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X-ROKHLIN PROPERTY FOR ACTIONS OF COUNTABLE DISCRETE GROUPS

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Abstract

We extend the definition of the X-Rokhlin property to countable discrete groups and prove some permanence properties. If the action of a countable discrete group on X is free and minimal and the action of this group on the separable simple C^*-algebra has the X-Rokhlin property, then the reduced crossed product is simple.

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1. Introduction

Given a C^*-dynamical system (G, A, α), the Rokhlin property is an important condition on α that ensures good properties transfer from A to $A \rtimes_\alpha G$. In [8, 9], Izumi began his work on finite group actions with the Rokhlin property. Since then, there have been many studies on the structure of crossed product C^*-algebras. However, actions with the Rokhlin property are rare; many C^*-algebras do not admit finite group actions with the Rokhlin property for any finite groups. To address this issue, Phillips introduced the tracial Rokhlin property in [11]. Nevertheless, both definitions have the disadvantage that they involve C^*-algebras with many projections. However, some important examples of C^*-algebras are projectionless. There is another way to extend the Rokhlin property. Hirshberg, Winter and Zacharias introduced the concept of the Rokhlin dimension for actions of finite groups and integers in [7]. This definition uses contractive positive elements instead of projections. For actions of finite groups, the Rokhlin dimension zero is equivalent to the Rokhlin property. Subsequently, the Rokhlin dimension has been extended to actions of more general groups: for compact groups by Gardella in [3, 4], for residually finite groups by Szabó, Wu and Zacharias in [13], and for the group of real numbers by Hirshberg *et al.* in [6].

In [5], Hirshberg and Phillips defined a new Rokhlin-type property for compact group actions, which is called the X-Rokhlin property. They showed that if an action

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of a finite group G has finite Rokhlin dimension with commuting towers, then there exists a compact metric G-space X such that the action has the X-Rokhlin property. In [3], Gardella also proved that if an action of a compact group G has finite Rokhlin dimension with commuting towers, then there exists a compact metric G-space X such that the action has the X-Rokhlin property. In [12], Sureshkumar and Vaidyanathan proved that if an action of a finitely generated residually finite group G has finite Rokhlin dimension with commuting towers, then there also exists a unital equivariant homomorphism from $C(\bar{G}^{*(d+1)})$ to the central sequence algebra of the original C^*-algebra. It seems that the Rokhlin dimension can be regarded as a special case of the X-Rokhlin property. The definition of the X-Rokhlin property does not depend on the compactness of the group. If G is a countable discrete group and we define the X-Rokhlin property for the action of G, what properties can we obtain? This is the motivation for this paper.

The goal of this paper is to study the actions of countable discrete groups with the X-Rokhlin property. The main theorem of this paper is as follows.

THEOREM 1.1. *Let G be a discrete countable group and X a compact metric space endowed with a free and minimal action of G. Let A be a separable C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. If α has the X-Rokhlin property, then α is pointwise outer.*

Moreover, applying the above theorem, we can prove that the reduced crossed product is simple if the coefficient C^*-algebra is simple.

This paper is organised as follows. In Section 2, we provide some basic notions and preliminaries. In Section 3, we introduce the definition of the X-Rokhlin property for actions of countable discrete groups. In Section 4, we prove some permanence properties of the X-Rokhlin property. In Section 5, we study the simplicity of reduced crossed products and prove the main theorem of this paper.

2. Preliminaries

2.1. Some basic notation. We use the following notation in this paper.

NOTATION 2.1. Let A be a C^*-algebra. We denote by $M(A)$ the multiplier algebra of A and by $\mathrm{Aut}(A)$ the automorphism group of A. The centre of A is denoted by $\mathcal{Z}(A)$.

Let G be a group and X a compact metric space endowed with a free action of G. We always denote the induced action of G on $C(X)$ by λ.

The abbreviation c.p.c. stands for completely positive contractive.

2.2. Central sequence algebras. Let A be a C^*-algebra. We denote by $\ell^\infty(\mathbb{N}, A)$ the set of all bounded sequences in A. Then, it is a C^*-algebra with the supremum norm and pointwise operations. Let

$$c_0(\mathbb{N}, A) = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\};$$

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$c_0(\mathbb{N}, A)$ is an ideal in $\ell^\infty(\mathbb{N}, A)$. We denote by A_∞ the quotient $\ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ and write $\eta_A : \ell^\infty(\mathbb{N}, A) \to A_\infty$ for the quotient map. The C^*-algebra A can be embedded in A_∞ as constant sequences and we denote it by $\iota_A : A \to A_\infty$.

Let $D \subseteq A$ be a subalgebra. Denote

$$A_\infty \cap D' = \{a \in A_\infty \mid ad = da \text{ for all } d \in D\}.$$

It is the relative commutant of D in A_∞. Let

$$\mathrm{Ann}(D, A_\infty) = \{a \in A_\infty \mid ad = da = 0 \text{ for all } d \in D\};$$

$\mathrm{Ann}(D, A_\infty)$ is a closed two-sided ideal in $A_\infty \cap D'$. We set

$$F(D, A_\infty) = (A_\infty \cap D')/\mathrm{Ann}(D, A_\infty),$$

the central sequence algebra of A relative to D. We write $\kappa_{D,A} : A_\infty \cap D' \to F(D, A_\infty)$ for the quotient map. If D is σ-unital, then $F(D, A_\infty)$ is unital. If $D = A$, we write $F_\infty(A) = F(A, A_\infty)$. Moreover, if A is unital, then $F_\infty(A) = A_\infty \cap A'$.

Let G be a countable discrete group, $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on A, and D be an α-invariant subalgebra of A. Since every automorphism on A naturally induces an automorphism on $\ell^\infty(\mathbb{N}, A)$ by component-wise application, α induces an action $\ell^\infty(\alpha)$ of G on $\ell^\infty(\mathbb{N}, A)$. Furthermore, $c_0(\mathbb{N}, A)$ is an $\ell^\infty(\alpha)$-invariant ideal of $\ell^\infty(\mathbb{N}, A)$. Thus, $\ell^\infty(\alpha)$ induces an action on A_∞, which we denote by α_∞. Then, α_∞ can be restricted to an action on $A_\infty \cap D'$. For convenience, we still denote it by α_∞. Moreover, $\mathrm{Ann}(D, A_\infty)$ is an α_∞-invariant ideal of $A_\infty \cap D'$. It also induces an action on $F(D, A_\infty)$, which we denote by α_F.

3. *X*-Rokhlin property

In this section, we introduce the X-Rokhlin property for discrete group actions. We begin with the definition of the X-Rokhlin property for compact group actions in [5]. Let G be a locally compact group, A, B be C^*-algebras and $\alpha : G \to \mathrm{Aut(A)}$, $\beta : G \to \mathrm{Aut(B)}$ be point-norm continuous actions. We say that a homomorphism $\varphi : A \to B$ is equivariant if $\beta_g(\varphi(a)) = \varphi(\alpha_g(a))$ for all $a \in A$ and $g \in G$.

DEFINITION 3.1 [5, Definition 1.6]. Let G be a compact group, let A be a unital separable C^*-algebra and let α be an action of G on A. Let X be a compact metric space endowed with a free action of G. We say that α has the X-Rokhlin property if there exists an equivariant unital homomorphism φ from $C(X)$ to $A_\infty \cap A'$.

It is also shown in [5] that the X-Rokhlin property is indeed a generalisation of the finite Rokhlin dimension for finite group actions.

LEMMA 3.2 [5, Lemma 1.7]. *For any finite group G and every nonnegative integer d, there is a compact metrisable free G-space X such that an action $\alpha : G \to \mathrm{Aut}(A)$ on a unital C^*-algebra A has $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) \leq d$ if and only if α has the X-Rokhlin property.*

In [3], Gardella introduced the Rokhlin dimension for the actions of compact groups. He also proved that the finite Rokhlin dimension for compact group actions is

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a special case of the X-Rokhlin property. In [13], Szabó *et al.* introduced the Rokhlin dimension for actions of residually finite groups. Sureshkumar and Vaidyanathan show a similar theorem as follows.

THEOREM 3.3 [12]. *Let G be a finitely generated residually finite group, let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then, $\dim^{c}_{\mathrm{Rok}}(\alpha) \leq d$ if and only if for each α-invariant separable C^*-subalgebra $D \subseteq A$, there is a unital equivariant homomorphism $\phi\colon C(\bar{G}^{*(d+1)}) \to F(D, A_\infty)$.*

For general groups, the X-Rokhlin property seems to be a good way to extend the Rokhlin dimension. Now, we introduce the definition of the X-Rokhlin property for the actions of countable discrete groups.

DEFINITION 3.4. Let G be a countable discrete group and X a compact metric space endowed with a free action of G. Let A be a C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α has the X-Rokhlin property if for any separable, α-invariant C^*-subalgebra $D \subseteq A$, there exists a unital $\lambda - \alpha_F$ equivariant homomorphism $\varphi\colon C(X) \to F(D, A_\infty)$.

REMARK 3.5. If A is separable, then α has the X-Rokhlin property if and only if there exists a unital $\lambda - \alpha$ equivariant homomorphism from $C(X)$ to $F_\infty(A)$. In fact, there is always a conclusion $A_\infty \cap A' \subseteq A_\infty \cap D'$.

We give some examples that have the X-Rokhlin property.

EXAMPLE 3.6. If G is a finite group and α has finite Rokhlin dimension with commuting towers, then there exists X such that α has the X-Rokhlin property (Lemma 3.2).

If G is a finitely generated residually finite group and α has finite Rokhlin dimension with commuting towers, then there exists X such that α has the X-Rokhlin property (Theorem 3.3).

Let X be a compact Hausdorff space. Recall that a unital $C(X)$-algebra is a unital C^*-algebra A endowed with a unital $*$-homomorphism $\eta\colon C(X) \to \mathcal{Z}(A)$.

EXAMPLE 3.7. Let X be a compact metric space and let A be a unital $C(X)$-algebra. Let G be a countable discrete group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action and let $T\colon G \to \mathrm{Homeo(X)}$ be a free action on X. If α lies over T, that is, $\eta(f \circ T_g^{-1}) = \alpha_g(\eta(f))$ for all $f \in C(X)$ and $g \in G$, then η is a unital equivariant homomorphism from $C(X)$ to A. Thus, α has the X-Rokhlin property.

In the following proposition, we provide an equivalent local characterisation of the X-Rokhlin property.

PROPOSITION 3.8. *Let G be a countable discrete group and X a compact metric space with a free action of G. Let A be a C^*-algebra and $\alpha\colon G \to \mathrm{Aut(A)}$ be an action. Then, α has the X-Rokhlin property if and only if for any finite subsets $F \subseteq A$, $M \subseteq G$, $S \subseteq C(X)$ and $\varepsilon > 0$, there exist a c.p.c. order zero map $\psi\colon C(X) \to A$ satisfying:*

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(1) $\|\psi(f_1)\psi(f_2)a - \psi(f_1f_2)a\| < \varepsilon$ *for all* $f_1, f_2 \in S$ *and* $a \in F$;
(2) $\|\psi(f)a - a\psi(f)\| < \varepsilon$ *for all* $f \in S$ *and* $a \in F$;
(3) $\|\alpha_g(\psi(f))a - \psi(\lambda_g(f))a\| < \varepsilon$ *for all* $f \in S, a \in F$ *and* $g \in M$;
(4) $\|\psi(1)a - a\| < \varepsilon$ *for all* $a \in F$.

PROOF. Assume α has the X-Rokhlin property. For any finite subsets $F \subseteq A$, $M \subseteq G$, $S \subseteq C(X)$ and $\varepsilon > 0$, let D be the C^*-subalgebra generated by $\{\alpha_g(a)\colon a \in F, g \in G\}$. Obviously, D is a separable, α-invariant C^*-subalgebra with $F \subseteq D$. Moreover, since α has the X-Rokhlin property, there exists a unital equivariant homomorphism $\varphi\colon C(X) \to F(D, A_\infty)$. Since $C(X)$ is nuclear, we can use the Choi–Effros lifting theorem [2] and find a c.p.c. map $\phi\colon C(X) \to A_\infty \cap D'$ such that $\kappa_{D,A} \circ \phi = \varphi$. We claim that ϕ satisfies:

(a1) $\phi(f_1)\phi(f_2)a = \phi(f_1f_2)a$ for all $f_1, f_2 \in C(X)$ and $a \in D$;
(a2) $\alpha_{\infty,g}(\phi(f))a = \phi(\lambda_g(f))a$ for all $f \in C(X), g \in G$ and $a \in D$;
(a3) $\phi(1)a = a$ for all $a \in D$.

For claim (a1), since φ is a homomorphism, for any $f_1, f_2 \in C(X)$, we have $\varphi(f_1)\varphi(f_2) = \varphi(f_1f_2)$. It follows that, $\kappa_{D,A}(\phi(f_1))\kappa_{D,A}(\phi(f_2)) = \kappa_{D,A}(\phi(f_1f_2))$ and so $\phi(f_1)\phi(f_2) - \phi(f_1f_2) \in \mathrm{Ann}(D, A_\infty)$. That is,

$$\phi(f_1)\phi(f_2)a = \phi(f_1f_2)a$$

for any $f_1, f_2 \in C(X)$ and $a \in D$.

For claim (a2), since φ is equivariant, for any $f \in C(X)$ and $g \in G$, we have $\alpha_{F,g}(\varphi(f)) = \varphi(\lambda_g(f))$. Then, $\alpha_{F,g}(\kappa_{D,A}(\phi(f))) = \kappa_{D,A}(\phi(\lambda_g(f)))$. Moreover, $\kappa_{D,A}(\alpha_{\infty,g}(\phi(f))) = \kappa_{D,A}(\phi(\lambda_g(f)))$. Thus,

$$\alpha_{\infty,g}(\phi(f))a = \phi(\lambda_g(f))a$$

for all $f \in C(X), g \in G$ and $a \in D$.

For claim (a3), recall that $1_{F(D,A_\infty)} = \kappa_{D,A}(\eta_A((e_n)_{n\in\mathbb{N}}))$, where $(e_n)_{n\in\mathbb{N}}$ is the countable approximation unit of D. It is easy to see $\phi(1)a = a$ for all $a \in D$. The claim is proved.

In fact, ϕ is a c.p.c. map from $C(X)$ to A_∞, so we can use the Choi–Effros lifting theorem again and obtain a c.p.c. map $\Psi = (\psi_n)\colon C(X) \to \ell^\infty(\mathbb{N}, A)$ such that $\eta_A \circ \Psi = \phi$. We claim that Ψ satisfies:

(b1) $\|\psi_n(f_1)\psi_n(f_2)a - \psi_n(f_1f_2)a\| \to 0, n \to \infty$ for all $f_1, f_2 \in C(X)$ and $a \in D$;
(b2) $\|\psi_n(f)a - a\psi_n(f)\| \to 0, n \to \infty$ for all $f \in C(X)$ and $a \in D$;
(b3) $\|\alpha_g(\psi_n(f))a - \psi_n(\lambda_g(f))a\| \to 0, n \to \infty$ for all $f \in C(X), g \in G$ and $a \in D$;
(b4) $\|\psi_n(1)a - a\| \to 0, n \to \infty$ for all $a \in D$.

Similarly to the proof above, claim (b1) comes from claim (a1), claim (b3) comes from claim (a2), and claim (b4) comes from claim (a3). For claim (b2), since ϕ is a map from $C(X)$ to $A_\infty \cap D'$, we have $\phi(f)a = a\phi(f)$ for any $f \in C(X)$ and $a \in D$. Then, $\eta_A(\Psi(f))a = a\eta_A(\Psi(f))$. Moreover, $\eta_A(a) = a$ for any $a \in D$. Thus, $\psi_n(f)a - a\psi_n(f) \in$

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$c_0(\mathbb{N}, A)$ for any $f \in C(X)$ and $a \in D$. The claim is proved. Since F, M, S are all finite sets, for any $\varepsilon > 0$, we can find a c.p.c. map ψ_{n_0} with the required properties.

Now, if α satisfies the conditions above, we will show α has the X-Rokhlin property. For any α-invariant separable C^*-subalgebra of A, we can find an increasing sequence of finite subsets such that $\overline{\bigcup_{n\in\mathbb{N}} F_n} = D$. Since G is countable and $C(X)$ is separable, we can also find increasing sequences of subsets such that $\overline{\bigcup_{n\in\mathbb{N}} M_n} = G$ and $\overline{\bigcup_{n\in\mathbb{N}} S_n} = C(X)$. For every $n \in \mathbb{N}$, there exists a c.p.c. map $\psi_n \colon C(X) \to A$ satisfying conditions (1), (2), (3) and (4) for the quadruple $(1/n, F_n, M_n, S_n)$. Thus, we obtain a c.p.c. map $\Psi = (\psi_n) \colon C(X) \to \ell^\infty(\mathbb{N}, A)$ satisfying:

(c1) $\|\psi_n(f_1)\psi_n(f_2)a - \psi_n(f_1f_2)a\| \to 0, n \to \infty$ for all $f_1, f_2 \in \bigcup_{n\in\mathbb{N}} S_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(c2) $\|\psi_n(f)a - a\psi_n(f)\| \to 0, n \to \infty$ for all $f \in \bigcup_{n\in\mathbb{N}} S_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(c3) $\|\alpha_g(\psi_n(f))a - \psi_n(\lambda_g(f))a\| \to 0, n \to \infty$ for all $f \in \bigcup_{n\in\mathbb{N}} S_n, g \in \bigcup_{n\in\mathbb{N}} M_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(c4) $\|\psi_n(1)a - a\| \to 0, n \to \infty$ for all $a \in \bigcup_{n\in\mathbb{N}} F_n$.

Define $\phi = \eta_A \circ \Psi \colon C(X) \to A_\infty$. Then, ϕ satisfies:

(d1) $\phi(f_1)\phi(f_2)a = \phi(f_1f_2)a$ for all $f_1, f_2 \in \bigcup_{n\in\mathbb{N}} S_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(d2) $\phi(f)a = a\phi(f)$ for all $f \in \bigcup_{n\in\mathbb{N}} S_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(d3) $\alpha_{\infty,g}(\phi(f))a = \phi(\lambda_g(f))a$ for all $f \in \bigcup_{n\in\mathbb{N}} S_n, g \in \bigcup_{n\in\mathbb{N}} M_n$ and $a \in \bigcup_{n\in\mathbb{N}} F_n$;
(d4) $\phi(1)a = a$ for all $a \in \bigcup_{n\in\mathbb{N}} F_n$.

Since $\overline{\bigcup_{n\in\mathbb{N}} F_n} = D$, $\overline{\bigcup_{n\in\mathbb{N}} M_n} = G$ and $\overline{\bigcup_{n\in\mathbb{N}} S_n} = C(X)$, ϕ is actually a c.p.c. map from $C(X)$ to $A_\infty \cap D'$ satisfying:

(e1) $\phi(f_1)\phi(f_2)a = \phi(f_1f_2)a$ for all $f_1, f_2 \in C(X)$ and $a \in D$;
(e2) $\alpha_{\infty,g}(\phi(f))a = \phi(\lambda_g(f))a$ for all $f \in C(X), g \in G$ and $a \in D$;
(e3) $\phi(1)a = a$ for all $a \in D$.

Then, $\varphi = \kappa_{D,A} \circ \phi$ is a unital equivariant homomorphism from $C(X)$ to $F(D, A_\infty)$. Hence, α has the X-Rokhlin property. □

4. Permanence properties

In this section, we will prove some permanence properties for actions with the X-Rokhlin property analogous to those proved by Gardella for compact group actions with finite Rokhlin dimension.

THEOREM 4.1. *Let G be a countable discrete group and X a compact metric space with a free action of G. Let A be a C^*-algebra and $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Let I be an α-invariant ideal in A and $\bar{\alpha}$ be the induced action on A/I. If α has the X-Rokhlin property, then $\bar{\alpha}$ has the X-Rokhlin property.*

PROOF. We denote the canonical quotient map $A \to A/I$ by π and we apply Proposition 3.8 to prove the theorem. For any $\varepsilon > 0$ and finite subsets $F \subseteq A/I, M \subseteq G$

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and $S \subseteq C(X)$, we can choose a finite subset $F' \subseteq A$ such that $\pi(F') = F$. Then, for (ε, F', M, S), there exists a c.p.c. map $\varphi\colon C(X) \to A$ that meets conditions (1), (2), (3) and (4) in Proposition 3.8. Let $\psi = \pi \circ \varphi\colon C(X) \to A/I$. Then, ψ is a c.p.c. map. For any $f, f_1, f_2 \in S, g \in M$ and $a \in F$, there exists $b \in F'$ such that $\pi(b) = a$.

For condition (1),

$$\begin{aligned}\|\psi(f_1)\psi(f_2)a - \psi(f_1f_2)a\| &= \|\pi(\varphi(f_1)\varphi(f_2)b - \varphi(f_1f_2)b)\| \\ &\le \|\varphi(f_1)\varphi(f_2)b - \varphi(f_1f_2)b\| < \varepsilon.\end{aligned}$$

For condition (2),

$$\|\psi(f)a - a\psi(f)\| = \|\pi(\varphi(f)b - b\varphi(f))\| \le \|\varphi(f)b - b\varphi(f)\| < \varepsilon.$$

For condition (3),

$$\begin{aligned}\|\bar{\alpha}_g(\psi(f))a - \psi(\lambda_g(f))a\| &= \|\pi(\alpha_g(\varphi(f))b - \varphi(\lambda_g(f))b)\| \\ &\le \|\alpha_g(\varphi(f))b - \varphi(\lambda_g(f))b\| < \varepsilon.\end{aligned}$$

For condition (4),

$$\|\psi(1)a - a\| = \|\pi(\varphi(1)b - b)\| \le \|\varphi(1)b - b\| < \varepsilon.$$

Therefore, $\bar{\alpha}$ has the X-Rokhlin property. □

THEOREM 4.2. *Let G be a countable discrete group and X a compact metric space with a free action of G. Let A be a C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.*

(a) *Let $E \subseteq A$ be an α-invariant hereditary subalgebra and δ be the induced action. If α has the X-Rokhlin property, then δ has the X-Rokhlin property.*
(b) *Let B be a C^*-algebra and $\beta\colon G \to \mathrm{Aut}(B)$ an action. Let $A \otimes B$ be any C^*-algebra completion of the algebraic tensor product for which the tensor product action $\alpha \otimes \beta$ is defined. If α has the X-Rokhlin property, then $\alpha \otimes \beta$ has the X-Rokhlin property.*

PROOF. (a) Let $\varepsilon > 0$ and consider finite subsets $F \subseteq E, M \subseteq G$ and $S \subseteq C(X)$. Without loss of generality, we assume F and S both consist of contractions. Then we can apply [1, Lemma 4.9] and choose an approximate unit $e \in E$ such that $\|\alpha_g(e) - e\| < \varepsilon/6, \|ea - a\| < \varepsilon/6, \|ae - a\| < \varepsilon/6$, and $\|eae - a\| < \varepsilon/6$ for all $g \in M$ and $a \in F$. Since α has the X-Rokhlin property, for $\varepsilon/6, F \cup \{e\}, M$ and S, there exists a c.p.c. map $\varphi\colon C(X) \to A$ satisfying the conditions in Proposition 3.8. Define $\psi\colon C(X) \to E$ by $\psi(f) = e\varphi(f)e$ for all $f \in C(X)$. Choose any $f, f_1, f_2 \in S, g \in M$ and $a \in F$.

For condition (1),

$$\begin{aligned}\|\psi(f_1)\psi(f_2)a - \psi(f_1f_2)a\| &= \|e\varphi(f_1)ee\varphi(f_2)a - e\varphi(f_1f_2)ea\| \\ &\le \|\varphi(f_1)\varphi(f_2)eea - \varphi(f_1f_2)ea\| + 2\varepsilon/6 \\ &\le \|\varphi(f_1)\varphi(f_2)a - \varphi(f_1f_2)a\| + 5\varepsilon/6 < \varepsilon.\end{aligned}$$

For condition (2),

$$\|\psi(f)a - a\psi(f)\| = \|e\varphi(f)ea - ae\varphi(f)e\| \le \|\varphi(f)a - a\varphi(f)\| + 2\varepsilon/6 < \varepsilon.$$

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For condition (3),

$$\begin{aligned}\|\delta_g(\psi(f))a - \psi(\lambda_g(f))a\| &= \|\delta_g(e\varphi(f)e)a - e\varphi(\lambda_g(f))ea\| \\ &\leq \|e\alpha_g(\varphi(f))ea - e\varphi(\lambda_g(f))ea\| + 2\varepsilon/6 \\ &\leq \|\alpha_g(\varphi(f))a - \varphi(\lambda_g(f))a\| + 4\varepsilon/6 < \varepsilon.\end{aligned}$$

For condition (4),

$$\|\psi(1)a - a\| = \|e\varphi(1)ea - a\| \leq \|\varphi(1)a - a\| + 2\varepsilon/6 < \varepsilon.$$

Hence, δ has the X-Rokhlin property.

(b) Let $\varepsilon > 0$ and consider finite subsets $F \subseteq A \otimes B, M \subseteq G$ and $S \subseteq C(X)$. Without loss of generality, we assume F and S both consist of contractions. We may also assume $F \subseteq A \odot B$ consists of elementary tensors, that is,

$$F = \{a_i \otimes b_i \mid a_i \in A, b_i \in B, i = 1, \ldots, n\}$$

for some $n \in \mathbb{N}$. Let $F_B = \{b_i \mid i = 1, \ldots, n\}$. By [1, Lemma 4.9], we can choose an approximate unit $e \in B$ such that $\|\alpha_g(e) - e\| < \varepsilon/2, \|eb - b\| < \varepsilon/2, \|be - b\| < \varepsilon/2$ and $\|eb - be\| < \varepsilon/2$ for all $g \in M$ and $b \in F_B$. Let $F_A = \{a_i \mid i = 1, \ldots, n\}$. For $(\varepsilon/2, F_A, M, S)$, there is a c.p.c. map $\varphi\colon C(X) \to A$ satisfying the conditions in Proposition 3.8. Then, $\psi\colon C(X) \to A \otimes B$, defined by $\psi(f) = \varphi(f) \otimes e$, is a c.p.c. map. Let $f, f_1, f_2 \in S, g \in M$ and $a_i \otimes b_i \in F$.

For condition (1),

$$\begin{aligned}\|\psi(f_1)\psi(f_2)(a_i \otimes b_i) - \psi(f_1 f_2)(a_i \otimes b_i)\| &= \|\varphi(f_1)\varphi(f_2)a_i \otimes eeb_i - \varphi(f_1 f_2)a_i \otimes eb_i\| \\ &\leq \|(\varphi(f_1)\varphi(f_2)a_i - \varphi(f_1 f_2)a_i) \otimes eb_i\| + \varepsilon/2 < \varepsilon.\end{aligned}$$

For condition (2),

$$\begin{aligned}\|\psi(f)(a_i \otimes b_i) - (a_i \otimes b_i)\psi(f)\| &= \|\varphi(f)a_i \otimes eb_i - a_i\varphi(f) \otimes b_i e\| \\ &\leq \|(\varphi(f)a_i - a_i\varphi(f)) \otimes eb_i\| + \varepsilon/2 < \varepsilon.\end{aligned}$$

For condition(3),

$$\begin{aligned}&\|(\alpha \otimes \beta)_g(\psi(f))(a_i \otimes b_i) - \psi(\lambda_g(f))(a_i \otimes b_i)\| \\ &\qquad = \|\alpha_g(\varphi(f))a_i \otimes \beta_g(e)b_i - \varphi(\lambda_g(f))a_i \otimes eb_i\| \\ &\qquad \leq \|(\alpha_g(\varphi(f))a_i - \varphi(\lambda_g(f))a_i) \otimes eb_i\| + \varepsilon/2 < \varepsilon.\end{aligned}$$

For condition (4),

$$\|\psi(1)(a_i \otimes b_i) - a_i \otimes b_i\| = \|\varphi(f)a_i \otimes eb_i - a_i \otimes b_i\| \leq \|(\varphi(f)a_i - a_i) \otimes b_i\| + \varepsilon/2 < \varepsilon.$$

Hence, $\alpha \otimes \beta$ has the X-Rokhlin property. □

THEOREM 4.3. *Let G be a countable discrete group and X a compact metric space with a free action of G. Let (A_n, ι_n) be a direct system of C^*-algebras and let α^n be the action on A_n such that $\iota_n \circ \alpha^n = \alpha^{n+1} \circ \iota_n$ for all $n \in \mathbb{N}$. Let $A = \lim_{n\to\infty} A_n$ and α be the induced action on A. If α^n has the X-Rokhlin property for every $n \in \mathbb{N}$, then α has the X-Rokhlin property.*

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PROOF. Let $\varepsilon > 0$, and $F \subseteq A, M \subseteq G$ and $S \subseteq C(X)$ be finite sets. Then we can choose a subset $F' \subseteq A_n$ for some $n \in \mathbb{N}$ such that, for any $a \in F$, there exists $b \in F'$ such that $\|a - \iota_{n,\infty}(b)\| < \varepsilon/3$. Since α^n has the X-Rokhlin property, for $\varepsilon/3, F', M$ and S, there is a c.p.c. map $\varphi\colon C(X) \to A_n$ satisfying the conditions of Proposition 3.8. Then, $\psi = \iota_{n,\infty} \circ \varphi$ is a c.p.c. map. Let $f, f_1, f_2 \in S, g \in M$ and $a \in F$.

For condition (1),

$$\begin{aligned}\|\psi(f_1)\psi(f_2)a - \psi(f_1f_2)a\| &= \|\iota_{n,\infty}(\varphi(f_1)\varphi(f_2)b - \varphi(f_1f_2)b)\| + 2\varepsilon/3\\ &\le \|\varphi(f_1)\varphi(f_2)b - \varphi(f_1f_2)b\| + 2\varepsilon/3 < \varepsilon.\end{aligned}$$

For condition (2),

$$\begin{aligned}\|\psi(f)a - a\psi(f)\| &= \|\iota_{n,\infty}(\varphi(f)b - b\varphi(f))\| + 2\varepsilon/3\\ &\le \|\varphi(f)b - b\varphi(f)\| + 2\varepsilon/3 < \varepsilon.\end{aligned}$$

For condition (3),

$$\begin{aligned}\|\alpha_g(\psi(f))a - \psi(\lambda_g(f))a\| &= \|\alpha_g(\iota_{n,\infty}(\varphi(f)))a - \iota_{n,\infty}(\varphi(\lambda_g(f)))a\|\\ &= \|\iota_{n,\infty}(\alpha_g^n(\varphi(f)))a - \iota_{n,\infty}(\varphi(\lambda_g(f)))a\|\\ &\le \|\alpha_g^n(\varphi(f))b - \varphi(\lambda_g(f))b\| + 2\varepsilon/3 < \varepsilon.\end{aligned}$$

For condition (4),

$$\|\psi(1)a - a\| = \|\iota_{n,\infty}(\varphi(1)a - a)\| \le \|\varphi(1)b - b\| + 2\varepsilon/3 < \varepsilon.$$

Therefore, $\bar{\alpha}$ has the X-Rokhlin property. □

THEOREM 4.4. *Let G be a countable discrete group and X a compact metric space with a free action of G. Let A be a C^*-algebra and $\alpha\colon G \to \mathrm{Aut(A)}$ be an action. Let H be a subgroup of G and $\alpha|_H$ be the restricted action on A. If α has the X-Rokhlin property, then $\alpha|_H$ has the X-Rokhlin property.*

PROOF. We can immediately prove this from the definition. □

5. Simplicity of the reduced crossed product

Simplicity is an important property of C^*-algebras. In C^*-dynamical systems, if the coefficient C^*-algebra is simple, when is the crossed product also simple? In this section, we study the simplicity of the reduced crossed product formed by actions with the X-Rokhlin property. First, we recall some definitions.

DEFINITION 5.1. Let A be a C^*-algebra and $\varphi \in \mathrm{Aut}(A)$. Then, φ is inner if there is a unitary $u \in M(A)$ such that $\varphi(a) = uau^*$ for all $a \in A$. We say φ is outer if it is not inner. Let G be a discrete group. An action $\alpha\colon G \to \mathrm{Aut}(A)$ is pointwise outer if α_g is outer for every $g \in G\backslash\{1_G\}$.

Now, we present the main theorem of this paper as follows.

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THEOREM 5.2. *Let G be a discrete countable group and X a compact metric space endowed with a free and minimal action of G. Let A be a separable C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. If α has the X-Rokhlin property, then α is pointwise outer.*

PROOF. We suppose α is not pointwise outer, that is, there exists $g \in G\backslash\{1_G\}$ such that α_g is inner. Moreover, there exists a unitary $u \in M(A)$ such that $\alpha_g(a) = uau^*$ for all $a \in A$. Since α can naturally induce an action α_∞ on $A_\infty \cap A'$, for all $\eta_A((x_n)_{n\in\mathbb{N}}) \in A_\infty \cap A'$ where $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$ and $d \in A$,

$$\begin{aligned}\alpha_{\infty,g}(\eta_A((x_n)_{n\in\mathbb{N}}))d &= \eta_A((\alpha_g(x_n))_{n\in\mathbb{N}})d = \eta_A((ux_nu^*d)_{n\in\mathbb{N}})\\ &= \eta_A((uu^*dx_n)_{n\in\mathbb{N}}) = \eta_A((x_n)_{n\in\mathbb{N}})d.\end{aligned}$$

Therefore, for the induced action α_F, we have $\alpha_{F,g} = \mathrm{id}_{F_\infty(A)}$.

Since α has the X-Rokhlin property, there exists a unital equivariant homomorphism $\varphi\colon C(X) \to F_\infty(A)$. It is easy to see that $\mathrm{Ker}(\varphi)$ is a λ-invariant ideal of $C(X)$, since $\varphi(\lambda_g(f)) = \alpha_g(\varphi(f)) = 0$ for any $f \in \mathrm{Ker}(\varphi)$ and $g \in G$. It is well known that there is a one-to-one correspondence between the ideals of $C(X)$ and the closed subsets of X. Since the action on X is minimal, that is, X does not have nontrivial closed subsets, $\mathrm{Ker}(\varphi)$ must be a trivial ideal of $C(X)$. Furthermore, φ is unital, so $\mathrm{Ker}(\varphi) \neq X$. Hence, $\mathrm{Ker}(\varphi) = \emptyset$. Thus, φ is injective.

Since the action of G on X is free, for any $g \in G\backslash\{1_G\}$ and $x \in X$, we have $gx \neq x$. Since X is Hausdorff, we can always find an open subset of X such that $g^{-1}U \cap U = \emptyset$ for any $g \in G\backslash\{1_G\}$. Then, there exists a nonzero positive function $f \in C(X)$ such that $\mathrm{supp}(f) \subseteq U$ and $\lambda_g(f)f = 0$. Hence,

$$0 = \varphi(\lambda_g(f)f) = \varphi(\lambda_g(f))\varphi(f) = \alpha_{F,g}(\varphi(f))\varphi(f) = \varphi(f)\varphi(f) = \varphi(f^2).$$

Since φ is injective, $f^2 = 0$. Thus, $f = 0$, which is a contradiction. □

Applying [10, Theorem 3.1], we can immediately obtain the following corollary.

COROLLARY 5.3. *Let G be a countable discrete group and X a compact metric space endowed with a free and minimal action of G. Let A be a simple separable C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action that has the X-Rokhlin property. Then, the reduced crossed product $A \rtimes_{\alpha,r} G$ is simple.*

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