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On cyclic maps

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Introduction. Throughout X will denote a connected, finite C.W. complex.

Let G be a subgroup of Σ_n, the symmetric group, which acts transitively on the Cartesian product, X^n, of the space X. A map $f: X^n \to X$ is G-symmetric if it commutes with the action of G. If $x_0 \in X$ is a base point let $i: X \to X^n$ denote the inclusion, $i(x) = (x, x_0, \ldots, x_0)$. In ((6); (7); (11)–(13)) the following problem is posed: if X is an orientable topological n-manifold, what is the set of integers which may be obtained as the degree of $(f.i)$ where f is a G-symmetric map? The degree of $(f.i)$ is called the James number of f. If $G = \Sigma_n (G = Z_n)$ a G-symmetric map will be called a symmetric map (a cyclic map). If $X = S^n$, the n-sphere, this problem has been studied in ((6)–(8), (11)–(13)).

In this note we consider the degrees of the composition, $(f.d)$, where f is a cyclic map and d is the diagonal. This gives restrictions on the set of James numbers.

Our main theorem is

THEOREM 3·3. *Suppose that $H^*(X; Z)$ is free of p-torsion for the prime p and let $n = p^u$. Let $Z_{(p)}$ be the ring of rationals with denominators prime to p.*

If $f: X^n \to X$ is a cyclic map then

$$(f.d)^* H^{2s}(X; Z_{(p)}) \subset n^s . H^{2s}(X; Z_{(p)}) \quad (s > 0),$$

and

$$(f.d)^* H^{2s+1}(X; Z_{(p)}) \subset n^s . p^{[\frac{1}{2}(u+1)]} . H^{2s+1}(X; X_{(p)}) \quad (s \geqslant 0).$$

The method is to exploit the relation between cyclic products and the Adams operations, ψ^n. Section 1 is concerned with the computation of $\mathrm{Cotor}^{*,*}_{K^*(Z_n)}(K^*(X^n), Z)$, which is used in section 2 to describe generators for $K^*_{Z_n}(X^n)^\wedge$. In section 3 these generators are related to the Adams operations and the main theorem is proved.

1. Let Z_n denote the cyclic group of integers modulo n. In this section we study $\mathrm{Cotor}^{*,*}_{K^*(Z_n)}$, where K^* is Z_2-graded, complex K-theory. The general definition and properties of the (relative) derived functor Cotor are given in (1; 4; 5; 10). Put

$$A = K^*(Z_n).$$

This is the Z-coalgebra dual to the integral group ring and is given by

$$A = Z[b_k] \Big/ \left(b_i . b_j - \delta_{ij} . b_i ; \sum_1^n b_j - 1 \right),$$

$(\deg(b_k) \equiv 0(2))$, where i, j, k are integers modulo n and δ_{ij} is the Kronecker delta.

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The coalgebra structure of A is given by

$$\nabla: A \to A \otimes A,\ \nabla(b_j) = \sum_1^n b_{j-k} \otimes b_k$$

$$\eta: Z \to A, \quad \eta(1) = 1$$

and

$$\epsilon: A \to Z, \quad \epsilon(b_j) = \delta_{jn}.$$

Dualizing the standard free resolution of the integers as a module over the group ring, we obtain the following periodic, Z_2-graded, split injective resolution of Z as a left A-comodule,

$$Z \underset{\epsilon}{\overset{\eta}{\rightleftarrows}} A = C_0 \underset{t_1}{\overset{d_1}{\rightleftarrows}} A = C_1 \underset{t_2}{\overset{d_2}{\rightleftarrows}} A = C_2 \underset{t_1}{\overset{d_1}{\rightleftarrows}} \dots \quad (*).$$

The homomorphisms are given by

$$d_1(b_j) = b_j - b_{j-1}, \quad d_2(b_j) = 1,$$

$$t_1(b_j) = (1-\delta_{jn}).\left(\sum_1^j b_k\right) \quad \text{and} \quad t_2(b_j) = \delta_{jn}.b_n.$$

If Y is a right A-comodule with comultiplication $\nabla_Y: Y \to Y \otimes A$, $\mathrm{Cotor}_A^{*,*}(Y, Z)$ is given by the homology of the complex

$$Y \underset{\delta_1}{\to} Y \underset{\delta_2}{\to} Y \underset{\delta_1}{\to} Y \underset{\delta_2}{\to} Y \dots$$

where for $(i = 1, 2)$ δ_i is given by the composition

$$Y \underset{\nabla_Y}{\longrightarrow} Y \otimes A \underset{1 \otimes d_i}{\longrightarrow} Y \otimes A \underset{1 \otimes \epsilon}{\longrightarrow} Y.$$

The algebra $\mathrm{Cotor}_A^{*,*}(Z, Z)$ is non-zero only in

$$\mathrm{Cotor}_A^{2j,0}(Z, Z) = \begin{cases} Z & (j = 0) \\ Z_n & (j > 0) \end{cases}$$

and $\mathrm{Cotor}_A^{2j,0}(Z, Z) \otimes \mathrm{Cotor}_A^{2k,0}(Z, Z) \to \mathrm{Cotor}_A^{2(j+k),0}(Z, Z)$ is an epimorphism.

Now let M be a finitely generated, torsion free, Z_2-graded Z-algebra. Denote by M^n the n-fold tensor product of M. Let $b_1^*: M^n \to M^n$ be the signed permutation

$$b_1^*(m_1 \otimes \dots \otimes m_n) = (-1)^e.m_n \otimes m_1 \otimes \dots \otimes m_{n-1},$$

where

$$e = (\deg m_n).\left(\sum_1^{n-1} \deg m_j\right)$$

and put $b_j^* = (b_1^*)^j$. M^n is a right A-comodule with

$$\nabla_{M^n}(x) = \sum_1^n b_k^*(x) \otimes b_k$$

and the homomorphisms $\delta_i: M^n$ are given by

$$\delta_1(x) = x - b_1^*(x), \quad \delta_2(x) = \sum_1^n b_k^*(x).$$

Let $\{x_i\}$ be a free base of homogeneous elements for M as an Abelian group. If

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$z \in \ker \delta_i \subset M^n$ then z is a linear combination of cycles of the form $\Sigma\lambda_k . b_k^*(x_1 \otimes \ldots \otimes x_n)$ where k runs through a set of coset representatives in Z_n/H and

$$H = \{k \in Z_n : b_k^*(x_1 \otimes \ldots \otimes x_n) = \pm x_1 \otimes \ldots \otimes x_n\}.$$

If $H = Z_{n/q} \subset Z_n$, $x_1 \otimes \ldots \otimes x_n$ is

$$w = (x_1 \otimes \ldots \otimes x_q)^{n/q} = x_1 \otimes \ldots \otimes x_q \otimes x_1 \otimes \ldots \otimes x_q \in M^n.$$

Suppose
$$z = \sum_0^{q-1} \lambda_k . b_k^*(w) \in \ker \delta_1$$

then
$$b_q^*(w) = (-1)^{(n/q+1)(\sum_1^q \deg x_j)} w,$$

thus $\lambda_0 = \lambda_1 = \ldots = \lambda_{q-1}$ and either n/q is odd or $\sum_1^q \deg x_j$ is even.

Put
$$r = (n/q+1) . \left(\sum_1^q \deg x_j\right)$$

and suppose $z \in \ker \delta_2$. Thus

$$\sum_1^n b_j^*(z) = \left(\sum_1^{n/q-1} (-1)^{r.j}\right) . \left(\sum_0^{q-1} \lambda_k\right) . \left(\sum_0^{q-1} b_k^*(w)\right) = 0.$$

If $\Sigma\lambda_k = 0$ then $z \in \operatorname{im} \delta_1 \subset \ker \delta_2$, thus we must have n/q even and $\sum_1^q \deg x_j$ odd. This cycle is equivalent to
$$\left(\sum_1^{q-1} \lambda_k\right) . w \text{ modulo } \operatorname{im} \delta_1.$$

Let $\sum_1^q \deg x_j = s$ and [ ] denote a homology class then we have proved:

PROPOSITION 1·1. *As a module over* $\operatorname{Cotor}_A^{*,*}(Z, Z)$, $\operatorname{Cotor}_A^{*,*}(M^n, Z)$ *is generated by the elements*

$$\left[\sum_0^{q-1} b_k^*((x_1 \otimes \ldots \otimes x_q)^{n/q})\right] \in \operatorname{Cotor}^{0,(n/q),s} \quad (n/q \text{ odd or } s \text{ even})$$

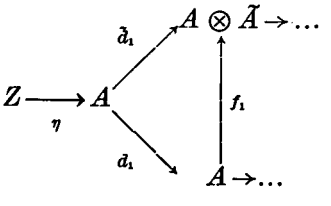
and
$$[(x_1 \otimes \ldots \otimes x^q)^{n/q}] \in \operatorname{Cotor}^{1,0} \quad (n/q \text{ even and } s \text{ odd}).$$

Remark 1·2. Let M be a finitely generated, Z_2-graded Z-algebra whose torsion is coprime to n and let $Z_{(n)}$ be the ring of rationals whose denominators are coprime to n. In this case $\operatorname{Cotor}^{*,*}_{A \otimes Z_{(n)}}((M \otimes Z_{(n)})^n, Z_{(n)})$ is generated as a module over

$$\operatorname{Cotor}^{*,*}_{A \otimes Z_{(n)}}(Z_{(n)}, Z_{(n)})$$

by the classes in $\operatorname{Cotor}^{0,*}$ and $\operatorname{Cotor}^{1,0}$ described in Proposition 1·1.

Let $\tilde{A} = \ker \epsilon \cong \operatorname{coker} \eta$ and let $Z \xrightarrow{\eta} A \xrightarrow{\tilde{d}_1} A \otimes \tilde{A}$ be the start of the cobar resolution ((5), section 3), then $\tilde{d}_1(b_k) = \sum_1^n (b_k - b_{k-j}) \otimes b_j$. Suppose that



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is a mapping of the resolution (*) into the cobar resolution. If $z \in M^n \cong M^n \square_A A$ then

$$(1 \square_A f_1)(z) \in M^n \square_A (A \otimes \tilde{A}) \cong M^n \otimes \tilde{A}$$

is the element $$\left\{ -\left(\sum_{j=1}^{n-1} \left(\sum_{k=0}^{j-1} b_k^*(z) \right) \otimes b_j \right) \right\}.$$

In particular, if n is even and $\deg x_1$ is odd, the homology class $[(x_1)^n]$ is represented by

$$\left\{ -\left(\sum_{q=0}^{\frac{1}{2}n-1} (x_1)^n \otimes b_{2q+1} \right) \right\} \in M^n \otimes \tilde{A}$$

in terms of the cobar resolution.

2. In this section we construct a set of generators for $K^*_{Z_n}(X^n)^\wedge$, where $(\ )^\wedge$ denotes the $I(Z_n)$-adic completion.

If $x \in K^0(X)$ let $x^{\otimes n} \in K^0_{Z_n}(X^n)$ be the element constructed in ((2), §§2, 3), this restricts to $x^n \in K^0(X^n)$.

Suppose $n = 2m$ and $R(Z_n) = Z[y]/(y^n - 1)$ is the complex representation ring of Z_n. The real representation of Z_n given by the cyclic permutation of R^n can be written

$$R^n = \Delta(R) \oplus R_\tau \oplus \sum_{j=1}^{m-1} r(y^j)$$

where $r(y^j)$ is the underlying real representation of y^j, R_τ is the one-dimensional, non-trivial representation induced by $Z_n \to Z_2$ and $\Delta(R)$ is the one-dimensional trivial representation.

If $x \in K^{-1}(X) = K(X \times (I, \partial I))$ form

$$x^{\otimes n} \in K_{Z_n}(X^n \times (I, \partial I)^n) \cong K_{Z_n}(X^n \times R^n) \cong K_{Z_n}(X^n \times (\Delta(R) \oplus R_\tau)),$$

where this last isomorphism is given by multiplication by the Thom element, Λ_{-1}, ((3), Ch. 4). Let $z(x) \in K_{Z_n}(X^n \times (\Delta(R) \oplus R_\tau))$ be the element such that $z(x).\Delta_{-1} = x^{\otimes n}$. If $B \in K^{-2}(pt)$ is the Bott periodicity element then

$$z(x).B^{m-1} = x^n \in K^{-n}(X^n).$$

Let $$w(x) = d^*(z(x)) \in K_{Z_n}(X^n \times \Delta(R)) \cong K^{-1}_{Z_n}(X^n).$$

Suppose $n = 2m + 1$ and $x \in K^{-1}(X)$. In this case the real representatior

$$R^n = \Delta(R) \oplus \sum_{j=1}^{m} r(y^j).$$

Define

$$w(x) \in K^{-1}_{Z_n}(X^n) \cong K_{Z_n}(X^n \times \Delta(R)) \quad \text{by} \quad w(x).\Lambda_{-1} = x^{\otimes n} \in K_{Z_n}(X^n \times R^n).$$

The restriction of this element satisfies $w(x).B^m = x^n \in K^{-n}(X^n)$.

We now show that the elements constructed above give rise to a system of generators for $(K^*_{Z_n}(X^n) \otimes Z_{(n)})^\wedge$.

Let $Z_n \to EZ_n \to BZ_n$ be the universal Z_n-principal bundle. There is an isomorphism

$$(K^*_{Z_n}(X^n) \otimes Z_{(n)})^\wedge \cong \lim_{\leftarrow} (K^*(X^n \times_{Z_n} EZ_n) \otimes Z_{(n)}) \quad ((3), \text{Ch. } 7; (10)).$$

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EZ_n is filtered by Z_n-subspaces $\{(EZ_n)_k\}$ ((1), § 2; (9), Ch. 3; (10), § 2), and this filtration begins

$$Z_n = (EZ_n)_0 \subset CZ_n \times Z_n \cup Z_n = (EZ_n)_1 \subset EZ_n.$$

Let $n = 2m$ and let $\Delta(I)$ and I_r be the unit balls in $\Delta(R)$ and R_r respectively. Let $z(x)$ be represented by a complex, (E, d), of Z_n product bundles, $\{E_i \times \Delta(I) \times I_r\}$, over

$$X^n \times (\Delta(I), \partial\Delta(I)) \times (I_r, \partial I_r)$$

then $w(x)$ is represented by $(\tilde{E}, \tilde{d})$, the complex restricted to $X^n \times (\Delta(I), \partial\Delta(I)) \times (0)$. The complex $(\tilde{E}, \tilde{d}) \times_{Z_n} EZ_n$ represents an element in $\varprojlim K^{-1}(X^n \times_{Z_n} EZ_n)$ which corresponds to $w(x) \in K^{-1}(X^n)$.

Let $h\colon \Delta(I) \times I \to \Delta(I) \times I_r$ be a (non-equivariant) homotopy, $h(s,t) = (h_1(s,t), h_2(s,t))$, satisfying

$$h(\partial\Delta(I) \times I) \subset \partial(\Delta(I) \times I_r),$$

$$h(s,0) = (s,0) \quad \text{and} \quad h(s,1) \in \partial(\Delta(I) \times I_r).$$

The homotopy, h, gives a (non-equivariant) homotopy of the complex

$$(\tilde{E}, \tilde{d}) \times_{Z_n} (EZ_n)_0 \cong (\tilde{E}, \tilde{d})$$

to an exact complex and hence $w(x) = 0 \in K^{-1}(X^n)$. Thus $w(x) \in \varprojlim K^{-1}(X^n \times_{Z_n} EZ_n)$ pulls back to an element of $\varprojlim K^{-1}(X^n \times_{Z_n} (EZ_n, Z_n))$ which maps into

$$K^{-1}(X^n \times_{Z_n} ((EZ_n)_1, (EZ_n)_0)) \cong K^{-1}(X^n \times (CZ_n, Z_n)),$$

((1), § 2; (9), Ch. 3, (5); (10), § 2),

to an element represented by a complex described in the following manner. The underlying family of vector bundles is that of $\tilde{E} \times (CZ_n)$ over $X^n \times \Delta(I) \times (CZ_n)$, where CZ_n is the cone on Z_n, $([-1,1] \times Z_n)/(\{-1\} \times Z_n)$. Over the point $(u_1, \ldots, u_n, s, t, a)$,

$$(u_i \in X; s \in \Delta(I); t \in [-1,1]; a \in Z_n),$$

the differential is given by

$$(-1 \leqslant t \leqslant 0), \quad d_{(u_1, \ldots, u_n, s, 0)}(_)$$

$$(0 \leqslant t \leqslant 1), \quad (d_{(u_{a(1)}, \ldots, u_{a(n)}, h_1(s,t), h_2(s,t))}(_ \cdot a^{-1})) \cdot a.$$

We wish to identify this element in

$$K(X^n \times (\Delta(I), \partial\Delta(I)) \times (CZ_n, Z_n)) \cong K(X^n \times (\Delta(I), \partial\Delta(I)) \otimes \tilde{A},$$

where $A = K^0(Z_n)$. It suffices to determine the pre-image of this element in

$$K(X^n \times (\Delta(I), \partial\Delta(I)) \times (I_1, \partial I_1) \times Z_n), \quad (I_1 = [-1,1]),$$

which is represented in the following manner. Over the point $(u_1, \ldots, u_n, s, t, a)$ the differential is given by

$$(-1 \leqslant t \leqslant 0), \quad d_{(u_1, \ldots, u_n, h_1(s,-t), h_2(s,-t))}(_)$$

$$(0 \leqslant t \leqslant 1), \quad (d_{(u_{a(1)}, \ldots, u_{a(n)}, h_1(s,t), h_2(s,t))}(_ \cdot a^{-1})) \cdot a.$$

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The complex given above is homotopic to an acyclic one over $X^n \times (I, \partial I)^2 \times \{a\}$, if $a \in \ker(Z_n \to Z_2)$ and is homotopic to (E, d) otherwise. Hence $w(x)$ pulls back to

$$\left(\sum_{q=0}^{m-1} x^n . (B^{m-1})^{-1} \otimes b_{2q+1}\right) \in K^*(X^n \times_{Z_n}((EZ_n)_1, (EZ_n)_0) \cong K^*(X^n) \otimes \tilde{A}.$$

PROPOSITION 2·1. *Suppose $H^*(X; Z)$ has only torsion coprime to n, then*
(*a*) *the Rothenberg–Steenrod spectral sequence*

$$E_2^{*,*} = \mathrm{Cotor}^{*,*}_{K^*(Z_n) \otimes Z_{(n)}}((K^*(X) \otimes Z_{(n)})^n, Z_{(n)}) \Rightarrow (K^*_{Z_n}(X^n) \otimes Z_{(n)})^\wedge$$

collapses and
(*b*) *as a module over $(R(Z_n) \otimes Z_{(n)})^\wedge$, $(K^*_{Z_n}(X^n) \otimes Z_{(n)})^\wedge$ is generated by elements*

$$x^{\otimes n} \quad (x \in K^0(X)),$$
$$w(x) \quad (x \in K^{-1}(X)),$$

and elements in the image of the transfer homomorphisms.

Proof. This spectral sequence is derived from the filtration $\{X^n \times_{Z_n}(EZ_n)_k\}$ of $X^n \times_{Z_n}(EZ_n)$ and the E_1 – term is the cobar resolution ((1), § 2; (5); (9), Ch. 4 & 5; (10) § 2. In Proposition 1·1 a system of generators for the E_2-term, as a module over

$$\mathrm{Cotor}^{*,*}_{A \otimes Z_{(n)}}(Z_{(n)}, Z_{(n)})$$

was described. The preceding discussion shows that the classes $[(x_1)^n]$ in $\mathrm{Cotor}^{0,*}$ and $\mathrm{Cotor}^{1,0}$ represent permanent cycles convergent to the elements described above. The other generators

$$[\Sigma b_k^*((x_1 \otimes \ldots \otimes x_q)^{n/q})] \in \mathrm{Cotor}^{0,*}$$

and

$$[(x_1 \otimes \ldots \otimes x_q)^{n/q}] \in \mathrm{Cotor}^{1,0}$$

are permanent cycles which converge to the images of the elements

$$(x_1 \otimes \ldots \otimes x_q)^{\otimes n/q} \quad \text{and} \quad w((x_1 \otimes \ldots \otimes x_q))$$

under the transfer homomorphism

$$\pi_*: K^*_{Z_{n/q}}((X^q)^{n/q}) \to K^*_{Z_n}(X^n) \quad ((3), \text{Ch. 1}).$$

Finally $\mathrm{Cotor}^{*,*}_{A \otimes Z_{(n)}}(Z_{(n)}, Z_{(n)})$ is generated in even total degree and hence produces only permanent cycles.

3. Suppose that $H^*(X; Z)$ is free of p-torsion, where p is a prime and let $n = p^u$. Write K^*_G for $K^*_G \otimes Z_{(p)}$. For $(0 \leqslant j \leqslant k \leqslant u)$ let $\pi(j,k): K^*_{Z_{p^j}} \to K^*_{Z_{p^k}}$ be the transfer homomorphism ((3), Ch. 1). Proposition 2·1 states that $K^0_{Z_n}(X^n)^\wedge$ is generated over $R(Z_n)^\wedge$ by elements

$$\pi(j,u)((x_1 \otimes \ldots \otimes x_{p^{u-j}})^{\otimes p^j}), \quad (\Sigma \deg x_k \equiv 0 \pmod 2),$$

where

$$(x_1 \otimes \ldots \otimes x_{p^{u-j}})^{\otimes p^j} \in K^0_{Z_{p^j}}((X^{p^{u-j}})^{p^j}).$$

If $x \in K^0(X)$, a simple calculation of the type given in ((2), 2·5) shows that the element $x^{\otimes n}$ restricts to

$$\sum_{j=0}^{u} \ominus_{j,u}(x) \otimes \pi(j,u)(1) \in K^0(X) \otimes R(Z_n)$$

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under the homomorphism induced by the diagonal $d: X \to X^n$. If ψ^k is the Adams operation then $\ominus_{j,u}$ is an operation satisfying

$$p^{u-j}.\theta_{j,u} = (\psi^{p^j})^{p^{u-j}} - (\psi^{p^{j+1}})^{p^{u-j-1}} \quad (0 \leqslant j < u)$$

and $\ominus_{u,u} = \psi^n$.

Define $$\ominus_u(x) = x^{\otimes n} - \sum_{j=0}^{u-1} \pi(j,u)\,(\ominus_j(\ominus_{0,u-j}(x) \otimes 1 \otimes \ldots \otimes 1))$$

in $K^0_{Z_n}(X^n)$, $(\deg x \equiv 0 \pmod 2)$.

LEMMA 3·1. *If $x \in K^0(X)$ then $d^*(\ominus_u(x)) = \psi^n(x) \otimes 1$.*

Proof. By induction on u assume the result for all spaces X and all $u' < u$. Thus

$$\begin{aligned} &d^*(\pi(j,u)\,(\ominus_j(\ominus_{0,u-j}(x) \otimes 1 \otimes \ldots \otimes 1))) \\ &= \pi(j,u) d_1^*.d_2^*.\ominus_j(\ominus_{0,u-j}(x) \otimes 1 \otimes \ldots \otimes 1) \\ &= \pi(j,u) d_1^*.\psi^{p^j}(\ominus_{0,u-j}(x) \otimes 1 \otimes \ldots \otimes 1) \\ &= \psi^{p^j}(\ominus_{0,u-j}(x)) \otimes \pi(j,u)\,(1)\,(j < u), \end{aligned}$$

where d_1^* and d_2^* are the obvious diagonals. However, $\psi^{p^j}(\ominus_{0,u-j}(x)) = \ominus_{j,u}(x)$ and hence

$$d^*(\ominus_u(x)) = \psi^n(x) \otimes 1 \in K^0(X) \otimes R(Z_n).$$

THEOREM 3·2. *Suppose $H^*(X; Z)$ is free of p-torsion for the prime p and let $n = p^u$. If $f: X^n \to X$ is a cyclic map then*

(i) $(f.d)^*(H^{2s}(X; Z_{(p)})) \subset n^s.H^{2s}(X; Z_{(p)})\,(s > 0)$

and

(ii) $(f.d)^*(H^{2s+1}(X; Z_{(p)})) \subset n^s.p^{[\frac{1}{2}(u+1)]}.H^{2s+1}(X; Z_{(p)})\,(s \geqslant 0)$.

Proof. We have

$$\{\mathrm{Im}\,(f.d)^*: K^0(X) \subset K^0_{Z_n}(X)^\wedge \to K^0_{Z_n}(X^n)^\wedge \to K^0_{Z_n}(X)^\wedge = K^0(X) \otimes R(Z_n)^\wedge\}$$

is contained in $K^0(X)$. However, $K^0_{Z_n}(X^n)^\wedge$ is generated over $R(Z_n)^\wedge$ by the elements $\ominus_u(x)$, $(\deg x = 0 \pmod 2)$, and elements induced from $K^0_{Z_{p^j}}(X^n)^\wedge\,(j < u)$. Hence $\mathrm{im}\,d^* \cap K^0(X) \subset \psi^n(K^0(X))$.

There exist inclusions $K^0(X) \subset K^0(X) \otimes Q$ ($Q \equiv$ rational field), and

$$H^*(X; Z_{(p)}) \subset H^*(X; Q).$$

Also $K^0(X) \otimes Q \cong H^{ev}(X; Q)$, via the Chern character. Thus $K^0(X)$ has a free basis consisting of eigenvectors for the Adams operation, ψ^n, such that the eigenspace corresponding to the eigenvalue n^s is $H^{2s}(X; Z_{(p)})$, and hence (i) is proved.

For (ii), if $x \in H^{2s+1}(X; Z_{(p)})$ is not p-divisible and $p^t.x \in \mathrm{im}\,(f.d)^*$ then

$$p^{2t}.(x \otimes x) \in \mathrm{im}\,((f \times f).d)^* \subset H^{4s+2}(X \times X; Z_{(p)}) \quad \text{and} \quad 2t \geqslant (2s+1).u.$$

COROLLARY 3·3. *Let $f: X^n \to X$ be a cyclic map as in Theorem 3·2 and let*

$$v \in H^j(X; Z_{(p)})\,(j > 0),$$

be primitive with respect to f then $j = 1$ or 2.

Proof. If $f^*(v) = v \otimes 1 \otimes \ldots \otimes 1 \otimes v \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes v$ then $(f.d)^*(v) = n.v$.

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