denote the modular function of G by Δ.

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REMARK 3.2. We need the following fact in what follows. Let P be a closed subsemigroup of a locally compact group G. Assume that $e \in P$ and $\mathrm{Int}(P)$ is dense in P. Let $A \subseteq G$ be nonempty, closed and $AP \subseteq A$. Then:

(1) the interior of A, $\mathrm{Int}(A)$, is dense in A; and
(2) the boundary of A has measure zero.

Hilgert and Neeb proved this statement for Lie semigroups [8, Lemma II.12]. In [18], Renault and Sundar noted that the statement remains valid for Ore semigroups. We should remark here that neither the smooth conditions on P nor the Ore condition is required. The proof given in Lemma II.12 carries over line by line.

LEMMA 3.3. *Let (a_k) be a sequence in G such that $a_k \to a \in G$. Then $1_{Pa_k} \to 1_{Pa}$ almost everywhere. Also, we have $1_{\Omega a_k} \to 1_{\Omega a}$ almost everywhere.*

PROOF. Suppose that $a_k \to a$. Let us denote the complement of the boundary of P by $\mathcal{B}$. Let $x \in \mathcal{B}$. Then $xa_k^{-1} \to xa^{-1} \in \mathcal{B}$. Then 1_{Pa_k} converges pointwise to 1_{Pa} on $\mathcal{B}$. Note that the boundary of P has measure zero by Remark 3.2. Hence, $1_{Pa_k} = 1_{\Omega a_k}$ almost everywhere. Therefore, $1_{Pa_k} \to 1_{Pa}$ almost everywhere. The proof is now completed. □

PROPOSITION 3.4. *Let $\alpha = \{\alpha_x\}_{x\in\Omega}$ be a semigroup over Ω on M such that for $x \in \Omega$, α_x is a unital normal $*$-endomorphism on M. Assume that for $T \in M_*$ and $A \in M$, the map $\Omega \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is measurable. Suppose that α is faithful. Then α is an E_0-semigroup over Ω on M.*

PROOF. Let $\{\beta_x\}_{x\in\Omega}$ be the family of linear maps on M_* such that $\langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $x \in \Omega$, $T \in M_*$ and $A \in M$. Then $\beta_{xy} = \beta_y\beta_x$ for all $x, y \in \Omega$. Let $T \in M_*$ and $A \in M$ be given. Since the map $\Omega \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is measurable, the map $\Omega \ni x \to \langle \beta_x(T), A\rangle \in \mathbb{C}$ is measurable. We claim that α is an E_0-semigroup over Ω on M. It is enough to prove that:

(1) the endomorphism α_e is the identity; and
(2) for $T \in M_*$ and $A \in M$, the map $\Omega \ni x \to \langle \beta_x(T), A\rangle \in \mathbb{C}$ is continuous.

The faithfulness of α implies that α_e is the identity map on M. This proves (1). Observe that $\beta = \{\beta_x\}_{x\in P}$ is norm bounded. To prove (2), it is enough to check the weak continuity of β on some norm-dense set in M_*.

For $f \in C_c(\Omega)$ and $T \in M_*$, let

$$T(f) := \int_\Omega f(x)\beta_x(T)\, d\mu(x).$$

The integral $T(f)$ exists as M_* is separable.

We claim that the linear span of $\{T(f) : T \in M_*, f \in C_c(\Omega)\}$ is norm dense in M_*.

Suppose that the linear span of $\{T(f) : T \in M_*, f \in C_c(\Omega)\}$ is not norm dense in M_*. Then, by the Hahn–Banach theorem, there exists a nonzero $A \in M$ such that

$$\langle T(f), A\rangle = \int_\Omega f(x)\langle \beta_x(T), A\rangle\, d\mu(x) = 0 \quad \text{for all } f \in C_c(\Omega).$$

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By the density of $C_c(\Omega)$ in $L^1(P)$, there exists a subset $E_T \subseteq \Omega$, depending on T, of measure zero such that $\langle \beta_x(T), A\rangle = 0$ almost everywhere for all $x \in \Omega\backslash E_T$. Let D be a countable dense subset of M_*. For $T \in D$, there exists a subset $E_T \subseteq \Omega$ of measure zero such that $\langle \beta_x(T), A\rangle = 0$ for all $x \in \Omega\backslash E_T$. Set $E = \bigcup_{T\in D} E_T$. Observe that the measure of E is zero and that $0 = \langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $x \in \Omega\backslash E$, $T \in D$. Since D is dense in M_*, $\langle T, \alpha_x(A)\rangle = 0$ for all $T \in M_*$ and $x \in \Omega\backslash E$. Therefore, $\alpha_x(A) = 0$ for all $x \in \Omega\backslash E$.

Let $y \in \Omega$ be given. Note that $\Omega^{-1}y$ is an open subset of G containing the identity e. Since Ω is dense in P, it follows that $\Omega^{-1}y \cap \Omega$ is a nonempty open subset of G. The fact that E has measure zero implies that $\Omega^{-1}y \cap \Omega \cap \Omega\backslash E$ is a nonempty subset of Ω. Let $x \in \Omega^{-1}y \cap \Omega \cap \Omega\backslash E$. Then $y = zx$ for some $z \in \Omega$. Since $\alpha_x(A) = 0$, $\alpha_y(A) = 0$. The faithfulness of α implies that $A = 0$, which is a contradiction. Hence, the linear span of $\{T(f) : T \in M_*, f \in C_c(\Omega)\}$ is norm dense in M_*.

Let $T \in M_*$ and $f \in C_c(\Omega)$ be given. Let $x \in \Omega$ and $\{x_k\}$ be a sequence in Ω such that $x_k \to x$. We calculate as follows to observe that

$$\begin{aligned}\beta_{x_k}(T(f)) &= \int_\Omega f(y)\beta_{yx_k}(T)\, d\mu(y)\\ &= \int_G f(ux_k^{-1})1_{\Omega x_k}(u)\beta_u(T)\, d\mu(ux_k^{-1})\\ &= \Delta(x_k^{-1})\int_G f(ux_k^{-1})1_{\Omega x_k}(u)\beta_u(T)\, d\mu(u).\end{aligned}$$

By the dominated convergence theorem (DCT) and by Lemma 3.3, we have that $\beta_{x_k}(T(f))$ converges weakly to $\beta_x(T(f))$. Thus, the map $\Omega \ni x \to \langle \beta_x(T(f)), A\rangle \in \mathbb{C}$ is continuous. Hence, α is an E_0-semigroup over Ω on M. □

Proposition 3.5. *Let $\alpha = \{\alpha_x\}_{x\in\Omega}$ be an E_0-semigroup over Ω on M. Assume that α is faithful. Then there exists a unique E_0-semigroup over P on M, say $\widetilde{\alpha} = \{\widetilde{\alpha}_x\}_{x\in P}$, such that $\widetilde{\alpha}_x = \alpha_x$ for all $x \in \Omega$.*

Proof. Let $\{\beta_x\}_{x\in\Omega}$ be the family of linear maps on M_* such that $\langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $x \in \Omega$, $T \in M_*$ and $A \in M$. Then $\beta_{xy} = \beta_y\beta_x$ for all $x, y \in \Omega$. Let $T \in M_*$ and $A \in M$ be given. Since the map $\Omega \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is continuous, the map $\Omega \ni x \to \langle \beta_x(T), A\rangle \in \mathbb{C}$ is continuous.

We know that the linear span of $\{T(f) : T \in M_*, f \in C_c(\Omega)\}$ is norm dense in M_* and the family β is norm bounded. Let $T \in M_*$ and $f \in C_c(\Omega)$ be given. Let $x \in P$ be given. Suppose that (x_k) is a sequence in Ω such that $x_k \to x \in P$.

Now, by a change of variable,

$$\begin{aligned}\beta_{x_k}(T(f)) &= \int_\Omega f(y)\beta_{yx_k}(T)\, d\mu(y)\\ &= \int_G f(ux_k^{-1})1_{\Omega x_k}(u)\beta_u(T)\, d\mu(ux_k^{-1})\\ &= \Delta(x_k^{-1})\int_G f(ux_k^{-1})1_{\Omega x_k}(u)\beta_u(T)\, d\mu(u).\end{aligned}$$

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By Lemma 3.3 and by DCT, $\beta_{x_k}(T(f))$ converges to

$$\Delta(x^{-1})\int_P f(ux^{-1})1_{\Omega x}(u)\beta_u(T)\,d\mu(u).$$

The above integral expression is independent of the chosen sequence (x_k).

For $x \in P$, define $\widetilde{\beta}_x(T) = \lim_{k\to\infty}\beta_{x_k}(T)$, where (x_k) is any sequence in Ω such that $x_k \to x \in P$ and the limit is taken in the weak sense. Clearly, $\widetilde{\beta}_x = \beta_x$ for all $x \in \Omega$. Hence, $\widetilde{\beta} = \{\widetilde{\beta}_x\}_{x\in P}$ is a family of linear maps on M_* such that for $x, y \in P, \widetilde{\beta}_{xy} = \widetilde{\beta}_y\widetilde{\beta}_x$ and for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle\widetilde{\beta}_x(T), A\rangle \in \mathbb{C}$ is continuous.

Fix $x \in P$ and $A \in M$. Let (x_k) be a sequence in Ω such that $x_k \to x \in P$. We claim that $\alpha_{x_k}(A)$ converges σ-weakly. Note that $\alpha_{x_k}(A) \in \{T \in M : \|T\| \le \|A\|\}$. Since $\{T \in M : \|T\| \le \|A\|\}$ is σ-weak compact, it is enough to show that any convergent subsequence of $(\alpha_{x_k}(A))$ converges to the same limit.

Claim. Suppose that (x_k), (y_k) are sequences in Ω and $B, C \in M$ are such that $x_k \to x$, $y_k \to x$, $\alpha_{x_k}(A) \to B$ and $\alpha_{y_k}(A) \to C$. Then $B = C$.

Let $s \in \Omega$ be given. Note that $sx_k \to sx$, $sy_k \to sx$ and $sx \in \Omega$. Observe that $\alpha_{sx_k}(A) = \alpha_s(\alpha_{x_k}(A))$. Taking limits, we obtain $\alpha_{sx}(A) = \alpha_s(B)$. Similarly, we obtain $\alpha_{sx}(A) = \alpha_s(C)$. Thus, $\alpha_s(B - C) = 0$ for every $s \in \Omega$. The faithfulness of α implies that $B = C$. This proves our claim. Hence, any convergent subsequence of $(\alpha_{x_k}(A))$ converges to the same limit. This also implies that the limit of $(\alpha_{x_k}(A))$ is independent of the chosen sequence (x_k).

For $x \in P$, let $\widetilde{\alpha}_x(A) := \lim_{k\to\infty}\alpha_{x_k}(A)$, where the limit is taken in the σ-weak sense. Clearly, $\widetilde{\alpha}_x = \alpha_x$ for all $\in \Omega$. Since α_y is unital for all $y \in \Omega$, $\widetilde{\alpha}_x(1) = \lim_{k\to\infty}\alpha_{x_k}(1) = 1$.

Let $x \in P$ be given. We claim that $\widetilde{\alpha}_x(U)$ is unitary whenever U is unitary in M. Let $U \in M$ be unitary and set $V := \widetilde{\alpha}_x(U)$. Choose a sequence (x_k) in Ω such that $x_k \to x$. Let $s \in \Omega$ be given. Note that $\alpha_{sx_k}(U) \to \alpha_{sx}(U)$, which is unitary. But $\alpha_{sx_k}(U) = \alpha_s(\alpha_{x_k}(U))$ converges, by definition, to $\alpha_s(\widetilde{\alpha}_x(U)) = \alpha_s(V)$. Now $\alpha_s(V^*V) = \alpha_s(V)^*\alpha_s(V) = \alpha_{sx}(U)^*\alpha_{sx}(U) = 1$. Similarly, $\alpha_s(VV^*) = 1$. Since α is faithful, it follows that $VV^* = V^*V = 1$. Therefore, $\widetilde{\alpha}_x(U)$ is unitary.

Claim. $\widetilde{\alpha}_x$ is a $*$-endomorphism of M. By definition, it is clear that $\widetilde{\alpha}_x$ is a $*$-linear map on M. Note that weak operator and strong operator topologies are same on the set of unitaries in M and the linear span of unitaries in M is strongly dense in M. Let U and V be given unitaries. We have

$$\begin{aligned}\widetilde{\alpha}_x(UV) &= \lim_{k\to\infty}\alpha_{x_k}(UV)\\ &= \lim_{k\to\infty}\alpha_{x_k}(U)\alpha_{x_k}(V)\\ &= \lim_{k\to\infty}\alpha_{x_k}(U)\lim_{k\to\infty}\alpha_{x_k}(V)\\ &= \widetilde{\alpha}_x(U)\widetilde{\alpha}_x(V).\end{aligned}$$

Hence, $\widetilde{\alpha}$ is a $*$-endomorphism of M.

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We claim that $\widetilde{\alpha}_x$ is normal. Note that $\langle\widetilde{\beta}_x(T), A\rangle = \langle T, \widetilde{\alpha}_x(A)\rangle$ for all $T \in M_*$ and $A \in M$. It follows that $\widetilde{\alpha}_x$ is normal. Thus, $\widetilde{\alpha} = \{\widetilde{\alpha}_x : x \in P\}$ is a family of unital normal $*$-endomorphisms of M.

Claim. $\widetilde{\alpha} = \{\widetilde{\alpha}_x\}_{x\in P}$ is an E_0-semigroup on M. Let $x, y \in P$ be given. Choose sequences (x_k) and (y_k) in Ω such that $x_k \to x$ and $y_k \to y$. Let $A \in M$ be given. Fix m and let k tend to infinity in the equation $\alpha_{x_k y_m}(A) = \alpha_{x_k} \circ \alpha_{y_m}(A)$ to obtain the equation $\alpha_{xy_m}(A) = \widetilde{\alpha}_x \circ \alpha_{y_m}(A)$. Now let m tend to infinity to arrive at the equation $\widetilde{\alpha}_{xy} = \widetilde{\alpha}_x \circ \widetilde{\alpha}_y$. This proves our claim. The faithfulness of α implies that $\widetilde{\alpha}_e$ is the identity map on M.

The σ-weak continuity of $\widetilde{\alpha} = \{\widetilde{\alpha}_x : x \in P\}$ is immediate from Proposition 3.1. Uniqueness follows from the density of Ω. This completes the proof. □

THEOREM 3.6. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of unital normal $*$-endomorphisms of M such that for $T \in M_*$ and $A \in M$, the map $P \ni x \to T(\alpha_x(A)) \in \mathbb{C}$ is measurable. Assume that α is faithful in the sense that $\bigcap_{x\in\Omega} \ker(\alpha_x) = \{0\}$. Then $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on M.*

PROOF. By Proposition 3.4, $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over Ω on M. By Proposition 3.1, $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on M. □

PROPOSITION 3.7. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P on M. Assume that α is faithful. Suppose that $\{U_x : x \in \Omega\}$ is a σ-weakly measurable family of isometries in M such that $U_{xy} = U_x\alpha_x(U_y)$ for all $x, y \in \Omega$. Then there exists a unique σ-weakly continuous family of isometries $\{\widetilde{U}_x\}_{x\in P}$ in M such that $\widetilde{U}_x = U_x$ for $x \in \Omega$. Moreover, if U_x, for $x \in \Omega$, is unitary in M, then $\widetilde{U}_x$, $x \in P$, is unitary in M.*

PROOF. The proof of σ-weak continuity of the family $\{U_x : x \in \Omega\}$ is the same as in [7, Proposition 2.3.1], in which Arveson used Connes' 2×2 matrix trick.

Fix $x \in P$. Let (x_k) be any sequence in Ω such that $x_k \to x \in P$. We claim that U_{x_k} converges σ-weakly to an isometry. Since the closed unit ball of M is σ-weak compact, it is enough to show that any convergent subsequence of (U_{x_k}) converges to the same limit.

Claim. Let (x_k) and (y_k) be sequences in Ω such that $x_k \to x$, $y_k \to x$, $U_{x_k} \to U$ and $U_{y_k} \to V$, where the convergence is in σ-weak topology. We claim that $U = V$ and U is an isometry. Moreover, if U_{x_k} and U_{y_k} are unitary, then U is unitary.

Choose a point $s \in \Omega$. Letting k tend to infinity in the equation $U_s\alpha_s(U_{x_k}) = U_{sx_k}$, we arrive at the equation $U_s\alpha_s(U) = U_{sx}$. Similarly, we obtain $U_s\alpha_s(V) = U_{sx}$. Thus, $\alpha_s(U) = \alpha_s(V)$ for $s \in \Omega$. The faithfulness of α implies that $U = V$. We calculate as follows to observe that

$$\begin{aligned}\alpha_s(U^*U) &= \alpha_s(U^*)\alpha_s(U)\\ &= \alpha_s(U^*)U_s^*U_s\alpha_s(U)\\ &= U_{sx}^*U_{sx}\\ &= 1\\ &= \alpha_s(1).\end{aligned}$$

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Since α is faithful, $U^*U = 1$ and hence U is an isometry. Also, the equation $U_s\alpha_s(U) = U_{sx}$ tells us that if U_x, for $x \in \Omega$, is unitary, then $\alpha_s(UU^*) = \alpha_s(1) = 1$. The faithfulness of α implies that $UU^* = 1$. Hence, U is unitary. This proves our claim. Hence, any convergent subsequence of (U_{x_k}) converges to the same limit. This also implies that the limit of (U_{x_k}) is independent of the chosen sequence (x_k).

For $x \in P$, let $\widetilde{U_x} := \lim_{k\to\infty} U_{x_k}$, where (x_k) is any sequence in Ω such that $x_k \to x \in P$.

Let (x_k) and (y_k) be sequences in Ω such that $x_k \to x$, $y_k \to y$. Fix k and m. Letting k tend to infinity in the equation $U_{x_k y_m} = U_{x_k}\alpha_{x_k}(U_{y_m})$, we get the equation $U_{xy_m} = \widetilde{U_x}\alpha_x(U_{y_m})$. Again letting m tend to infinity in the equation $U_{xy_m} = \widetilde{U_x}\alpha_x(U_{y_m})$, we arrive at the equation $\widetilde{U_{xy}} = \widetilde{U_x}\alpha_x(\widetilde{U_y})$. Thus, the family $\{\widetilde{U_x}\}_{x\in P}$ satisfies the α-cocycle equation. Uniqueness follows from the density of Ω. The proof is now complete. □

4. Applications of Theorem 3.6

Example 4.1 (CAR flow). Fix a Hilbert space $\mathcal{H}$. For $n \geq 1$, let $\mathcal{H}^{\otimes^n}$ be the n-fold tensor product of the Hilbert space $\mathcal{H}$. Let S_n be the group of all permutations of the set $\{1, 2, 3, \ldots, n\}$. For $\sigma \in S_n$, let U_σ be the unitary defined on the product vectors in $\mathcal{H}^{\otimes^n}$ by the relation

$$U_\sigma(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) = \zeta_{\sigma^{-1}(1)} \otimes \zeta_{\sigma^{-1}(2)} \otimes \cdots \otimes \zeta_{\sigma^{-1}(n)},$$

where σ^{-1} is the inverse of σ. Let $\mathcal{H}^{@^n} = \{\xi \in \mathcal{H}^{\otimes^n} : U_\sigma(\xi) = \epsilon(\sigma)\xi \text{ for all } \sigma \in S_n\}$, where $\epsilon(\sigma) = \pm 1$ according to whether the permutation σ is even or odd. Call $\mathcal{H}^{@^n}$ the n-fold antisymmetric tensor product of $\mathcal{H}$. For $n \geq 1$, let Q_n be the projection of $\mathcal{H}^{\otimes^n}$ onto $\mathcal{H}^{@^n}$. The image of $\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n$ under the projection Q_n will be denoted by $\zeta_1 \wedge \zeta_2 \wedge \cdots \wedge \zeta_n$.

Let $\Gamma_a(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{@^n}$ be the antisymmetric Fock space over $\mathcal{H}$. Note that the set $\{\bigwedge_{k=1}^n \zeta_k | n \in \mathbb{N}, \zeta_k \in \mathcal{H}\}$ is total in $\Gamma_a(H)$. Denote the vector $1 \oplus 0 \oplus 0 \oplus \cdots$ by Θ.

For each $\zeta \in \mathcal{H}$, we define a bounded linear operator $a(\zeta)$ on $\Gamma_a(\mathcal{H}\oplus)$, where $a(\zeta) : \mathcal{H}_n \to \mathcal{H}_{n+1}$, by the relation

$$a(\zeta)\left(\bigwedge_{k=1}^{n}\right) = \sqrt{n+1}\,\zeta \wedge \zeta_1 \wedge \cdots \wedge \zeta_n \quad \text{for all } \zeta_k \in \mathcal{H},$$
$$1 \leq k \leq n, \quad n \in \mathbb{N}, \quad a(\zeta)(\Phi) = \zeta.$$

One can find the adjoint $a^*(\zeta)$ of $a(\zeta)$ which maps $\mathcal{H}_{n+1}$ into $\mathcal{H}_n$ as follows:

$$a^*(\zeta)\left(\bigwedge_{k=1}^{n+1}\right) = (n+1)^{-1/2} \sum_{k=1}^{n+1} (-1)^{k-1} \langle \zeta, \zeta_k \rangle \zeta \wedge \zeta_1 \wedge \cdots \wedge \hat{\zeta}_k \wedge \cdots \wedge \zeta_{n+1},$$

where the hat ˆ over ζ_k indicates its omission.

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The family $\{a(\zeta)\}_{\zeta\in\mathcal{H}}$ of bounded operators satisfies the following CAR relation. For $\zeta, \eta \in \mathcal{H}$ and $\lambda \in \mathbb{C}$:

(1) $a(\lambda\zeta + \eta) = \lambda a(\zeta) + a^{\dagger}(\eta)$;
(2) $a(\zeta)a(\eta) + a(\eta)a(\zeta) = 0$;
(3) $a(\zeta)a^{*}(\eta) + a^{*}(\eta)a(\zeta) = \langle\zeta, \eta\rangle 1$.

By [14, Proposition 23.4], the set $\{a(\zeta), a^{*}(\eta) : \zeta, \eta \in \mathcal{H}\}$ is irreducible in $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Remark 4.2. Suppose that V is an isometry acting on a Hilbert space $\mathcal{H}$. Then there exists a unique normal $*$-endomorphism of $\mathcal{B}(\Gamma_a(\mathcal{H}))$, say α_V, such that $\alpha_V(a(\zeta)) = a(V(\zeta))$ for all $\zeta \in \mathcal{H}$. For a proof of this fact, the reader is referred to the monograph [7] of Arveson.

Let $V = \{V_x\}_{x\in P}$ be a semigroup of strongly continuous isometries acting on a Hilbert space $\mathcal{H}$. For every fixed $x \in P$, let $\alpha_x : \mathcal{B}(\Gamma_a(\mathcal{H})) \to \mathcal{B}(\Gamma_a(\mathcal{H}))$ be the normal $*$-endomorphism such that

$$\alpha_x(a(\zeta)) = a(V_x\zeta) \quad \text{for all } \zeta \in \mathcal{H}.$$

Since the family $V = \{V_x\}_{x\in P}$ is a semigroup, the family $\{\alpha_x\}_{x\in P}$ forms a semigroup of unital, normal $*$-endomorphisms on $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Note that the linear span of a finite product of $\{a(\zeta), a^{*}(\eta) : \zeta, \eta \in \mathcal{H}\}$ is strongly dense in $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

For every normal linear functional T on $\mathcal{B}(\Gamma_a(\mathcal{H}))$ and A belonging to the linear span of $\{a^{\dagger}(\zeta), a(\eta) : \zeta, \eta \in \mathcal{H}\}$, the map $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is continuous.

Suppose that T is a normal linear functional on $\mathcal{B}(\Gamma_a(\mathcal{H}))$. Let $A \in \mathcal{B}(\Gamma_a(\mathcal{H}))$ be given. By the Kaplansky density theorem, there exists a sequence $\{A_k\}$ in a linear span of finite products of $\{a(\zeta), a^{*}(\eta) : \zeta, \eta \in \mathcal{H}\}$ such that $\|A_k\| \le \|A\|$ and $(A_k) \to A$ in strong operator topology (SOT). Therefore, the map $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is measurable. Hence, by Theorem 3.6, $\{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Remark 4.3. Let α and β be normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. Then there exists a unique normal $*$-endomorphism from $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ into $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$, denoted by $\alpha\otimes\beta$, such that $\alpha\otimes\beta(A\otimes B) = \alpha(A)\otimes\beta(B)$ for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. For a proof this fact, the reader is referred to the monograph [7] of Arveson.

Example 4.4 (Tensor product). Let $\alpha = \{\alpha_x\}_{x\in P}$ and $\beta = \{\beta_x\}_{x\in P}$ be E_0-semigroups on $B(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. The tensor product $\alpha\otimes\beta$ is the semigroup $\{(\alpha\otimes\beta)_x\}_{x\in P}$ of unital normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$, where $(\alpha\otimes\beta)_x : \mathcal{B}(\mathcal{H}\otimes\mathcal{K}) \to \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is defined by the formula $(\alpha\otimes\beta)_x(A\otimes B) = \alpha_x(A)\otimes\beta_x(B)$ for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$.

Observe that the linear span of the set $\{A\otimes B : A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K})\}$ is strongly dense in $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$.

Let $\xi_1, \xi_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{K}$ be given. Suppose that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})\}$ are given. Then the map $P \ni x \to \langle\alpha_x(A)(\xi_1), \xi_2\rangle\langle\beta_x(B)(\eta_1), \eta_2\rangle \in \mathbb{C}$ is continuous.

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By a Kaplansky density theorem argument, it follows that the map $P \ni x \to (\alpha \otimes \beta)_x \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is weakly measurable. Hence, by Theorem 3.6, $\{(\alpha \otimes \beta)_x\}_{x \in P}$ is an E_0-semigroup over P on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Now we shall see that every strongly measurable isometric representation of P on $\mathcal{H}$ is strongly continuous.

Definition 4.5. An isometric representation of P on a separable Hilbert space $\mathcal{H}$ is a map $V : P \to \mathcal{B}(\mathcal{H})$ such that:

(1) for $x \in P$, V_x is an isometry;
(2) for $x, y \in P$, $V_{x+y} = V_x V_y$; and
(3) for $\xi \in \mathcal{H}$, the map $P \ni x \to V_x(\xi) \in \mathcal{H}$ is continuous.

Proposition 4.6. *Let $V = \{V_x\}_{x \in P}$ be a semigroup of isometries on $\mathcal{H}$ such that for $\xi \in \mathcal{H}$, the map $P \ni x \to V_x(\xi) \in \mathcal{H}$ is measurable. Then, for $\xi \in \mathcal{H}$, the map $P \ni x \to V_x(\xi) \in \mathcal{H}$ is continuous.*

Proof. Let $\Gamma_a(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{@n}$ be the antisymmetric Fock space over $\mathcal{H}$. Denote the vector $1 \oplus 0 \oplus \cdots$ by Θ. For every fixed $x \in P$, let $\alpha_x : \mathcal{B}(\Gamma_a(\mathcal{H})) \to \mathcal{B}(\Gamma_a(\mathcal{H}))$ be the normal $*$-endomorphism such that

$$\alpha_x(a(\zeta)) = a(V_x \zeta) \quad \text{for all } \zeta \in \mathcal{H}.$$

Since the family $V = \{V_x\}_{x \in P}$ is a semigroup, the family $\{\alpha_x\}_{x \in P}$ forms a semigroup of unital, normal $*$-endomorphisms on $\mathcal{B}(\Gamma_a(\mathcal{H}))$. Note that $\alpha = \{\alpha_x\}_{x \in P}$ forms a measurable E_0-semigroup. By Theorem 3.6, $\alpha = \{\alpha_x\}_{x \in P}$ is an E_0-semigroup. Hence, for $\xi \in \mathcal{H}$, the map $P \ni x \to \alpha_x(a(\xi))\Theta = a(V_x \xi)\Theta = V_x(\xi) \in \mathcal{H}$ is continuous. □

Remark 4.7. Proposition 4.4 for the case of a closed convex cone $\mathbb{R}^d$ is known (see [9, page 303]). The author could not find any other reference in the literature which handles general semigroups.

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