Canad. J. Math. 2026, pp. 1–21
http://dx.doi.org/10.4153/S0008414X26102090
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Connections and symplectic forms on the cotangent bundle of flag manifolds

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Abstract. We investigated the symplectic geometry of homogeneous spaces associated with semisimple Lie groups, focusing on cotangent bundles of maximal flag manifolds. Our work provides an explicit description of the canonical symplectic structure on these spaces using connections and curvatures of principal bundles naturally associated with the underlying Lie groups. We extend classical results concerning the exactness of symplectic forms on adjoint orbits, previously known for specific Lie algebras, to arbitrary simple Lie groups. In particular, we identify conditions under which the Kostant–Kirillov–Souriau form on a regular adjoint orbit coincides with the canonical symplectic form of the cotangent bundle, yielding exact symplectic structures. The approach combines differential-geometric techniques with Lie-theoretic constructions, offering a unifying framework that connects the geometry of coadjoint orbits with symplectic structures on homogeneous spaces.

1 Introduction and basic notations

1.1 Context and motivation

The main topic of this work is the study of symplectic forms on homogeneous spaces of the form $G/MA \cong T^*\mathbb{F}$, where $\mathbb{F}$ denotes the maximal flag manifold associated with a semisimple Lie group G. We provide an explicit description of the symplectic structure on $T^*\mathbb{F}$ by analyzing the curvature 2-form of specific principal bundle realizations of G/MA, combined with appropriate linear functionals.

In [4], several geometric realizations of $T^*\mathbb{F}$ were established, including its interpretation as:

- the homogeneous space G/MA, and
- a (co)adjoint orbit through a regular element in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$.

The geometry of (co)adjoint orbits is a classical topic in Lie theory and symplectic geometry, with numerous applications (see, for instance, [2, 3, 5–8, 11]).

Received by the editors August 25, 2025; revised January 9, 2026; accepted February 9, 2026.
Published online on Cambridge Core February 16, 2026.
L.G. research is partially supported by São Paulo Research Foundation FAPESP grants 2023/13131-8, and CNPq Grant No. 306021/2024-2. L.A.S.M. research is partially supported by São Paulo Research Foundation FAPESP grants 2018/13481-0.
AMS subject classification: **14M15, 17B08, 53D05.**
Keywords: Coadjoint orbits, flag manifolds, exact symplectic forms.

https://doi.org/10.4153/S0008414X26102090 Published online by Cambridge University Press

While [4] investigated various aspects of the symplectic geometry of $T^*\mathbb{F}$, including the study of Lagrangian submanifolds, it left open the following fundamental question:

When is the standard Kostant–Kirillov—Souriau (KKS) symplectic form on a complex adjoint orbit exact?

Our investigation begins with the special case of $\mathfrak{sl}(n+1,\mathbb{C})$. Recall that for a regular coadjoint orbit $\mathcal{O}$ equipped with the imaginary part $\operatorname{Im}\Omega$ of the standard complex symplectic form, Arnold established the following theorem.

Theorem 1.1 [1] *The orbit $(\mathcal{O}, \operatorname{Im}\Omega)$ is symplectomorphic to $(T^*\mathbb{F}, \omega_{Liouville})$ if and only if all eigenvalues of some (hence any) representative matrix are real. Here, Ω denote the standard complex symplectic form on the coadjoint orbit $\mathcal{O}$.*

In this article, we extend this result to arbitrary simple Lie groups.

Our approach is based on describing the Liouville 1-form and the canonical symplectic form on $T^*\mathbb{F}$ in terms of connections and curvatures of principal bundles diffeomorphic to $T^*\mathbb{F}$. The identification of $T^*\mathbb{F}$ with the appropriate principal bundle arises naturally from the Lie-theoretic description of coadjoint orbits of G (specifically, via the fibration $G \to G/MA \cong T^*\mathbb{F}$ – see the next section for notation).

More precisely, as is usual in reductive homogeneous spaces, the choice of complementary subspace induces a left-invariant connection (horizontal distribution) in the principal bundle $G \to G/MA$. The curvature of this connection yields an $\mathfrak{m} \oplus \mathfrak{a}$-valued 2-form on G. Composing this curvature form with invariant linear functionals on $\mathfrak{m} \oplus \mathfrak{a}$ produces 2-forms that descend to closed 2-forms on G/MA. In many important cases, these descended forms are in fact symplectic.

The main results are the following (see Section 5 for details): let G be a simple Lie group with Lie algebra $\mathfrak{g}$, and fix the identification $T^*\mathbb{F} \cong G/MA \cong \operatorname{Ad}(G)H_0$, where H_0 is a regular element in the center of $\mathfrak{m} \oplus \mathfrak{a}$ satisfying $\operatorname{Ad}(K)H_0 = \mathbb{F}$.

Theorem 1.2 *Let the identification of $T^*\mathbb{F}$ with G/MA be made by equating both to an adjoint orbit $\operatorname{Ad}(G)H_0$, where H_0 is in the center of $\mathfrak{m} \oplus \mathfrak{a}$ and regular in the sense that $\operatorname{Ad}(K)H_0 = \mathbb{F}$. Then, the Liouville form on $T^*\mathbb{F} = G/MA$ is given by*

$$\lambda^*_{H_0,0}\left(\omega - \omega_{K\times K}\right)$$

where $\lambda_{H_0}(\cdot) = \langle H_0, \cdot\rangle$ is the "height function." Therefore, the canonical symplectic form on $T^*\mathbb{F}$ is

$$\Gamma = \lambda^*_{H_0,0}\left(\Omega - \Omega_{K\times K}\right)$$

where Ω and $\Omega_{K\times K}$ are the curvatures. In this expression, the component $\lambda^*_{H_0,0}\Omega = \Omega_{\lambda_{H_0}}$ is the KKS form of the adjoint orbit by H_0.

Corollary 1.3 *If $H_0 \in \mathfrak{a}$ is a regular real element, then the KKS form κ_{H_0} of the adjoint orbit $\operatorname{Ad}(G)H_0$ coincides with the canonical symplectic form Γ of $T^*\mathbb{F}$ and is therefore an exact form. In terms of connections,*

$$\kappa_{H_0} = d\left(\lambda^*_{H_0}\left(\omega - \omega_{K\times K}\right)\right)$$

where ω is the G-invariant connection on $G \to G/MA$.

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These results provide new insights into the interplay between Lie group symmetries, principal bundle geometry, and exact symplectic forms, with potential applications in geometric mechanics and mathematical physics.

In the next section, we recall some well-known facts from the semisimple Lie theory.

1.2 Basic notations and preliminaries concepts

Let us fix the notation we use throughout this article. We will closely follow the references [9, 10, 12].

Let us fix a (global) *Cartan involution* of a connected reductive group G, that is, an automorphism θ of G with $\theta^2 = 1$ and let $K = \{g \in G \mid \theta g = g\}$ be the subgroup of elements in G left fixed by θ.

We abuse the notation and denote also by θ the differential of θ at $e \in G$. This is an automorphism of $\mathfrak{g}$ with $\theta^2 = 1$. It is called the *Cartan involution* of $\mathfrak{g}$.

Decompose the Lie algebra $\mathfrak{g}$ of G as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{1.1}$$

for the corresponding eigenspace decomposition for θ, associated with the eigenvalues $+1$ and -1, respectively.

This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$. The decomposition has the following properties:

- $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.
- $\mathfrak{k}$ is the Lie algebra of K.

We remark that the subgroup K is the maximal compact subgroup of G.

A *reductive Lie algebra* is a Lie algebra that is the direct sum of two ideals, one equal to a semisimple Lie algebra and the other equal to an abelian Lie algebra.

Fix a linear connected reductive group G, and let $\mathfrak{a}$ be any maximal abelian subspace of $\mathfrak{p}$.

For $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda$ be the corresponding simultaneous eigenspace, namely,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, then λ is a *restricted root*, and any $X \in \mathfrak{g}_\lambda$ is called a *restricted-root vector*. Let Π be the set of all restricted roots. The result of the previous paragraph is that we obtain a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Pi} \mathfrak{g}_\lambda.$$

This is called the *restricted-root space decomposition* of $\mathfrak{g}$.

It is well known that $\mathfrak{g}_0$ decomposes into $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $\mathfrak{a}$ is the maximal abelian subspace of $\mathfrak{p}$, fixed at the begging. Moreover, the sum is orthogonal with respect to the Cartan–Killing form.

Let Π^+ be the set of positive roots of Π. Now we consider the Lie subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{n} = \bigoplus_{\lambda \in \Pi^+} \mathfrak{g}_\lambda.$$

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Let A and N be the subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. We can now state the Iwasawa decomposition, first on the level of Lie algebras and then on the level of Lie groups. In the next proposition, $\mathfrak{k}$ is as in Equation (1.1), while $\mathfrak{a}$ denotes, as before, the maximal abelian subspace of $\mathfrak{p}$ fixed at the beginning.

Proposition 1.4 (Iwasawa Decomposition) *For G connected reductive, $\mathfrak{g}$ is a direct sum*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here:

- $\mathfrak{a}$ *is abelian;*
- $\mathfrak{n}$ *is nilpotent;*
- $\mathfrak{a} \oplus \mathfrak{n}$ *is solvable;*
- $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Theorem 1.5 (Iwasawa decomposition for Lie groups) *For a connected reductive G, let A and N be subgroups with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then:*

- *A, N, and AN are simply connected closed subgroups of G;*
- *the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto G.*

Next, we define $M = Z_K(\mathfrak{a})$ to be the *centralizer* of $\mathfrak{a}$ in K, that is, the set of all $k \in K$ such that $\mathrm{Ad}(k)H = H$ for all $H \in \mathfrak{a}$. Let us enumerate some basic properties of M:

(1) M is a closed subgroup of K, hence compact.
(2) M centralizes $\mathfrak{a}$ and normalizes each $\mathfrak{g}_\lambda$.
(3) M centralizes A and normalizes N. In fact, $\mathrm{Ad}(m)\mathfrak{g}_\lambda \subseteq \mathfrak{g}_\lambda$ for all $\lambda \in \Pi$ and $m \in M$.
(4) MAN is a closed subgroup of G.

The subgroup MAN is called a *minimal parabolic subgroup* of G. The homogeneous space G/MAN is called *maximal flag manifold* of G, and will be denoted by $\mathbb{F}$.

2 Invariant G-connections and symplectic forms in $T^*\mathbb{F}$

Keep the notation of the previous section. The main topic of this work is the study of symplectic forms in homogeneous space $G/MA = T\mathbb{F}$, where $\mathbb{F}$ denotes the maximal flag manifold of the semisimple Lie group G. By endowing the flag manifold with a Riemannian metric, one can identify $T\mathbb{F}$ with $T^*\mathbb{F}$. In what follows, we abuse notation and will denote $G/MA = T^*\mathbb{F}$. We will define symplectic forms as the curvatures of connections in the principal bundle $G \to G/MA$ with structure group MA.

The homogeneous space G/MA is reducible in the sense that the isotropy algebra $\mathfrak{m} \oplus \mathfrak{a}$ admits a complementary subspace that is invariant under the adjoint representation of MA. As usually happens in a reducible homogeneous space, the complementary subspace defines a connection (horizontal distribution) in the principal bundle $G \to G/MA$, and such connection is left invariant. The curvature of this connection is a 2-form on G with values in $\mathfrak{m} \oplus \mathfrak{a}$. The composition of this 2-form with invariant linear functionals on $\mathfrak{m} \oplus \mathfrak{a}$ defines 2-forms that project onto closed 2-forms on G/MA. In many cases, these 2-forms on G/MA are symplectic forms.

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This construction of connections and symplectic forms will be discussed in detail below.

The isotropy Lie algebra $\mathfrak{m} \oplus \mathfrak{a}$ admits the complementary space $\mathfrak{n}^+ \oplus \mathfrak{n}^-$, which is invariant under the adjoint representation of MA, since if $mh \in MA$, then $\mathrm{Ad}(mh)$ leaves each root space invariant.

By taking left translations, define

$$\mathcal{H}_g = \left(dL_g\right)_1 (\mathfrak{n}^+ \oplus \mathfrak{n}^-)$$

which is a left-invariant distribution on G. This distribution is a *left-invariant connection* on the principal bundle $G \to G/MA$ because if $mh \in MA$, then

$$\begin{aligned} \left(dR_{mh}\right)_g \mathcal{H}_g &= \left(dR_{mh}\right)_g \left(dL_g\right)_1 (\mathfrak{n}^+ \oplus \mathfrak{n}^-) \\ &= \left(dL_{gmh}\right)_1 (\mathfrak{n}^+ \oplus \mathfrak{n}^-) \\ &= \mathcal{H}_{gmh}. \end{aligned}$$

The *connection form* ω corresponding to $\mathcal{H}_g$ is a 1-form with values in the isotropy algebra $\mathfrak{m} \oplus \mathfrak{a}$ and is given by

$$\omega_g (v) = \mathrm{proj}_{\mathfrak{m}\oplus\mathfrak{a}} \left(\left(dL_{g^{-1}}\right)_g v\right) \qquad v \in T_g G$$

where $\mathrm{proj}_{\mathfrak{m}\oplus\mathfrak{a}}$ is the projection onto $\mathfrak{m} \oplus \mathfrak{a}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$.

The values of ω on invariant vector fields are given as follows:

(1) If $X^l (g) = \left(dL_g\right)_1 (X)$ is a left-invariant vector field, then $\omega \left(X^l\right)$ is constant and given by

$$\omega \left(X^l\right) = \mathrm{proj}_{\mathfrak{m}\oplus\mathfrak{a}} X.$$

In particular, if $X \in \mathfrak{n}^+ \oplus \mathfrak{n}^-$, then X^l is a horizontal vector field.

(2) If $X^r (g) = \left(dR_g\right)_1 (X)$ is a right-invariant vector field, then

$$\omega_g \left(X^r (g)\right) = \mathrm{proj}_{\mathfrak{m}\oplus\mathfrak{a}} \left(\mathrm{Ad}\,(g)\, X\right) \tag{2.1}$$

since $X^r (g) = \left(\mathrm{Ad}\,(g)\, X\right)^l (g)$.

The *curvature* Ω of the connection is obtained from the structure equation

$$d\omega (X, Y) = -\frac{1}{2} [\omega (X), \omega (Y)] + \Omega (X, Y)$$

where $d\omega (X, Y) = X\omega (Y) - Y\omega (X) - \omega [X, Y]$. If X and Y are horizontal vector fields, then $\omega (X) = \omega (Y) = 0$ and thus

$$\Omega (X, Y) = -\omega [X, Y] \qquad X, Y \in \mathcal{H}.$$

Since the connection is left-invariant, both the connection form ω and the curvature are left-invariant differential forms, that is, $L_g^* \omega = \omega$ and $L_g^* \Omega = \Omega$, for all $g \in G$.

It is possible to write the connection and its curvature using a basis of 1-forms that are left-invariant on G. This basis is chosen starting with a basis $\{X_1, \ldots, X_r, Y_1, \ldots, Y_s\}$ of $\mathfrak{g}$ such that the first r elements form a basis of $\mathfrak{m} \oplus \mathfrak{a}$ and the remaining elements form a basis of $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. Thus, the basis of 1-differential forms,

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written as $\{\varepsilon_1, \ldots, \varepsilon_r, \delta_1, \ldots, \delta_s\}$, is such that at the origin it is the dual basis of $\mathfrak{g}^*$ to the basis of $\mathfrak{g}$.

With this choice, the invariant connection ω is defined on a left-invariant vector field X by

$$\omega(X) = \varepsilon_1(X) X_1 + \cdots + \varepsilon_r(X) X_r \tag{2.2}$$

since this is the vertical component of the invariant vector field. It follows from this and the expression for the curvature that if X and Y are left-invariant horizontal vector fields, then

$$\Omega(X, Y) = -\left(\varepsilon_1([X, Y]) X_1 + \cdots + \varepsilon_r([X, Y]) X_r\right). \tag{2.3}$$

An alternative way to view the connection and curvature is through the *extension of the Cartan–Killing form* $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. This form is symmetric and invariant under the adjoint representation. Therefore, it extends to a pseudo-Riemannian (nonpositive-definite) metric on G. Let this metric be denoted by $\mathcal{K}$. Since $\mathfrak{m} \oplus \mathfrak{a}$ is orthogonal to $\mathfrak{n}^+ \oplus \mathfrak{n}^-$ with respect to the Cartan–Killing form, the left-invariance of the horizontal and vertical spaces of the connection implies that they are orthogonal with respect to $\mathcal{K}$. That is, the horizontal spaces of the connection are the $\mathcal{K}$-orthogonal subspaces to the vertical spaces.

Thus, if X is a left-invariant vector field, then $\omega(X)$ is the element of $\mathfrak{m} \oplus \mathfrak{a}$ whose left-invariant vector field is the orthogonal projection of X (with respect to $\mathcal{K}$) onto the vertical subspace $\mathfrak{m} \oplus \mathfrak{a}$, which is orthogonal to the horizontal subspace $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. The curvature $\Omega[X, Y]$ is the orthogonal projection of $-[X, Y] = [Y, X]$.

The curvature Ω, when composed with linear functionals, can be "projected" onto the quotient space, often resulting in simple forms. This is done by the following construction.

A linear functional $\lambda \in (\mathfrak{m} \oplus \mathfrak{a})^*$ is MA-invariant if $\lambda \circ \mathrm{Ad}(g) = \lambda$ for all $g \in MA$. Let λ be an MA-invariant linear functional and define on G the 2-form $(\lambda\Omega)(X, Y) = \lambda(\Omega(X, Y))$ where Ω is the curvature of ω. This 2-form is right-invariant by MA since if $g \in MA$, then

$$R_g^*(\lambda\Omega) = \lambda\left(R_g^*\Omega\right) = \lambda\left(\mathrm{Ad}\left(g^{-1}\right)\Omega\right) = \lambda\Omega. \tag{2.4}$$

Moreover, Ω is a tensorial 2-form, that is, $\Omega(X, Y) = 0$ if either X or Y is vertical (this is a general property of curvatures in principal bundles). It follows that $\lambda\Omega$ is the pull-back $\lambda\Omega = \pi^*\Omega_\lambda$ by the projection $\pi : G \to G/MA$ of a 2-form Ω_λ on G/MA.

The Bianchi identity for a curvature states that $d\Omega(X, Y, Z) = 0$ if X, Y, Z are horizontal. Applying the functional λ, the same identity holds for $\lambda\Omega$ in place of Ω. This implies that $d\Omega_\lambda = 0$, that is, Ω_λ is a closed 2-form for any $\lambda \in (\mathfrak{m} \oplus \mathfrak{a})^*$ that is $\mathrm{Ad}(MA)$-invariant. Therefore, if λ is such that Ω_λ is non-degenerate, then Ω_λ is a symplectic form.

Example 2.1 (The KKS symplectic form of an adjoint orbit) A general situation in which Ω_λ is a symplectic form is obtained by functionals $\lambda = \lambda_H$ defined by $\lambda_H(\cdot) = \langle H, \cdot \rangle$, $H \in \mathfrak{m} \oplus \mathfrak{a}$. These linear functionals are $\mathrm{Ad}(MA)$-invariant if and only if H commutes with M. This occurs if $H \in \mathfrak{a}$ or if H belongs to the center of $\mathfrak{m}$.

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In this case, Ω_λ at the origin is given by

$$\Omega_\lambda(X,Y) = \lambda(\Omega(X,Y)) = -\lambda_H(\omega[X,Y]). \tag{2.5}$$

That is,

$$\Omega_\lambda(X,Y) = -\langle H,[X,Y]\rangle. \tag{2.6}$$

This is the expression at the origin of the KKS symplectic form on the adjoint orbit $\mathrm{Ad}(G)(-H)$. The two symplectic forms Ω_{λ_H} and KKS are invariant under $\mathrm{Ad}(G)$. Since they coincide at the origin, they are equal.

The discussion above can be summarized in the following result.

Proposition 2.2 *Let $H \in \mathfrak{m}\oplus\mathfrak{a}$ be a regular element (i.e., $\alpha(H)\neq 0$, for all $\alpha\in\Pi$). Then Ω_{λ_H} is the KKS symplectic form of the adjoint orbit* $\mathrm{Ad}(G)(-H)$.

3 Orbits in the product $\mathbb{F}\times\mathbb{F}^* = \mathbb{F}\times\mathbb{F}$

The homogeneous space G/MA is a special case of an orbit in the flag product $\mathbb{F}\times\mathbb{F}^*$.

When $\mathbb{F}$ is the maximal flag, its dual $\mathbb{F}^*$ is $\mathbb{F}$ itself with a translated origin, viewed as the homogeneous space G/P^- where $P^- = MAN^- = P^{w_0}$, with $P^{w_0} = w_0Pw_0^{-1}$ and w_0 representing the principal involution in K.

The diagonal action of G on $\mathbb{F}\times\mathbb{F}^*$ is given by $g(x,y) = (gx,gy)$. This action is not transitive – its orbits are parameterized by the Weyl group $\mathcal{W}$. Let x_0 be the origin of G/P and define $x_w = wx_0$ (the origin of G/P^w where $P^w = wPw^{-1} = MAN^w$). The orbits are

$$\mathcal{O}_w = G\cdot(x_0,x_w).$$

Since G acts transitively on $\mathbb{F}$, every orbit intersects the fiber $\{x_0\}\times\mathbb{F}^*$. The intersection with a diagonal orbit is a P-orbit because $g(x_0,y) = (x_0,z)$ iff $g\in P$ and y,z lie in the same P-orbit.

The stabilizer satisfies $g(x_0,x_w) = (x_0,x_w)$ iff $g\in P\cap P^w$, yielding the homogeneous space:

$$G\cdot(x_0,x_w) = G/(P\cap P^w).$$

For $w=1$, $P^w = P$ and the orbit is compact (coinciding with the flag). For $w=w_0$ (principal involution), $P^{w_0} = P^-$ and $P\cap P^{w_0} = MA$, making the orbit G/MA – the first case is closed (compact), and the second is open and dense.

Let $p_1:\mathbb{F}\times\mathbb{F}^*\to\mathbb{F}$ and $p_2:\mathbb{F}\times\mathbb{F}^*\to\mathbb{F}^*$ be the projections. Their restrictions to $\mathcal{O}_w$ are surjective, with fibers being Bruhat cells of type w.

Specifically:

- the fiber of p_1 is $\{x_0\}\times(P\cdot x_w) = \{x_0\}\times(N\cdot x_w)$, a Bruhat w-cell;
- for other fibers: $p_2(p_1^{-1}\{x\}) = g(N\cdot x_w)$ covers all type-w Bruhat cells;
- similarly, $p_1(p_2^{-1}\{y\}) = g(N^w\cdot x_0)$ is also a type-w cell.

For the closed orbit $\mathcal{O}_1$, the corresponding Bruhat cells reduce to points, making both projections diffeomorphisms. For the open orbit $\mathcal{O}_{w_0}$, the Bruhat cells are open and dense.

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The Lie algebra structure is

$$\mathfrak{p}^w = \mathrm{Ad}(w)\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^w$$
$$\mathfrak{n}^w = \mathrm{Ad}(w)\mathfrak{n}^+ = \sum_{\alpha>0} \mathrm{Ad}(w)\mathfrak{g}_\alpha$$
$$\mathfrak{p}^w \cap \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus (\mathfrak{n}^w \cap \mathfrak{n}^+).$$

Defining $\Pi_w^+ = w\Pi^+ \cap \Pi^+$, we can express

$$\mathfrak{n}^w \cap \mathfrak{n}^+ = \sum_{\alpha \in \Pi_w^+} \mathfrak{g}_\alpha$$

and consequently,

$$\mathfrak{p}^w \cap \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Pi_w^+} \mathfrak{g}_\alpha.$$

4 $K \times K$-invariant connections

The group G acts on $\mathbb{F} \times \mathbb{F} = \mathbb{F} \times \mathbb{F}^*$ via the diagonal action $g\,(x, y) = (gx, gy)$. This action is the restriction to the diagonal subgroup

$$\Delta_G = \{(g, g) : g \in G\} \tag{4.1}$$

of the action of $G \times G$ on $\mathbb{F} \times \mathbb{F}$. The isotropy group of the orbit of (x_0, x_{w_0}) is the intersection of $P = MAN$ with $P^- = MAN^-$, and therefore it is MA. This means that $G \cdot (x_0, x_{w_0}) = G/MA$, and this homogeneous space embeds in $\mathbb{F} \times \mathbb{F}$. Bundles, connections, symplectic forms, etc. on $\mathbb{F} \times \mathbb{F}$ restrict to geometric objects on G/MA, which are generally not invariant under G, but are invariant under the group $K \times K$, which acts transitively on $\mathbb{F} \times \mathbb{F}^*$, the maximal flag manifold of the group $G \times G$. Here, the subgroup K denotes the maximal compact subgroup of G (cf. Section 1.2), and we remark that K also acts transitively on the flag manifold $\mathbb{F}$ (see [4]).

Next, we will discuss connections on principal bundles over $\mathbb{F} \times \mathbb{F}^*$ whose restrictions to the open orbit $G \cdot (x_0, x_{w_0}) = G/MA$ provide connections on principal bundles over G/MA.

Initially, connections on a general flag of a semisimple group will be described. Subsequently, these connections will be specialized to the flags $\mathbb{F} \times \mathbb{F}^*$ of $G \times G$.

4.1 General case

The homogeneous space G/N is a principal bundle $G/N \to G/MAN = \mathbb{F}$ with structural group MA because N is normal in MAN and $MAN/N \approx MA$. The left action of G on G/N commutes with the right action of MA, meaning that G acts on the principal bundle $G/N \to \mathbb{F}$ by bundle automorphisms.

If $y_0 = 1N$ is the origin of G/N, then the orbit Ky_0 is the homogeneous space $K/(K \cap N) = K/\{1\} = K$. This orbit is a subbundle (reduction) of G/N with structural subgroup M, and $K \to K/M$ is the canonical projection associated with the action of K on $\mathbb{F}$.

The right action of A allows us to write $G/N = (Ky_0)\,A$, and in this way, G/N becomes a principal A-bundle over the K-orbit Ky_0. This principal bundle can be

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viewed in terms of homogeneous spaces through the canonical fibration $G/N \to G/AN = K$, which is a principal bundle with *structural group* $AN/N = A$.

This bundle is trivial (unlike the bundle over $\mathbb{F}$) because, by the Iwasawa decomposition, G/N is diffeomorphic to $K \times A$, which allows us to write

$$G/N = K \times A,$$

that is, G/N is identified with $K \times A$, where in this identification, the origin $1 \cdot N$ of G/N is identified with $(1,1) \in K \times A$.

The bundle $K \to K/M = \mathbb{F}$ has a K-invariant (left) connection where the horizontal space $\mathcal{H}_1$ at the origin is the space orthogonal to the isotropy algebra $\mathfrak{m}$:

$$\mathcal{H}_1 = \sum_{\alpha} \mathfrak{k}_\alpha$$

with $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$.

This connection extends to a connection on $G/N \to \mathbb{F}$ since $K \to \mathbb{F}$ is a subbundle. The extension is also K-invariant. The connection form is denoted by ω_K. The same notation is used for both bundles $K \to \mathbb{F}$ and $G/N \to \mathbb{F}$. On K, this form takes values in $\mathfrak{m}$, while on G/N, the form ω_K takes values in $\mathfrak{m} \oplus \mathfrak{a}$.

The curvature Ω_K of the connection on K is given by $\Omega_K(A,B) = [A,B]_{\mathfrak{m}}$, where A and B are left-invariant vector fields on K, and the subscript $\mathfrak{m}$ indicates the component in the direction of $\mathfrak{m}$. Note that Ω_K is a 2-form on K with values in $\mathfrak{m}$.

The curvature on G/N is obtained from the curvature on K by right translations by elements of A. In general, a curvature satisfies the property

$$R_h^* \Omega_K = \mathrm{Ad}\left(h^{-1}\right) \Omega_K.$$

The group A commutes with $\mathfrak{m}$, so $\mathrm{Ad}\left(h^{-1}\right)$ is the identity on $\mathfrak{m}$. Hence, on G/N, the curvature Ω_K also takes values in $\mathfrak{m}$.

The component in the direction of $\mathfrak{a}$ of the curvature Ω_K vanishes. The same does not happen with the $\mathfrak{a}$-component of the connection ω_K. Next, we will derive an expression for this component by composing the connection with linear functionals of $\mathfrak{a}$.

If $\lambda \in (\mathfrak{m} \oplus \mathfrak{a})^*$ is a linear functional, then the composition $\lambda(\omega_K)$ is a 1-form on G/N. The following proposition provides a description in terms of the Cartan–Killing form for a composition $\lambda_H(\omega_K)$ when $\lambda_H(\cdot) = \langle H, \cdot \rangle$ is the linear functional dual to $H \in \mathfrak{m} \oplus \mathfrak{a}$. The formula presented in the next proposition holds for an MA-invariant functional λ_H, which occurs if H is in the center of $\mathfrak{m} \oplus \mathfrak{a}$ (e.g., if $H \in \mathfrak{a}$ or any H in the case of a complex algebra, because in this case $\mathfrak{m} \oplus \mathfrak{a}$ is a Cartan subalgebra and thus abelian).

Proposition 4.1 *Consider $G = KAN$ with its corresponding Iwasawa decomposition, and for $g \in G$, write $g = khn$ for the Iwasawa decomposition of g, where $k \in K$, $h \in A$, and $n \in N$. Let $Z \in \mathfrak{g}$, and denote by $\widetilde{Z}$ the vector field on G/N induced by Z. Let $x_0 = 1 \cdot N$ be the origin of G/N. Then,*

$$\lambda_H(\omega_K(\widetilde{Z}(gx_0))) = \langle \mathrm{Ad}(k)H, Z \rangle. \tag{4.2}$$

Proof The formula is checked first at the origin $x_0 = 1 \cdot N$ and with $g = 1$. Write the decomposition $Z = X + A + Y \in \mathcal{H}_1 \oplus (\mathfrak{m} \oplus \mathfrak{a}) \oplus \mathfrak{n}^+$ where, as

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mentioned above, $\mathcal{H}_1 = \sum_\alpha \mathfrak{k}_\alpha$ and $\mathfrak{k} = \mathcal{H}_1 \oplus \mathfrak{m}$. By definition, $\omega_K(\widetilde{Z}(x_0)) = A$, and thus $\lambda_H \omega_K(\widetilde{Z}(x_0)) = \lambda_H(A) = \langle H, A\rangle$.

However, in the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$, the $\mathfrak{a}$ component is orthogonal (with respect to the Cartan–Killing form) to $\mathcal{H}_1 \oplus \mathfrak{n}^+$. Therefore, $\langle H, A\rangle = \langle H, Z\rangle$, showing that $\lambda_H \omega_K(\widetilde{Z}(x_0)) = \langle H, Z\rangle$.

Still at the origin x_0, but with $g = n \in N$, the equality continues to hold because in the Iwasawa decomposition of g, the component $k = 1$.

Now, consider a point $x = gx_0$ with $g = ma \in MA$. This point is also given by $R_g(x_0)$, since the left and right actions of MA on the fiber over the origin coincide. Furthermore, the equality $\widetilde{Z}(R_g x_0) = (R_g)_*(\widetilde{Z}(x_0))$ holds because the left action of G commutes with the right action of MA. Therefore,

$$\omega_K(\widetilde{Z}(gx_0)) = \omega_K((R_g)_*(\widetilde{Z}(x_0))) = (R_g^* \omega_K)(\widetilde{Z}(x_0)). \tag{4.3}$$

But $R_g^* \omega_K = \mathrm{Ad}(g^{-1})\omega_K$, and since λ_H is MA-invariant, it follows that $\lambda_H(R_g^* \omega_K) = \lambda_H(\omega_K)$ for $g \in MA$. Hence,

$$\lambda_H(\omega_K(\widetilde{Z}(gx_0))) = \lambda_H(\omega_K(\widetilde{Z}(x_0))) = \langle H, Z\rangle, \tag{4.4}$$

where the last equality follows from the calculation at x_0. If g_1 is such that $x = g_1 x_0 = gx_0$, then $g_1 = gn = man$ with $n \in N^+$, and as in the case of the origin, multiplication on the right by $n \in N^+$ does not affect the terms of the equality. This concludes the proof of (4.2) for points in the fiber over the origin x_0.

For other fibers, the result is obtained by translating by $k \in K$. From $\widetilde{Z}(kx) = k_* \widetilde{\mathrm{Ad}(k^{-1})Z}(x)$, it follows that

$$\omega_K(\widetilde{Z}(kx)) = \omega_K(\widetilde{\mathrm{Ad}(k^{-1})Z}(x)).$$

Therefore, if x is in the fiber over the origin, then

$$\lambda_H(\omega_K(\widetilde{Z}(kx))) = \lambda_H(\omega_K(\widetilde{\mathrm{Ad}(k^{-1})Z}(x))) = \langle H, \mathrm{Ad}(k^{-1})Z\rangle = \langle \mathrm{Ad}(k)H, Z\rangle,$$

which proves the equality (4.2) for points in the fiber over kx_0. Since K acts transitively on the fibers, this concludes the proof. ∎

The second term $\langle \mathrm{Ad}(k)H, Z\rangle$ in (4.2) is the "height function" f_Z with respect to Z, defined by the Cartan–Killing form on the adjoint orbit $\mathrm{Ad}(K)H$. More precisely, for each $Z \in \mathfrak{g}$, we define the *height function* on the adjoint orbit $f_Z : \mathrm{Ad}(K)H \to \mathbb{C}$ by

$$f_Z(\mathrm{Ad}(k)H) = \langle \mathrm{Ad}(k)H, Z\rangle.$$

If $H \in \mathfrak{a}$, then this adjoint orbit is a flag manifold of $\mathfrak{g}$. However, if $H \in \mathfrak{m}$, then $\mathrm{Ad}(K)H$ is an adjoint orbit of K in $\mathfrak{k}$. For example, if $\mathfrak{g}$ is a complex algebra, then $\mathfrak{m} = i\mathfrak{h}_{\mathbb{R}}$ is the imaginary part of the Cartan subalgebra, and if $H \in i\mathfrak{h}_{\mathbb{R}}$, then the K-adjoint orbit is also a flag manifold of $\mathfrak{g}$.

From these observations about the K-adjoint orbits and formula (4.2), we obtain the following vanishing properties of $\lambda_H \omega_K$.

Corollary 4.2 *Let H be in the center of $\mathfrak{m} \oplus \mathfrak{a}$. Then, the following properties hold:*

(1) *If $Z \in \mathfrak{s}$ and $H \in \mathfrak{m}$, then $\lambda_H \omega_K\left(\widetilde{Z}(x)\right) = 0$.*

(2) *If $Z \in \mathfrak{k}$ and $H \in \mathfrak{a}$, then $\lambda_H \omega_K\left(\widetilde{Z}(x)\right) = 0$.*

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Proof In both cases, Z is orthogonal (with respect to the Cartan–Killing form) to the adjoint orbit of H, and therefore the results follow from

$$\lambda_H\left(\omega_K\left(\widetilde{Z}(x)\right)\right) = \langle \mathrm{Ad}(k)H, Z\rangle$$

as stated in (4.2). ∎

The hypothesis that H is in the center of $\mathfrak{m}\oplus\mathfrak{a}$ is automatically satisfied in cases where $\mathfrak{m}$, and therefore $\mathfrak{m}\oplus\mathfrak{a}$, is an abelian algebra. In these cases, the above corollary ensures that the $\mathfrak{m}$ or $\mathfrak{a}$ component of ω_K vanishes, and therefore the connection takes values in the other component.

Corollary 4.3 *If $\mathfrak{m}\oplus\mathfrak{a}$ is an abelian algebra (e.g., in the case where $\mathfrak{g}$ is a complex algebra), then*

(1) $\omega_K\left(\widetilde{Z}(x)\right)\in\mathfrak{a}$ *if* $Z\in\mathfrak{s}$, *and*

(2) $\omega_K\left(\widetilde{Z}(x)\right)\in\mathfrak{m}$ *if* $Z\in\mathfrak{k}$.

4.2 Specializing to $\mathbb{F}\times\mathbb{F} = \mathbb{F}\times\mathbb{F}^*$, the maximal flag of $G\times G$

Definition 4.4 Consider the Iwasawa decomposition of $G\times G$, given by $(K\times K)(A\times A)(N^+\times N^-)$. The maximal flag manifold is

$$\mathbb{F}\times\mathbb{F} = (G\times G)/(M\times M)(A\times A)(N^+\times N^-), \tag{4.5}$$

which is based at the point $(x_0, x_{w_0})\in\mathbb{F}\times\mathbb{F}$, where w_0 denotes the principal involution of the Weyl group $\mathcal{W}$.

Proposition 4.5 *The diagonal action of G on $\mathbb{F}\times\mathbb{F}^*$, given by $g(x,y) = (gx, gy)$, has a unique open orbit, namely, $G\cdot(x_0, x_{w_0})$. As a homogeneous space, this open orbit is G/MA.*

Proof The isotropy groups of x_0 and x_{w_0} are MAN^+ and MAN^-, respectively. Their intersection is therefore MA, as claimed. ∎

The homogeneous space $(G\times G)/(N^+\times N^-)$ is diffeomorphic to $G/N^+\times G/N^- = (K\times A)\times(K\times A)$. Moreover, the projection

$$(G\times G)/(N^+\times N^-)\to\mathbb{F}\times\mathbb{F}^* = (G\times G)/(M\times M)(A\times A)(N^+\times N^-) \tag{4.6}$$

defines a principal $(M\times M)(A\times A)$-bundle.

Lemma 4.6 *The projection*

$$(G\times G)/(N^+\times N^-)\to(G\times G)/(A\times A)(N^+\times N^-) = K\times K \tag{4.7}$$

is a principal bundle with fiber $A\times A$, which is trivial since

$$(G\times G)/(N^+\times N^-) = (K\times K)\times(A\times A), \tag{4.8}$$

where this identification is chosen so that the origin $(1,1)\cdot(N^+\times N^-)$ corresponds to $((1,1),(1,1))$.

Proposition 4.7 *The diagonal action of G on $(G\times G)/(N^+\times N^-)$, given by*

$$g(g_1N^+, g_2N^-) = (gg_1N^+, gg_2N^-),$$

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has trivial isotropy at the origin $o = (1,1)(N^+ \times N^-)$. *Hence, the orbit* $G \cdot o$ *is diffeomorphic to* G.

Proof Since $N^+ \cap N^- = \{1\}$, the isotropy group is trivial. ∎

For $z = (m_1 h_1, m_2 h_2)(N^+ \times N^-)$ with $m_i h_i \in MA$, the isotropy group of the G-action remains trivial, and its orbits project onto the open orbit $G/MA = G \cdot (x_0, x_{w_0}) \subset \mathbb{F} \times \mathbb{F}$.

The restriction of the principal bundle

$$(G \times G)/(N^+ \times N^-) \to \mathbb{F} \times \mathbb{F}$$

to the open orbit $G \cdot (x_0, x_{w_0})$ is an $(MA) \times (MA)$-principal bundle whose G-orbits (under the diagonal action) are diffeomorphic to G. Each orbit remains invariant under the right action of

$$\Delta_{MA} = \{(h,h) : h \in MA\}, \tag{4.9}$$

and hence these orbits form subbundles with structural group Δ_{MA}.

The group $G \times G$ acts on $K \times K$, which is identified with $G \times G/(AN^+ \times AN^-)$. Therefore, $G \approx \Delta_G$ acts on $K \times K$ via the diagonal action (restriction of the action of $G \times G$ to the diagonal Δ_G). If $\overline{w}_0$ is a representative in $\mathrm{Nor}_K(A)$ of the principal involution w_0, then the orbit $G \cdot (1, \overline{w}_0)$ is open in $K \times K$ and projects to the unique open orbit in $\mathbb{F} \times \mathbb{F}^*$.

The action of $G \times G$ on $(K \times K) \times (A \times A) = (G \times G)/(N^+ \times N^-)$ decomposes into the action on $K \times K$ and the cocycle $\rho^\# : (G \times G) \times (K \times K) \to A \times A$ defined by

$$\rho^\#((g,h),(k,u)) = (\rho(g,k), \rho(h,u)), \tag{4.10}$$

where $\rho : G \times K \to A$ is, as before, the cocycle defined by

$$gu = x\rho(g,u)\, n \in KAN^+, \tag{4.11}$$

which factors to a cocycle over $\mathbb{F}$ given by $\rho(g,x) = \rho(g,u)$.

With the cocycle notation (cf. Equation (4.11)), the action of $G \times G$ on $(K \times K) \times (A \times A)$ is determined by

$$(g,h) \cdot ((k,u),(1,1)) = ((g \cdot k, h \cdot u), (\rho(g,k), \rho(h,u))), \tag{4.12}$$

where in the first-ordered pair on the right-hand side, $\cdot$ denotes the action of G on K. (For an arbitrary point $((k,u),(a,b))$, the action is obtained by right invariance.)

In particular, the diagonal action of G is given by

$$(g,g) \cdot ((k,u),(1,1)) = ((g \cdot k, g \cdot u), (\rho(g,k), \rho(g,u))). \tag{4.13}$$

If $\overline{w}_0$ is a representative in $\mathrm{Nor}_K(A)$ of the principal involution w_0 of the Weyl group, then the orbit $G \cdot ((1,\overline{w}_0),(1,1))$ under the diagonal action is diffeomorphic to G because the isotropy group is $N^+ \cap N^- = \{1\}$.

An orbit of this type is the subset of $(K \times K) \times (A \times A)$ given by

$$\begin{aligned} G \cdot ((1,\overline{w}_0),(1,1)) &= \{((g \cdot 1, g \cdot \overline{w}_0), (\rho(g,1), \rho(g,\overline{w}_0))) : g \in G\} \\ &= \{((g \cdot 1, g \cdot \overline{w}_0), (\rho(g,x_0), \rho(g,x_{w_0}))) : g \in G\}. \end{aligned}$$

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That is, by writing the Iwasawa decompositions $g = kan^+ \in KAN^+$ and $g = uhn^- \in KAN^-$, one obtains $a = \rho(g,1)$, $h = \rho(g,\overline{w}_0)$, $k = g\cdot 1$, and $u = g\cdot\overline{w}_0$. Therefore, the orbit is given by the points

$$((k,u),(\rho(g,x_0),\rho(g,x_{w_0}))) \qquad g = kan^+ = uhn^-, \tag{4.14}$$

and these ordered pairs also provide the diffeomorphism between G and the orbit $G\cdot((1,\overline{w}_0),(1,1))$ from the chosen origin.

Summarizing, we have proven the following.

Proposition 4.8 *The orbit* $G\cdot((1,\overline{w}_0),(1,1))$ *is*

$$G\cdot((1,\overline{w}_0),(1,1)) = \left\{\left((g\cdot 1, g\cdot\overline{w}_0),(\rho(g,x_0),\rho(g,x_{w_0}))\right) : g\in G\right\},$$

and is diffeomorphic to G.

Since an orbit $G\cdot((1,\overline{w}_0),(1,1))$ is diffeomorphic to G, its admits the Cartan decomposition $G = SK$. Therefore, the orbit is diffeomorphic to the product $S\times K$.

To see this decomposition as subsets of $(K\times K)\times(A\times A)$ contained in the orbit, first consider the K component, whose orbit is

$$K\cdot((1,\overline{w}_0),(1,1)) = \{((k,k\overline{w}_0),(1,1)) : k\in K\} = (K\times K\overline{w}_0)\times\{(1,1)\}.$$

The image of an arbitrary element $g = sk \in SK$ is obtained by applying k and then s to the origin $((1,\overline{w}_0),(1,1))$. Using the expression in k and the equality (4.13), the result is

$$(s,s)\cdot((k,k),(1,1)) = ((s\cdot k, s\cdot k),(\rho(s,k),\rho(s,k\overline{w}_0))).$$

In particular, if $s = h = e^H \in A$ and $(k,k) = (1,1)$, then

$$\begin{aligned}(h,h)\cdot((1,1),(1,1)) &= ((1,1),(\rho(h,1),\rho(h,\overline{w}_0)))\\ &= \left((1,1),\left(h,\overline{w}_0^{-1}h\overline{w}_0\right)\right)\\ &= \left((1,1),\left(e^H,e^{w_0H}\right)\right).\end{aligned}$$

Let θ be the Cartan involution on G given by the Cartan decomposition $G = KS$.

An element $g\in G$ viewed in the open orbit of $(K\times K)\times(A\times A)$ is given by

$$g = ((k,u),(\rho(g,x_0),\rho(g,x_w))),$$

where $g = k\rho(g,x_0)\,n^+ = u\rho(g,x_w)\,n^-$ are Iwasawa decompositions.

Applying θ to these Iwasawa decompositions, we obtain

$$\theta(g) = \theta(k)\,\theta(\rho(g,x_0))\,\theta(n^+) = \theta(u)\,\theta(\rho(g,x_w))\,\theta(n^-).$$

The Cartan involution θ is the identity on K and is the inverse on S, and moreover, $\theta(N^+) = N^-$ and $\theta(N^-) = N^+$. Thus, we obtain the Iwasawa decompositions

$$\begin{aligned}\theta(g) &= k\rho(g,x_0)^{-1}\,\theta(n^+)\in KAN^-,\\ \theta(g) &= u\rho(g,x_w)^{-1}\,\theta(n^-)\in KAN^+.\end{aligned}$$

Hence, $\theta(g)$ viewed in the open orbit of $(K\times K)\times(A\times A)$ is

$$\theta(g) = \left((u,k),\left(\rho(g,x_w)^{-1},\rho(g,x_0)^{-1}\right)\right).$$

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This means that the involution θ viewed in the open orbit is given by

$$\theta : ((k,u),(h_1,h_2)) \longmapsto \left((u,k),\left(h_2^{-1},h_1^{-1}\right)\right). \tag{4.15}$$

In summary, we have the following lemma.

Lemma 4.9 *Let θ be the Cartan involution on G given by $G = KS$. For $g = k\rho(g,x_0)n^+ = u\rho(g,x_w)n^-$, the involution acts as*

$$\theta(g) = ((u,k),(\rho(g,x_w)^{-1},\rho(g,x_0)^{-1})). \tag{4.16}$$

Consequently, on the open orbit,

$$\theta : ((k,u),(h_1,h_2)) \mapsto ((u,k),(h_2^{-1},h_1^{-1})). \tag{4.17}$$

An application of formula (4.2) provides the following expressions for compositions of linear functionals with the connection $\omega_{K\times K}$ on the bundle $G \to G/MA$.

Proposition 4.10 *Let H_1, H_2 be elements of the center of $\mathfrak{m}\oplus\mathfrak{a}$ and consider the linear functional λ_{H_1,H_2} on $(\mathfrak{m}\oplus\mathfrak{a})\times(\mathfrak{m}\oplus\mathfrak{a})$ defined by*

$$\lambda_{H_1,H_2}(A,B) = \langle H_1, A\rangle + \langle H_2, B\rangle.$$

Then the composite $\lambda_{H_1,H_2}(\omega_{K\times K})$ is given by

$$\lambda_{H_1,H_2}\left(\omega_{K\times K}\left(\widetilde{Z}_1(g_1x_0),\widetilde{Z}_2(g_2x_w)\right)\right) = \langle \mathrm{Ad}(k_1)H_1, Z_1\rangle + \langle \mathrm{Ad}(k_2)H_2, Z_2\rangle$$

where $(g_1,g_2)\in G\times G$, $(Z_1,Z_2)\in\mathfrak{g}\times\mathfrak{g}$, and $g_1 = k_1a_1n^+ \in KAN^+$, $g_2 = k_2a_2n^- \in KAN^-$ are the components of g_1 and g_2 with respect to the opposite Iwasawa decompositions.

Proof This follows directly from the general expression (4.2) applied to $\omega_{K\times K}$. ■

In particular, taking $Z_1 = Z_2$, the vector field $\left(\widetilde{Z},\widetilde{Z}\right)$, $Z\in\mathfrak{g}$, comes from the diagonal action of G. By identifying G with the orbit $G\cdot o$, $\left(\widetilde{Z},\widetilde{Z}\right)$ becomes the right-invariant vector field Z^r, and the 1-form $\omega_{K\times K}$ on G is given by $\omega_{K\times K}(Z^r) = \omega_{K\times K}\left(\widetilde{Z},\widetilde{Z}\right)$.

Corollary 4.11 *If H_1, H_2 belong to the center of $\mathfrak{m}\oplus\mathfrak{a}$, then the composition $\lambda_{H_1,H_2}(\omega_{K\times K})$ evaluated at the right-invariant field Z^r on G, which coincides with $\left(\widetilde{Z},\widetilde{Z}\right)$, $Z\in\mathfrak{g}$, is given by*

$$\begin{aligned}\lambda_{H_1,H_2}(\omega_{K\times K}(Z^r(gx_0,gx_w))) &= \lambda_{H_1,H_2}\left(\omega_{K\times K}\left(\widetilde{Z}(gx_0),\widetilde{Z}(gx_w)\right)\right)\\ &= \langle \mathrm{Ad}(k_1)H_1, Z\rangle + \langle \mathrm{Ad}(k_2)H_2, Z\rangle\end{aligned}$$

where $g = k_1a_1n^+ \in KAN^+$ and $g = k_2a_2n^- \in KAN^-$ are the components of g with respect to the opposite Iwasawa decompositions. Note that $G = KAN^+ = KAN^-$.

5 Canonical symplectic form on $T^*\mathbb{F} = G/MA = \mathrm{Ad}(G)H_0$

The goal here is to describe the Liouville 1-form and the canonical symplectic form of the cotangent bundle $T^*\mathbb{F}$ in terms of the connections ω and $\omega_{K\times K}$ constructed in the previous sections (Proposition 4.1 and Corollary 4.11), using the fact that $T^*\mathbb{F}$ is diffeomorphic to the homogeneous space G/MA (see [4, Section 5.2]).

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5.1 Generalities

Let $\pi : T^*M \to M$ be a cotangent bundle.

(1) The Liouville 1-form Λ on T^*M is defined by $\Lambda_p(v) = p(\pi_* v)$, and the canonical symplectic form is $\Gamma = d\Lambda$.

(2) If g is a diffeomorphism of M, its lift is given by $g^\# \cdot \alpha = \alpha \circ \left(dg^{-1}\right)_{gx}$ for $\alpha \in T^*_x M$. The lift $g^\#$ is also a diffeomorphism and is linear on each fiber. The forms Λ and Γ are invariant under $g^\#$, that is, $\left(g^\#\right)^* \Lambda = \Lambda$ and $\left(g^\#\right)^* \Gamma = \Gamma$. The invariance of Γ under $g^\#$ means that $g^\#$ is a symplectic diffeomorphism. (A converse also holds: a diffeomorphism of T^*M is a lift if and only if Λ is invariant.)

(3) If X is a vector field on M, its lift $X^\#$ is the vector field whose flow is $\left(X_t\right)^\#$, where X_t is the flow of X.

The vector field $X^\#$ is the Hamiltonian field (with respect to Γ) of the function $h(\alpha) = \alpha(X(x))$, where $\alpha \in T^*_x M$.

By definition, $h(\alpha) = \Lambda(X^\#)$, that is,

$$X^\# = \mathrm{ham}(\Lambda(X^\#)).$$

Since the lifts $\left(X_t\right)^\#$ leave Λ invariant, the Lie derivative $L_{X^\#}\Lambda = 0$. (The converse also holds: $L_Y \Lambda = 0$ if and only if Y is a lift, $Y = X^\#$.)

(4) Given a function f on M, define $f^\# = f \circ \pi$, which is constant along the fibers. The Hamiltonian vector field $\mathrm{ham}(f^\#)$ is the "vertical" vector field that is constant on each fiber and is given by

$$\mathrm{ham}(f^\#)(p) = p + df_x, \qquad p \in T^*_x M.$$

The trajectories of $\mathrm{ham}(f^\#)$ are the straight lines $p + t\, df_x$ contained in the fibers.

By general properties of Hamiltonian fields,

$$\Gamma\left(X^\#, \mathrm{ham} f^\#\right) = \left(\mathrm{ham} f^\#\right) \Lambda\left(X^\#\right) = -X^\# f^\# = -Xf.$$

(5) *Lie group actions:* Let G be a Lie group acting on M, where the infinitesimal action of the Lie algebra $\mathfrak{g}$ is denoted by $X \in \mathfrak{g} \mapsto \widetilde{X}$. The lifts $g^\#$, $g \in G$, define an action of G on T^*M. This action is symplectic (with respect to Γ) because the lifts $g^\#$ are symplectic diffeomorphisms.

The corresponding infinitesimal action is given by the lifts $X^\#$ of the vector fields $\widetilde{X}$ on M with $X \in \mathfrak{g}$.

A lift $X^\#$ is the Hamiltonian field (with respect to Γ) of the function $\Lambda\left(X^\#\right)$. This implies that the action of G on T^*M is Hamiltonian (as well as symplectic).

The moment map $\mu : T^*M \to \mathfrak{g}^*$ is defined by

$$\mu(p)(X) = \Lambda_p\left(X^\#(p)\right) \qquad p \in T^*M,\ X \in \mathfrak{g}. \tag{5.1}$$

An alternative description of the moment map μ can be given more directly from the action: denote the action on M by $\phi : G \times M \to M$ with $\phi(g, x) = g \cdot x$. For each $x \in M$, let $\phi_x : G \to M$ be defined by $\phi_x(g) = g \cdot x$. With this notation,

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$\widetilde{X}(x) = (d\phi_x)_1(X)$, $X \in \mathfrak{g} = T_1G$. If $p \in T_x^*M$, then

$$\Lambda_p\left(X^{\#}(p)\right) = p\left(\widetilde{X}(x)\right) = p \circ (d\phi_x)_1(X).$$

However, by the definition of the adjoint linear transformation, $p \circ (d\phi_x)_1 = (d\phi_x)_1^*(p)$.

Therefore, in terms of the action ϕ, the moment map is given by

$$\mu_{|T_x^*M} = (d\phi_x)_1^*. \tag{5.2}$$

(6) *Equivariance of the moment map:* The moment map on T^*M defined via the Liouville form $\Lambda_p(X^{\#}(p))$ is equivariant, meaning

$$\mu\left(g^{\#}p\right) = \mathrm{Ad}^*(g)\,\mu(p) \qquad p \in T^*M,\ g \in G.$$

Indeed, if $p \in T_x^*M$, then $g^{\#}p \in T_{gx}^*M$, and we have

$$\mu\left(g^{\#}p\right)(X) = \Lambda_{g^{\#}p}\left(X^{\#}\left(g^{\#}p\right)\right) = g^{\#}p\left(\widetilde{X}(gx)\right) = p\left(g_*^{-1}\widetilde{X}(gx)\right) = p(\widetilde{\mathrm{Ad}(g^{-1})}$$
$$X(x)) = \mu(p)\left(\mathrm{Ad}\left(g^{-1}\right)X\right) = \left(\mathrm{Ad}^*(g)\,\mu(p)\right)(X).$$

The equivariance of μ implies that the image of an orbit $G^{\#} \cdot p$ under the action of G on T^*M is the coadjoint orbit $\mathrm{Ad}^*(G) \cdot \mu(p) \subset \mathfrak{g}^*$. Furthermore, if $\omega_{\mu(p)}$ is the KKS symplectic form on the orbit $\mathrm{Ad}^*(G) \cdot \mu(p)$, then the restriction of the symplectic form Γ to $G^{\#} \cdot p$ is the pullback $\mu^*\omega_{\mu(p)}$.

5.2 Flag manifolds

The maximal flag manifold $\mathbb{F}$ is given by a K-adjoint orbit $\mathrm{Ad}(K)H_0 = K/M$. The tangent bundle $T\mathbb{F}$ and cotangent bundle $T^*\mathbb{F}$ are realized as the G-adjoint orbit $\mathrm{Ad}(G)H_0 = G/MA$ with same H_0. For $\mathrm{Ad}(K)H_0$ to indeed represent the maximal flag manifold, H_0 must be chosen appropriately, which occurs, for instance, in the following cases.

(1) $H_0 \in \mathfrak{a}^+$ is a regular real element.
(2) $\mathfrak{g}$ is a complex Lie algebra where $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{h}$ is a (complex) Cartan subalgebra with $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}}$ as the real part and $\mathfrak{m} = i\mathfrak{h}_{\mathbb{R}}$ as the imaginary part. In this case, an orbit $\mathrm{Ad}(K)H_0$ represents the maximal flag manifold if H_0 is regular (real or not).

In both $\mathrm{Ad}(K)H_0 = K/M$ and $\mathrm{Ad}(G)H_0 = G/MA$, the origin x_0 is precisely H_0. The fibers of the bundles are in bijection with affine subspaces.

(1) The fiber of $T\mathbb{F}$ over $\mathrm{Ad}(k)H_0$ is $\mathrm{Ad}(k)(H_0 + \mathfrak{n}^-)$, $k \in K$. The bijection associates $\mathrm{Ad}(k)(H_0 + Y)$, $Y \in \mathfrak{n}^-$, with the tangent vector $\widetilde{Y}(kx_0)$.
(2) The fiber of $T^*\mathbb{F}$ over $\mathrm{Ad}(k)H_0$ is $\mathrm{Ad}(k)(H_0 + \mathfrak{n}^+)$, $k \in K$. The bijection associates $\mathrm{Ad}(k)(H_0 + X)$, $X \in \mathfrak{n}^+$, with the linear functional $p \in T_{kx_0}^*$ defined by $p\left(\widetilde{Y}(kx_0)\right) = \langle X, Y\rangle$, where $Y \in \mathfrak{n}^-$ and $\langle\cdot,\cdot\rangle$ is the Cartan–Killing form of $\mathfrak{g}$.

The affine subspaces of $\mathfrak{g}$ contained in $\mathrm{Ad}(G)H_0$, which appear as fibers of $T\mathbb{F}$ and $T^*\mathbb{F}$, are determined by the following Lie algebra decompositions.

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First, consider the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ arising from the root spaces. More generally, for $k \in K$, consider the conjugated decomposition:

$$\mathfrak{g} = \mathfrak{m}^k \oplus \mathfrak{a}^k \oplus (\mathfrak{n}^+)^k \oplus (\mathfrak{n}^-)^k,$$

where the superscript indicates conjugation by k ($\mathfrak{m}^k = \mathrm{Ad}(k)\,\mathfrak{m}$, etc.).

With this notation, the affine subspaces are given by:

$$\mathrm{Ad}(k)(H_0 + \mathfrak{n}^-) = \mathrm{Ad}(k)H_0 + (\mathfrak{n}^-)^k, \qquad \mathrm{Ad}(k)(H_0 + \mathfrak{n}^+) = \mathrm{Ad}(k)H_0 + (\mathfrak{n}^+)^k,$$

where the first is identified with the tangent space $T_x\mathbb{F}$ and the second with $T_x^*\mathbb{F}$ when $x = \mathrm{Ad}(k)H_0$.

The identifications of the tangent and cotangent spaces with affine subspaces of $\mathfrak{g}$ are used below to describe the Liouville and canonical forms of the cotangent bundle $T^*\mathbb{F}$. These forms are obtained from the moment map μ associated with the Hamiltonian action of G on $T^*\mathbb{F}$, derived from the lift of the action $\phi : G \times \mathbb{F} \to \mathbb{F}$ on $\mathbb{F}$. As shown in (5.2), the following equality holds:

$$\mu_{|T_x^*M} = (d\phi_x)_1^*.$$

The isotropy algebra of the action ϕ at the point $x = kx_0$ is given by $\mathfrak{m}^k \oplus \mathfrak{a}^k \oplus (\mathfrak{n}^+)^k$. Consequently, if $Z \in \mathfrak{g}$ decomposes as

$$Z = Z_{\mathfrak{m}} + Z_{\mathfrak{a}} + Z_{\mathfrak{n}^+} + Z_{\mathfrak{n}^-} \in \mathfrak{m}^k \oplus \mathfrak{a}^k \oplus (\mathfrak{n}^+)^k \oplus (\mathfrak{n}^-)^k,$$

then $\widetilde{Z}(x) = \widetilde{Z}_{\mathfrak{n}^-}(x)$, with $x = kx_0$. Thus,

$$(d\phi_x)_1(Z) = \widetilde{Z}_{\mathfrak{n}^-}(x) \qquad Z \in \mathfrak{g}.$$

Identifying $T_x\mathbb{F}$, where $x = kx_0$, with $(\mathfrak{n}^-)^k = \mathrm{Ad}(k)\mathfrak{n}^-$, the linear map $(d\phi_x)_1$ corresponds to the projection $\mathrm{proj}_{(\mathfrak{n}^-)^k}$ of $\mathfrak{g}$ onto $(\mathfrak{n}^-)^k$, with respect to the decomposition

$$\mathfrak{g} = \mathfrak{m}^k \oplus \mathfrak{a}^k \oplus (\mathfrak{n}^+)^k \oplus (\mathfrak{n}^-)^k.$$

The moment map μ is the adjoint $(d\phi_x)_1^*$, that is,

$$\mu(\alpha)(Z) = \alpha \circ (d\phi_x)_1^* \left(\widetilde{Z}(x)\right) = \alpha \circ (d\phi_x)_1^* \left(\widetilde{Z_{(\mathfrak{n}^-)^k}}(x)\right), \quad \alpha \in T_x^*\mathbb{F},\ Z \in \mathfrak{g}.$$

Therefore, $\mu(\alpha)(Z) = \mu(\alpha)\left(Z_{(\mathfrak{n}^-)^k}\right)$.

Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Cartan–Killing form $\langle \cdot, \cdot \rangle$, the subspace $\mathfrak{m}^k \oplus \mathfrak{a}^k$ vanishes on $(\mathfrak{n}^-)^k$, so $\mu(\alpha)$ lies in the nilpotent algebra $(\mathfrak{n}^+)^k$.

It is now possible to interpret the moment map using the identification of $T^*\mathbb{F}$ with $G/MA = \mathrm{Ad}(G)H_0$. This adjoint orbit is the union

$$\mathrm{Ad}(G)H_0 = \bigcup_{k \in K} \mathrm{Ad}(k)(H_0 + \mathfrak{n}^+).$$

An element $H_0^k + X^k = \mathrm{Ad}(k)(H_0 + X)$, with $X \in \mathfrak{n}^+$, is viewed as a linear functional on the tangent space $T_x\mathbb{F} = \mathrm{Ad}(k)(H_0 + \mathfrak{n}^-)$, where $x = \mathrm{Ad}(k)x_0$, and is expressed as $\langle X, Y \rangle$ with $Y \in \mathfrak{n}^-$.

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Summarizing, we have the following proposition.

Proposition 5.1 *Let $\mu : T^*\mathbb{F} \to \mathfrak{g}^*$ be the moment map induced by the linear action of G on $T^*\mathbb{F}$. Using the identifications $\mathfrak{g}^* \approx \mathfrak{g}$ (via the Cartan–Killing form $\langle \cdot, \cdot \rangle$) and $T^*\mathbb{F} \approx \mathrm{Ad}(G) H_0$, the map μ is given by*

$$\mu\left(H_0^k + X^k\right)(Z) = \langle X^k, Z \rangle, \tag{5.3}$$

where the superscript k indicates $X^k = \mathrm{Ad}(k) X$, and similarly for other terms.

This expression for the moment map μ provides descriptions of the canonical symplectic form on $T^*\mathbb{F}$.

One description is given in terms of *nilpotent orbits* in $\mathfrak{g}$. A key property of an equivariant moment map (as is the case here) is that the image of a group orbit is a coadjoint orbit of the representation. Furthermore, the restriction of the symplectic form to the orbit is the pullback of the KKS form on the coadjoint orbit.

In this case, the orbits of the linear action of G on $T^*\mathbb{F}$ are given by

$$G^{\#} \cdot p = \{p \circ dg_x^{-1} : g \in G\} \qquad p \in T_x^*\mathbb{F},$$

with $x \in \mathbb{F}$ being arbitrary, as the action of G on $\mathbb{F}$ is transitive.

By identifying $T^*\mathbb{F} \approx \mathrm{Ad}(G) H_0$, an element $\alpha \in T_{x_0}^*\mathbb{F}$, where $x_0 = H_0$, is given by $\alpha = H_0 + X$, with $X \in \mathfrak{n}^+$.

From (5.3), we have $\mu(H_0 + X) = \langle X, \cdot \rangle$. Hence, the following description of the canonical symplectic form Γ on $T^*\mathbb{F}$ is obtained.

Proposition 5.2 *Consider the identification $T^*\mathbb{F} \approx \mathrm{Ad}(G) H_0$. Then, the canonical symplectic form Γ on $T^*\mathbb{F}$, restricted to the orbit $G^{\#} \cdot (H_0 + X)$, $X \in \mathfrak{n}^+$, is the pullback of the KKS form on the coadjoint orbit of $\langle X, \cdot \rangle$.*

Via the identification $\mathfrak{g}^* \approx \mathfrak{g}$, the coadjoint orbit of $\langle X, \cdot \rangle$ corresponds to the adjoint orbit of the nilpotent element X.

Explicitly, if $p = \mathrm{Ad}(k)(H_0 + X)$, then

$$\Gamma_p\left(\widetilde{Y}, \widetilde{Z}\right) = \langle X, [Y, Z] \rangle \qquad Y, Z \in \mathfrak{g}, \tag{5.4}$$

where $\widetilde{Y} = \mathrm{ad}(Y)$ and $\widetilde{Z} = \mathrm{ad}(Z)$ are the linear vector fields on $\mathfrak{g}$ defined by Y and Z.

Another application of the moment map formula (5.3) is that it provides an expression for the Liouville form Λ in terms of the difference $\omega - \omega_{K \times K}$, where ω and $\omega_{K \times K}$ are the connections that are G-invariant and $K \times K$-invariant, respectively. From this, an expression for the canonical symplectic form in terms of the difference $\Omega - \Omega_{K \times K}$ of the respective curvatures is obtained.

For this interpretation, it must be assumed that H_0 lies in the center of $\mathfrak{m} \oplus \mathfrak{a}$ so that λ_{H_0} is MA-invariant. The condition $\mathrm{Ad}(K) H_0 = \mathbb{F}$ is still used, as $\mathrm{Ad}(G) H_0$ must be identified with $T^*\mathbb{F}$.

The relationship between Λ and μ is given by $\mu(Z) = \Lambda(Z^{\#})$, where $Z \in \mathfrak{g}$. Combining this equality with (5.3) yields

$$\Lambda_p\left(Z^{\#}(p)\right) = \langle X^k, Z \rangle \qquad Z \in \mathfrak{g}, \tag{5.5}$$

where $p = H_0^k + X^k = \mathrm{Ad}(k)(H_0 + X)$ is an element of the cotangent space $T_x^*\mathbb{F} = \mathrm{Ad}(k)(H_0 + \mathfrak{n}^+)$, with $x = H_0^k = \mathrm{Ad}(k) H_0$.

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The second term in the equality above can be rewritten as

$$\langle X^k, Z\rangle = \langle H_0^k + X^k, Z\rangle - \langle H_0^k, Z\rangle. \tag{5.6}$$

This expression is interpreted in terms of connections as follows.

(1) Let ω be the G-invariant connection on $G \to G/MA$. Then, the first term on the right-hand side of (5.6) is given by

$$\langle H_0^k + X^k, Z\rangle = \lambda_{H_0}\left(\omega_g Z^r(g)\right), \tag{5.7}$$

where Z^r is the right-invariant vector field defined by $Z \in \mathfrak{g}$, $g \in G$ is any element such that $\mathrm{Ad}(g)H_0 = H_0^k + X^k$, and $\lambda_{H_0}(\cdot) = \langle H_0, \cdot\rangle$.

Indeed, at the origin H_0 with $g = 1$, the equality holds because $\omega_1 Z^r(1)$ is the component of Z in the direction of $\mathfrak{m} \oplus \mathfrak{a}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$, which is orthogonal with respect to the Cartan–Killing form. At the origin, for $g \in MA$, the equality still holds since $H_0^k + X^k = \mathrm{Ad}(g)H_0 = H_0$.

For a general point, note that $\omega_g(Z^r(g)) = \omega_1((\mathrm{Ad}(g^{-1})Z)^r(1))$ because ω is left-invariant. Thus,

$$\lambda_{H_0}(\omega_g Z^r(g)) = \lambda_{H_0}\left(\omega_1((\mathrm{Ad}(g^{-1})Z)^r(1))\right),$$

which, using the equality at H_0, becomes

$$\begin{aligned}\lambda_{H_0}\left(\omega_1((\mathrm{Ad}(g^{-1})Z)^r(1))\right) &= \langle H_0, \mathrm{Ad}(g^{-1})Z\rangle = \langle \mathrm{Ad}(g)H_0, Z\rangle \\ &= \langle H_0^k + X^k, Z\rangle,\end{aligned}$$

if $\mathrm{Ad}(g)H_0 = H_0^k + X^k$. This concludes the proof of (5.7), which provides the first term on the right-hand side of (5.6) in terms of the G-invariant connection.

(2) The component $\langle H_0^k, Z\rangle = \langle \mathrm{Ad}(k)H_0, Z\rangle$ was computed in terms of the connection $\omega_{K\times K}$ in Corollary 4.11. The element $H_0 \in \mathfrak{a}$ lies in the center of $\mathfrak{m} \oplus \mathfrak{a}$, and thus $(H_0, 0)$ is in the center of $(\mathfrak{m} \oplus \mathfrak{a}) \times (\mathfrak{m} \oplus \mathfrak{a})$. Consequently, Corollary 4.11 applies to the linear functional $\lambda_{H_0,0}$, showing that

$$\begin{aligned}\lambda_{H_0,0}\left(\omega_{K\times K}\left(Z^r(gx_0, gx_w)\right)\right) &= \lambda_{H_0,0}\left(\omega_{K\times K}\left(\widetilde{Z}(gx_0), \widetilde{Z}(gx_w)\right)\right) \\ &= \langle \mathrm{Ad}(k)H_0, Z\rangle,\end{aligned}$$

where $g = khn^+$ are the components of g in the Iwasawa decomposition $G = KAN^+$. This computes $\langle \mathrm{Ad}(k)H_0, Z\rangle$ in terms of the connection $\omega_{K\times K}$.

The combination of the previous two items expresses the Liouville form of $T^*\mathbb{F}$ in terms of the difference $\omega - \omega_{K\times K}$ between the connections. If λ is an MA-invariant linear functional on $(\mathfrak{m} \oplus \mathfrak{a})^2$, then $\lambda\omega - \lambda\omega_{K\times K}$ is a right-invariant 1-form on the principal bundle $(G \times G)/(N^+ \times N^-) \to G/MA = T^*\mathbb{F}$, which vanishes on the vertical spaces (since this is the behavior of the difference between two connections). Therefore, $\lambda\omega - \lambda\omega_{K\times K}$ is the pullback of a 1-form on G/MA. This 1-form on G/MA is denoted by $\lambda^*(\omega - \omega_{K\times K})$.

Proposition 5.3 *Let the identification of $T^*\mathbb{F}$ with G/MA be made by equating both to an adjoint orbit $\mathrm{Ad}(G)H_0$, where H_0 is in the center of $\mathfrak{m} \oplus \mathfrak{a}$ and regular in the sense that $\mathrm{Ad}(K)H_0 = \mathbb{F}$. Then, the Liouville form on $T^*\mathbb{F} = G/MA$ is given by*

$$\lambda^*_{H_0,0}(\omega - \omega_{K\times K})$$

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where, as before, $\lambda_{H_0}(\cdot) = \langle H_0, \cdot \rangle$. *Therefore, the canonical symplectic form on* $T^*\mathbb{F}$ *is*

$$\Gamma = \lambda^*_{H_0,0}\left(\Omega - \Omega_{K\times K}\right)$$

where Ω *and* $\Omega_{K\times K}$ *are the curvatures. In this expression, the component* $\lambda^*_{H_0,0}\Omega = \Omega_{\lambda_{H_0}}$ *is the KKS form of the adjoint orbit by* H_0.

As previously seen, the $\mathfrak{a}\oplus\mathfrak{a}$-component of the curvature $\Omega_{K\times K}$ vanishes, that is, $\Omega_{K\times K}$ takes values in $\mathfrak{m}\oplus\mathfrak{m}$. Therefore, if H_1 and H_2 are in $\mathfrak{a}$, then $\lambda^*_{H_1,H_2}\Omega_{K\times K} = 0$. Taking, as in the previous proposition, $(H_1, H_2) = (H_0, 0)$ where $H_0 \in \mathfrak{a}$, it follows that the canonical symplectic form reduces to $\Gamma = \lambda^*_{H_0,0}\Omega$. The functional $\lambda_{H_0,0}$ restricted to the diagonal $\{(H,H) \in \mathfrak{a}\oplus\mathfrak{a}\}$ coincides with the functional λ_{H_0} on $\mathfrak{a}$. Therefore, $\lambda^*_{H_0,0}\Omega$ is seen as $\lambda^*_{H_0}\Omega$, which is in turn the KKS form of the adjoint orbit $\mathrm{Ad}(G)H_0$. This shows that this KKS form is exact.

Corollary 5.4 *If* $H_0 \in \mathfrak{a}$ *is a regular real element, then the KKS form* κ_{H_0} *of the adjoint orbit* $\mathrm{Ad}(G)H_0$ *coincides with the canonical symplectic form* Γ *of* $T^*\mathbb{F}$ *and is therefore an exact form. In terms of connections,*

$$\kappa_{H_0} = d\left(\lambda^*_{H_0}\left(\omega - \omega_{K\times K}\right)\right)$$

where ω *is the* G*-invariant connection on* $G \to G/MA$.

To conclude this section, we record some remarks regarding the preceding constructions.

(1) The fact that the canonical symplectic form Γ is written as the difference of two curvatures is in agreement with the Chern–Weil homomorphism, which shows that for an invariant functional λ and two curvatures Ω_1 and Ω_2, the 2-forms $\lambda^*\Omega_1$ and $\lambda^*\Omega_2$ are cohomologous. Therefore, the difference between them is an exact form, as is the case with the canonical form in a cotangent bundle.

(2) The cotangent bundle $T^*\mathbb{F}$ has been identified with the adjoint orbit $\mathrm{Ad}(G)H_0$ (with $\mathbb{F} = \mathrm{Ad}(K)H_0$), where the fibers are the orbits of kN^+k^{-1} with $k \in K$. The symmetric (dual) construction can be made with N^- instead of N^+. For this, the origin of the flag must be shifted, now viewed as the orbit $\mathrm{Ad}(K)(w_0H_0)$, where w_0 is the principal involution. This means that $\mathbb{F}$ is now seen as $\mathbb{F}^*$. In this new origin, the Iwasawa decomposition $G = KAN^-$ is considered, and the constructions work in the same way. This provides another diffeomorphism between $T^*\mathbb{F}$ and the adjoint orbit (in fact, $T^*\mathbb{F}^*$).

In terms of the immersion of $T^*\mathbb{F}$ in $\mathbb{F}\times\mathbb{F}^*$ (as an open orbit), in the first diffeomorphism, the fibers are the fibers $p_1^{-1}\{x\}$ with respect to the projection onto the first coordinate. In the second diffeomorphism, the fibers are $p_2^{-1}\{y\}$ with respect to the projection onto the second coordinate.

Changing the choice of diffeomorphism also changes the canonical symplectic form in the open orbit. For the first diffeomorphism, the linear functional $\lambda_{H_0,0}$ was used in the proposition above. For the second choice, one simply takes the functional λ_{0,w_0H_0}, when the canonical symplectic form is given by $\lambda^*_{0,w_0H_0}(\Omega - \Omega_{K\times K})$, with the Liouville form $\lambda^*_{0,w_0H_0}(\omega - \omega_{K\times K})$.

(3) The diffeomorphism between $T^*\mathbb{F}$ and the adjoint orbit $\mathrm{Ad}(G)H_0$ with $\mathbb{F} = \mathrm{Ad}(K)H_0$ is such that the cotangent space $T^*_x\mathbb{F}$ becomes the orbit $N^k\cdot x$, where

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$x = \mathrm{Ad}(k)H_0 \in \mathbb{F}$. In the diffeomorphism between $T^*\mathbb{F}$ and the open orbit of the diagonal action $G \cdot o = G \cdot (x_0, x_w)$, the base $\mathbb{F}$ becomes the K-orbit $K \cdot (x_0, x_w) = \{(kx_0, kx_w) : k \in K\}$, and a fiber $T^*_x\mathbb{F}$, $x = (kx_0, kx_w)$, is also the N^k-orbit $N^k \cdot x$. This orbit coincides with the fiber $p_1^{-1}\{kx_0\}$, where $p_1 : G \cdot o \to \mathbb{F}$ is the projection onto the first coordinate.

The fibers $T^*_x\mathbb{F}$ are Lagrangian submanifolds with respect to Γ, as happens in any cotangent bundle. This can also be seen in the expression for Γ in terms of the difference $\omega - \omega_{K\times K}$ of connections. To see this, take, for example, the fiber $N^+ \cdot o$ over the origin, which is the projection of $\pi(N^+)$ with respect to $\pi : G \to G/MA$. Given $X, Y \in \mathfrak{n}^+$, the curvature Ω computed on the left-invariant vector fields is $\Omega(X^l, Y^l)$, which is the $\mathfrak{m} \oplus \mathfrak{a}$ component of the bracket $[X, Y]$. Hence, the curvature vanishes since $[X, Y] \in \mathfrak{n}^-$. Therefore, the component $\lambda^*_{H_0,0}\Omega$ of the symplectic form vanishes on the fiber. The other component also vanishes on the fiber because $\Omega_{K\times K}$, computed on tangent vectors to the second coordinate, takes values in the second coordinate of $(\mathfrak{m} \oplus \mathfrak{a})^2$, where the functional $\lambda_{H_0,0}$ vanishes. The same argument works for any $x = k(x_0, x_w)$, by replacing N^+ with kN^+k^{-1}. This shows that the fibers $T^*_x\mathbb{F} = (\{x\} \times \mathbb{F}^*) \cap G \cdot o$ are Lagrangian submanifolds of the symplectic form $\lambda^*_{H_0,0}(\Omega - \Omega_{K\times K})$.

By switching the first and second coordinates, it is proven that the submanifolds $T^*_x\mathbb{F} = (\mathbb{F} \times \{y\}) \cap G \cdot o$ are Lagrangian for $\lambda^*_{0,w_0H_0}(\omega - \omega_{K\times K})$.

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