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A Combined Discontinuous Galerkin Method for Saltwater Intrusion Problem with Splitting Mixed Procedure

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Abstract. In this paper, a new combined method is presented to simulate saltwater intrusion problem. A splitting positive definite mixed element method is used to solve the water head equation, and a symmetric discontinuous Galerkin (DG) finite element method is used to solve the concentration equation. The introduction of these two numerical methods not only makes the coefficient matrixes symmetric positive definite, but also does well with the discontinuous problem. The convergence of this method is considered and the optimal L^2-norm error estimate is also derived.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Splitting mixed system, discontinuous Galerkin method, saltwater intrusion problem, convergence analysis.

1 Introduction

In recent years, saltwater intrusion has occurred in many countries and regions all over the world, and caused great damages to industrial and agricultural productions. In order to prevent and control this problem, more and more work on the problem has been done in past decades. This problem is often described by coupled density-dependent groundwater flow and advection-dispersion equations because of hydrodynamic dispersion and a wide transition zone [8, 12, 14, 23]. Numerical methods for solving the saltwater intrusion problem include the method of characteristics (MOC) [18, 19, 30], the finite element method (FEM) [22, 26] and the finite difference method (FDM) [11, 29], and so on.

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However, using the traditional finite element or finite difference method to solve the water head equation, we can not directly obtain the approximate flux which appears in the concentration equation. And differentiating the water head function to obtain the flux will result in extra error and reduce the accuracy. To obtain more accurate approximation of the flux function, the classical mixed finite element method [15] and the mixed hybrid finite element method (MHFE) [2,6] were introduced. But these techniques result in some saddle point problems whose numerical solutions have been quite difficult because of losing positive definite properties. In [28, 32–35], the authors of this paper proposed a class of splitting mixed finite element methods, in which the mixed system is symmetric positive definite and the flux equation is separated from the original equation.

Moreover, the concentration equation is parabolic and normally convection-dominated. The standard Galerkin method applied to the convection-dominated problems does not work well, and produces excessive numerical diffusion or nonphysical oscillation. A variety of numerical techniques have been introduced to obtain better approximations, such as higher-order Godunov method [4], streamline diffusion method [13], least-squares method [27], and the characteristic finite element methods [31,38,39]. However, these methods mentioned above are all continuous and can not deal with the discontinuous case. Discontinuous Galerkin finite element (DGFE) methods were introduced independently in [5,10,20]. Since then, they have become more and more powerful tools to deal with many types of discontinuous problems [3,21,24,25]. A combined classical mixed element scheme with DGFE method for incompressible miscible displacement problems was presented in [24, 25]. The similar method was extended in [9] to solve the compressible miscible displacement problem. The DGFE techniques have been applied by the authors of this paper [16,17,36,37], to nonlinear partial differential equations.

The main purpose of this paper is to construct a combined numerical method for solving saltwater intrusion problem: the splitting mixed element method is used to solve the water head equation and the symmetric interior penalty discontinuous Galerkin method is used to solve the concentration equation. The application of the splitting mixed element method results in a symmetric positive definite coefficient matrix of the mixed element system and separating the flux equation from the water head equation so that one can obtain an approximate solution of the flux function fast and independently by using various effective algorithms. Meantime, the discontinuous Galerkin method does well in handling the discontinuous areas since the trial and test functions across the finite element interfaces do not explicitly impose continuity constraints. In order to derive the optimal error estimate in L^2-norm, the classical mixed finite element spaces, which is not necessary in practice, will be used and a special projection of the concentration c, which imposes much additional condition to the penalty, is also used. In this paper, the restriction on the penalty is not our main work since it had been considered.

In order to illustrate our method, the following mathematical model of saltwater intrusion problem is considered: a coupled system composed of the water head equation and the concentration (of Cl^-) equation

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$$S_s\frac{\partial H}{\partial t}-\nabla\cdot(\widetilde{\kappa}(\nabla H-\eta c\mathbf{e}_3))=-\phi\eta\frac{\partial c}{\partial t}+\frac{\rho}{\rho_0}q, \tag{1.1a}$$

$$\phi\frac{\rho_0}{\rho}\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(\phi D\nabla c)=(\widetilde{c}-c)q, \tag{1.1b}$$

$$x\in\Omega,\qquad 0<t\leq T, \tag{1.1c}$$

with the initial-boundary conditions:

$$\mathbf{u}\cdot\boldsymbol{n}=0,\qquad D\nabla c\cdot\boldsymbol{n}=0,\qquad \text{on }\partial\Omega, \tag{1.2a}$$

$$H(x,0)=H^0(x),\qquad c(x,0)=c^0(x),\qquad x\in\Omega, \tag{1.2b}$$

where Ω is a convex bounded domain in R^3 with the boundary $\partial\Omega$, S_s is the specific retention, $H=\frac{p}{\rho_0 g}-z$ is water head function, p stand for pressure, ρ_0 represents the density of reference water (fresh water), g is gravitational acceleration, z is the height of water containing layer; ρ is density dependent only on the concentration of salt c, Hugakorn's linearization $\rho=\rho_0(1+\frac{c}{c_s})$ is adopted, c_s is the concentration corresponding to the maximum density, and ε is the density difference ratio $\varepsilon=\frac{\rho_s-\rho_0}{\rho_0}$. $\widetilde{\kappa}=\frac{\rho g}{\mu}\kappa$, μ is viscosity of the fluid,

$$\kappa=\begin{bmatrix}\kappa_1 & 0 & 0\\ 0 & \kappa_2 & 0\\ 0 & 0 & \kappa_3\end{bmatrix}$$

is the permeability; $\eta=\frac{\varepsilon}{c_s}$ is the density coupling coefficient; $\mathbf{e}_3=(0,0,1)^T$; ϕ is the porosity; and q is the source or sink term; c stands for the concentration of Cl^-, $\widetilde{c}$ is the salt concentration near the source well,

$$D=\begin{bmatrix}D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{bmatrix}$$

is the diffusion matrix, $\mathbf{u}=-\frac{\rho}{\rho_0}\kappa(\nabla H-\eta c\mathbf{e}_3)$ is Darcy velocity; $\boldsymbol{n}$ is the unit vector outer normal to $\partial\Omega$.

For convenience of analysis, we usually give the following assumptions

$$0<\phi_*\leq\phi\leq\phi^*,\quad 0<D_*\leq D\leq D^*,\quad 0<S_*\leq S_s\leq S^*, \tag{1.3a}$$

$$0<\alpha_*\leq\alpha\leq\alpha^*,\quad 0<\beta_*\leq\beta\leq\beta^*,\quad 0<a_*\leq a\leq a^*,\quad b_*\leq b(c)\leq\phi^*=b^*, \tag{1.3b}$$

$$\left|\frac{\partial\alpha(c)}{\partial c}\right|+\left|\frac{\partial\beta(c)}{\partial c}\right|+\left|\frac{\partial a(c)}{\partial c}\right|+\left|\frac{\partial^2 a(c)}{\partial c^2}\right|+\left|\frac{\partial b(c)}{\partial c}\right|\leq K^*. \tag{1.3c}$$

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We also assume the regularities of the solution of (1.1)-(1.2) as follows:

$$c\in L^\infty(H^{k+1})\cap L^2(W^1_\infty),\quad \frac{\partial c}{\partial t}\in L^2(H^{k+1})\cap L^\infty(L^\infty), \tag{1.4a}$$

$$\frac{\partial^2 c}{\partial t^2}\in L^2(L^2),\quad H\in L^\infty(H^{l+1})\cap H^2(L^2), \tag{1.4b}$$

$$\sigma\in L^\infty(H^{r+1}\cap W^1_\infty),\quad \frac{\partial \sigma}{\partial t}\in L^2(H^{r+1})\cap L^\infty(L^\infty),\quad \frac{\partial^2 \sigma}{\partial t^2}\in L^2(L^2). \tag{1.4c}$$

Throughout this paper, usual definitions, notations, and norms of Sobolev spaces as in [1] are used. K and ε indicate a generic constant and a small positive constant independent of mesh parameters, which may be different at their occurrences. Introduce the spaces $H(\mathrm{div};\Omega)=\{\mathbf{v}\in[L^2(\Omega)]^3;\ \nabla\cdot\mathbf{v}\in L^2(\Omega)\}$, $\mathcal{V}=\{\mathbf{v}\in H(\mathrm{div};\Omega);\ \mathbf{v}\cdot\nu=0 \text{ on } \partial\Omega\}$ and $\mathcal{W}=L^2(\Omega)$.

2 A combined discontinuous Galerkin method

In this section we give a splitting mixed element scheme to solve the water head equation and a discontinuous Galerkin scheme for the concentration equation. We begin with the partitions of Ω and a few notations.

Define a quasi-uniform family $\{\mathcal{T}_{h_c}\}$ of Ω with $\mathcal{T}_{h_c}=\{T_1,T_2,\cdots,T_{N_h}\}$, which will be used for the concentration equation. Let h_T and h_e denote the diameter of tetrahedron T and the measure of face e, respectively. Set $h_c=\max_{T\in\mathcal{T}_{h_c}}h_T$. We also define another two families $\mathcal{T}_{h_\sigma}$ and $\mathcal{T}_{h_p}$ of quasi-regular finite element partitions of the domain Ω, such that the elements in the partitions have the diameters bounded by h_σ, and h_p, respectively. These two partitions will be used for the flux equation and the water head equation.

2.1 Splitting mixed weak form for water head equation

The water head equation is a parabolic type equation, and we deal it with a splitting positive definite mixed finite element method. Define the flux σ as follows:

$$\sigma=-\widetilde{\kappa}(\nabla H-\eta c\mathbf{e}_3)=\frac{g\rho^2}{\rho_0\mu}\mathbf{u}.$$

So we have $\mathbf{u}=a(c)\sigma$, $a(c)=\frac{\rho_0\mu}{g\rho^2}$. A mixed weak form of the system (1.1a) is given by:

$$\left(\frac{\partial H}{\partial t},w\right)+(B\nabla\cdot\sigma,w)=(Bq\beta(c),w)-\left(B\phi\eta\frac{\partial c}{\partial t},w\right),\qquad \forall w\in\mathcal{W}, \tag{2.1a}$$

$$(\alpha(c)\sigma,\mathbf{v})-(H,\nabla\cdot\mathbf{v})=-(\eta c\mathbf{e}_3,\mathbf{v}),\qquad \forall\mathbf{v}\in\mathcal{V}, \tag{2.1b}$$

where $\alpha(c)=1/\widetilde{\kappa}$, $B=1/S_s$ and $\beta(c)=\rho(c)/\rho_0$.

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From (2.1b) we derive

$$\Big(\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}),\mathbf{v}\Big)-\Big(\frac{\partial H}{\partial t},\nabla\cdot\mathbf{v}\Big)=-\Big(\eta\frac{\partial c}{\partial t}\mathbf{e}_3,\mathbf{v}\Big). \tag{2.2}$$

Taking $w=\nabla\cdot\mathbf{v}$ in (2.1a) and substituting it into (2.2), we get the mixed system

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}),\mathbf{v}\Big)+(B\nabla\cdot\boldsymbol{\sigma},\nabla\cdot\mathbf{v})\\&=(B\beta q,\nabla\cdot\mathbf{v})-\Big(B\phi\eta\frac{\partial c}{\partial t},\nabla\cdot\mathbf{v}\Big)-\Big(\eta\frac{\partial c}{\partial t}\mathbf{e}_3,\mathbf{v}\Big),\qquad\forall\mathbf{v}\in\mathcal{V},\end{aligned}\tag{2.3a}$$

$$\Big(\frac{\partial H}{\partial t},w\Big)=(B[q\beta(c)-\nabla\cdot\boldsymbol{\sigma}],w)-\Big(B\phi\eta\frac{\partial c}{\partial t},w\Big),\qquad\forall w\in\mathcal{W}.\tag{2.3b}$$

Following Theorem 2.1 in [28], we easily know that the mixed formulation has a unique solution. And we notice that the flux equation is separated from the water head equation and then the water head function H, if required, can be obtained from (2.3b) straightly.

2.2 Discontinuous weak form for concentration equation

We present the weak formula for discontinuous Galerkin method for the concentration equation. Since discontinuous finite element is used, we introduce the piecewise Sobolev space given by

$$\mathcal{H}^s(\mathcal{T}_{h_c})=\big\{z\in L^2(\Omega):\ z|_T\in\mathcal{H}^s(T),\ \forall T\in\mathcal{T}_{h_c}\big\}$$

and it is equipped with the Sobolev norm

$$|||z|||_s=\Big(\sum_{T\in\mathcal{T}_{h_c}}\|z\|^2_{\mathcal{H}^s(T)}\Big)^{1/2}.$$

Denote Y_0 the set of the common faces of $T_i\cap T_j$ for all neighboring $T_i,T_j\in\mathcal{T}_{h_c}$. Correspondingly, let Y_∂ denote the set of the common faces of $T\cap\partial\Omega$ for all $T\in\mathcal{T}_{h_c}$ which has at least three points on the $\partial\Omega$. Set $\mathrm{Y}=\mathrm{Y}_0\cup\mathrm{Y}_\partial$. For $z\in\mathcal{H}^l(\mathcal{T}_{h_c})$, we define the mean value of z on γ and the jump of z across γ, respectively. For each $e\in\mathrm{Y}$, $[z],\{z\}\in L^2(e)$. If $e\in\mathrm{Y}_0$, then $e=T_i\cap T_j$ with $\boldsymbol{n}_e$ exterior to T_i for some pair of elements (T_i,T_j), set

$$\{z\}=\frac{1}{2}[(z|_{T_i})|_e+(z|_{T_j})|_e],\quad [z]=(z|_{T_i})|_e-(z|_{T_j})|_e.$$

If $e\in\mathrm{Y}_\partial$, then $[z]=\{z\}=z|_e$.

Now we define

$$\begin{aligned}A(\boldsymbol{\sigma};c,z)=&\sum_{T\in\mathcal{T}_{h_c}}\int_T\phi D\nabla c\cdot\nabla z+\sum_{T\in\mathcal{T}_h}\int_T a(c)(\boldsymbol{\sigma}\cdot\nabla c)z\\&-\sum_{e\in\mathrm{Y}_0}\int_e\{\phi D\nabla c\cdot\boldsymbol{n}_e\}[z]-\sum_{e\in\mathrm{Y}_0}\int_e\{\phi D\nabla z\cdot\boldsymbol{n}_e\}[c]+J_T^\gamma(c,z).\end{aligned}\tag{2.4}$$

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The function $J_T^{\gamma}(c,z)$ is introduced as the penalty term defined as follows:

$$J_T^{\gamma}(c,z)=\sum_{e\in\Upsilon_0}\int_e \gamma h_e^{-1}[c][z], \tag{2.5}$$

where γ is a differentiable function of t on each face e.

Since $[c]=0$, with the boundary condition we can get the weak form as follows: for $c\in\mathcal{H}^1(\mathcal{T}_{h_c})$,

$$\Big(b(c)\frac{\partial c}{\partial t},z\Big)+A(\sigma;c,z)=(q(\tilde{c}-c),z),\quad \forall z\in\mathcal{H}^1(\mathcal{T}_{h_c}), \tag{2.6}$$

where $b(c)=\phi\rho_o/\rho$.

2.3 The combined approximation procedure

In this section, we will present a combined splitting mixed finite element and discontinuous Galerkin method for saltwater intrusion problem, in which we use the splitting positive definite mixed element method to deal with water head equation and use the discontinuous Galerkin method to deal with concentration equation.

Let $\mathcal{V}_h\subset\mathcal{V}$ and $\mathcal{W}_h\subset\mathcal{W}$ be the rth and lth order finite element spaces defined on the partitions $\mathcal{T}_{h_\sigma}$ and $\mathcal{T}_{h_p}$, respectively. One is used to approximate the flux σ and the other is used to approximate the water head fucntion H. Next we introduce the discrete approximate space for the concentration c denoted by $\mathcal{M}_h$ as follows:

$$\mathcal{M}_h=\{z\in\mathcal{H}^k(\mathcal{T}_h):\ z|_T\in\mathcal{P}_k(T),\ T\in\mathcal{T}_{h_c}\},$$

where $\mathcal{P}_k(T)$ denotes the set of restrictions to T of polynomials of degree at most k for each $T\in\mathcal{T}_{h_c}$.

Combined the splitting positive definite mixed element method and discontinuous Galerkin method, a new numerical method can be established:

Scheme 2.1. Give an initial approximation $(\sigma_h^0,c_h^0,H_h^0)\in\mathcal{V}_h\times\mathcal{M}_h\times\mathcal{W}_h$ such that

$$(\alpha(c_h^0)\sigma_h^0,\mathbf{v}_h)=-(\nabla H^0-\eta c^0\mathbf{e}_0,\mathbf{v}_h),\qquad \forall\mathbf{v}_h\in\mathcal{V}_h, \tag{2.7a}$$
$$(c_h^0,z_h)=(c^0,z_h),\qquad \forall z_h\in\mathcal{M}_h, \tag{2.7b}$$
$$(H_h^0,w_h)=(H^0,w_h),\qquad \forall w_h\in\mathcal{W}_h. \tag{2.7c}$$

For $t>0$, seek $(\sigma_h(t),(c_h(t))\in\mathcal{V}_h\times\mathcal{M}_h$ such that

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\alpha(c_h)\sigma_h),\mathbf{v}_h\Big)+(B\nabla\cdot\sigma_h,\nabla\cdot\mathbf{v}_h)\\&=(Bq\beta(c_h),\nabla\cdot\mathbf{v}_h)-\Big(B\phi\eta\frac{\partial c_h}{\partial t},\nabla\cdot\mathbf{v}_h\Big)-\Big(\eta\frac{\partial c_h}{\partial t}\mathbf{e}_3,\mathbf{v}_h\Big),\qquad \forall\mathbf{v}_h\in\mathcal{V}_h,\end{aligned} \tag{2.8a}$$
$$\Big(b(c_h)\frac{\partial c_h}{\partial t},z_h\Big)+A(\sigma_h;c_h,z_h)=(q(\widetilde{c}-c_h),z_h),\qquad \forall z_h\in\mathcal{M}_h. \tag{2.8b}$$

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Seek $H_h(t)\in\mathcal{W}_h$ such that

$$\left(\frac{\partial H_h}{\partial t},w_h\right)=(B[q\beta(c_h)-\nabla\cdot\boldsymbol{\sigma}_h],w_h)-\left(B\phi\eta\frac{\partial c_h}{\partial t},w_h\right),\quad\forall w_h\in\mathcal{W}_h. \tag{2.9}$$

The nonlinear system (2.7)-(2.9) has a unique solution. In fact, we can change this nonlinear system into an initial value problem of a system of nonlinear first-order ordinary differential equations by the basis functions, as in [28]. By virtue of the theory of ordinary differential equations, we can easily get the existence and uniqueness of the solution of the nonlinear system.

3 Convergence analysis and error estimate

We assume that finite element spaces $\mathcal{V}_h$, $\mathcal{W}_h$ and $\mathcal{M}_h$ have the inverse property (see [7]) and approximate properties that there exist some integers $r,r_1,k>0$ and $l\geq 0$, such that, for $1\leq q\leq\infty$,

$$\inf_{\mathbf{v}_h\in\mathcal{V}_h}\|\mathbf{v}-\mathbf{v}_h\|_{[q}\leq K_1h_\sigma^{r+1}\|\mathbf{v}\|_{W^{r+1,q}},\qquad\forall\mathbf{v}\in H(\mathrm{div};\Omega)\cap W^{r+1,q}(\Omega); \tag{3.1a}$$

$$\inf_{\mathbf{v}_h\in\mathcal{V}_h}\|\nabla\cdot(\mathbf{v}-\mathbf{v}_h)\|_{L^q}\leq K_1h_\sigma^{r_1}\|\nabla\cdot\mathbf{v}\|_{W^{r_1,q}},\qquad\forall\mathbf{v}\in H(\mathrm{div};\Omega)\cap W^{r_1+1,q}(\Omega); \tag{3.1b}$$

$$\inf_{w_h\in\mathcal{W}_h}\|w-w_h\|_{L^q}\leq K_1h_p^{l+1}\|w\|_{W^{l+1,q}},\qquad\forall w\in L^2(\Omega)\cap W^{l+1,q}(\Omega), \tag{3.1c}$$

where $r_1=r$ in cases of BDDM, BDM, and BDFM elements, or $r_1=r+1$ in cases of RT and Nedelec elements.

It is well known that, in any one of the classical mixed finite element spaces, there exists an operator Π_h from $\mathcal{V}$ onto $\mathcal{V}_h$, see [7], such that, for any $1\leq q\leq+\infty$,

$$(\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v}),\nabla\cdot\mathbf{v}_h)=0,\quad\forall\mathbf{v}_h\in\mathcal{V}_h; \tag{3.2a}$$

$$\|\mathbf{v}-\Pi_h\mathbf{v}\|_{L^q}\leq Kh^{r+1}\|\mathbf{v}\|_{W^{r+1,q}}; \tag{3.2b}$$

$$\|\nabla\cdot(\mathbf{v}-\Pi_h\,\mathbf{v})\|_{L^q}\leq Kh^{r_1}\|\nabla\cdot\mathbf{v}\|_{W^{r_1,q}}. \tag{3.2c}$$

In order to derive the optimal order in L^2-norm, we introduce another projection operator $P_h:\mathcal{M}\to\mathcal{M}_h$ such that

$$A(\boldsymbol{\sigma};c-P_hc,z)+\lambda(c-P_hc,z)=0,\quad\forall z\in\mathcal{M}_h, \tag{3.3}$$

where λ is chosen large enough to insure the coercivity. Following the argument of Theorem 4.5 in [3], we can get the following estimates:

$$|||c-P_hc|||\leq Kh^j|||c|||_j,\qquad 2\leq j\leq k+1, \tag{3.4a}$$

$$\left|\left|\left|\frac{\partial}{\partial t}(c-P_hc)\right|\right|\right|\leq Kh^j\left(|||c|||_j+\left|\left|\left|\frac{\partial c}{\partial t}\right|\right|\right|_j\right),\qquad 2\leq j\leq k+1. \tag{3.4b}$$

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Meanwhile, we also introduce the L^2 projection operator Q_h from $L^2(\Omega)$ onto $\mathcal{W}_h$ such that

$$(H-Q_hH,w_h)=0,\quad \forall w_h\in\mathcal{W}_h. \tag{3.5}$$

It is well known that the a priori error estimate

$$\|H-Q_hH\|_{L^2}+\left\|\frac{\partial}{\partial t}(H-Q_hH)\right\|_{L^2}\leq Kh_p^{l+1}\|H\|_{H^{l+1}},\quad \forall w\in H^{l+1}(\Omega), \tag{3.6}$$

holds.

In order to drive the error estimate, the following lemmas will be used.

Lemma 3.1 (Trace inequalities [3,9]). *The trace inequalities hold for all $v\in H^1(T)$,*

$$\begin{aligned}&\|v\|_{0,e}^2\leq K(h_e^{-1}\|v\|_{0,T}^2+h_e\|v\|_{1,T}^2),\\&\|\nabla v\cdot\mathbf{n}_e\|_{0,e}^2\leq K(h_e^{-1}\|\nabla v\|_{1,T}^2+\|\nabla v\|_{0,T}^2\|\nabla^2 v\|_{0,T}^2).\end{aligned}$$

Lemma 3.2 (see [28]). *Assume that the finite element space $\mathcal{V}_h$ is any one of the classical mixed finite element spaces defined in [7]. The super-approximation, which for any function,*

$$\begin{aligned}&(\varphi\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v}),\nabla\cdot\mathbf{v}_h)\\&\leq Kh_\sigma\|\nabla\cdot\mathbf{v}_h\|_{L^2}\min(\|\varphi\|_{H^1}\|\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})\|_{L^\infty},\min(\|\varphi\|_{W^{1,\infty}},h_\sigma^{-\frac{3}{2}})\|\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})\|_{L^2})\end{aligned} \tag{3.7}$$

holds, for each function $\varphi\in W^{1,\infty}$, $\mathbf{v}\in\mathcal{V}$ and $\mathbf{v}_h\in\mathcal{V}_h$.

For Scheme 2.1, we have the following main result:

Theorem 3.1. *Under the assumptions* (1.3)*, if the solution of system* (1.1)-(1.2) *has the regular properties* (1.4)*, then there hold the priori error estimates*

$$|||c-c_h|||_{L^2(0,t;L^2(\Omega))}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(0,t;[L^2(\Omega)]^3)}\leq K\{h_c^{k+1}+h_\sigma^{r+1}\}, \tag{3.8a}$$

$$\|H-H_h\|_{L^2(0,t;L^2(\Omega))}\leq K\{h_c^{k+1}+h_\sigma^{r_1}+h_p^{l+1}\}. \tag{3.8b}$$

The proof of Theorem 3.1 will be divided into several lemmas.

Set $\xi_\sigma=\boldsymbol{\sigma}_h-\Pi_h\boldsymbol{\sigma}$, $\zeta_\sigma=\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}$, $\xi_c=c_h-P_hc$, $\zeta_c=c-P_hc$, $\xi_H=H_h-Q_hH$ and $\zeta_H=H-Q_hH$. Following (2.3), (2.6), (2.7)-(2.9), we can obtain the residual equations as follows:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(\alpha(c_h)\xi_\sigma),\mathbf{v}_h\right)+(B\nabla\cdot\xi_\sigma,\nabla\cdot\mathbf{v}_h)\\&=\left(\frac{\partial}{\partial t}(\alpha(c_h)\zeta_\sigma),\mathbf{v}_h\right)+\left(\frac{\partial}{\partial t}((\alpha(c)-\alpha(c_h))\boldsymbol{\sigma}),\mathbf{v}_h\right)+(B\nabla\cdot\zeta_\sigma,\nabla\cdot\mathbf{v}_h)\\&\quad+(Bq(\beta(c_h)-\beta(c)),\nabla\cdot\mathbf{v}_h)-\left(B\phi\eta\frac{\partial}{\partial t}(c_h-c),\nabla\cdot\mathbf{v}_h\right)-\left(\eta\mathbf{e}_3\frac{\partial}{\partial t}(c_h-c),\mathbf{v}_h\right),\end{aligned} \tag{3.9}$$

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and

$$\begin{aligned}&\Big(b(c_h)\frac{\partial\xi_c}{\partial t},z_h\Big)+\sum_{T\in\mathcal{T}_h}\int_T\phi D\nabla\xi_c\cdot\nabla z_h+J_T^\gamma(\xi_c,z_h)+(q\xi_c,z_h)\\&=\Big(b(c_h)\frac{\partial\zeta_c}{\partial t},z_h\Big)+\Big((b(c)-b(c_h))\frac{\partial c}{\partial t},z_h\Big)+((\lambda+q)\zeta_c,z_h)+\sum_{e\in\Upsilon_0}\int_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}[z_h]\\&\quad+\sum_{e\in\Upsilon_0}\int_e\{\phi D\nabla z_h\cdot\boldsymbol{n}_e\}[\xi_c]+((a(c)-a(c_h))\boldsymbol{\sigma}\cdot\nabla P_hc,z_h)\\&\quad+(a(c_h)(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\nabla P_hc,z_h)-(a(c_h)\boldsymbol{\sigma}_h\cdot\nabla\xi_c,z_h)\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}\Big(\frac{\partial\xi_H}{\partial t},\mathbf{v}_h\Big)&=\Big(\frac{\partial\zeta_H}{\partial t},\mathbf{v}_h\Big)+(Bq(\beta(c_h)-\beta(c)),w_h)\\&\quad+(B\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),w_h)-\Big(B\phi\eta\frac{\partial}{\partial t}(c_h-c),w_h\Big).\end{aligned}\tag{3.11}$$

Lemma 3.3. *Assume that $a(c_h)$, $a'(c_h)$, $b(c_h)$ and $b'(c_h)$ are bounded. For $t\geq0$, the estimate*

$$\begin{aligned}&\Big(b(c_h)\frac{\partial\xi_c}{\partial t},z_h\Big)+\sum_{T\in\mathcal{T}_h}\int_T\phi D\nabla\xi_c\cdot\nabla z_h+J_T^\gamma(\xi_c,z_h)+(q\xi_c,z_h)\\&\leq K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+|||z|||^2\\&\quad+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla z_h\cdot\boldsymbol{n}_e\}^2\Big\}\\&\quad+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(z_h,z_h)+J_T^\gamma(\xi_c,\xi_c)\big]\end{aligned}\tag{3.12}$$

holds, for any $z_h\in\mathcal{M}_h$.

Proof. We estimate terms in (3.10). It is easily seen that

$$\begin{aligned}&\Big|\Big(b(c_h)\frac{\partial\zeta_c}{\partial t},z_h\Big)\Big|+\Big|\Big((b(c)-b(c_h))\frac{\partial c}{\partial t},z_h\Big)\Big|+|((\lambda+q)\zeta_c,z_h)|\\&\quad+|((a(c)-a(c_h))\boldsymbol{\sigma}\cdot\nabla P_hc,z_h)|+|(a(c_h)(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\nabla P_hc,z_h)|\\&\leq K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+|||z|||^2\Big\}.\end{aligned}\tag{3.13}$$

In order to estimate the last term on the right hand side in (3.10), we make an induction hypothesis: There exists a positive constant K_1 such that

$$\|\xi_\sigma\|_{L^\infty}\leq K_1.\tag{3.14}$$

So, we have

$$\begin{aligned}|(a(c_h)\boldsymbol{\sigma}_h\cdot\nabla\xi_c,z_h)|&\leq|(a(c_h)\xi_\sigma\cdot\nabla\xi_c,z_h)|+|(a(c_h)\Pi_h\boldsymbol{\sigma}\cdot\nabla\xi_c,z_h)|\\&\leq K|||z|||^2+\varepsilon|||\nabla\xi_c|||^2.\end{aligned}\tag{3.15}$$

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Now we estimate the last two terms on the the right hand side of (3.10). we can easily get

$$\begin{aligned}&\Big|\sum_{e\in\Upsilon_0}\int_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}[z_h]\Big|+\Big|\sum_{e\in\Upsilon_0}\int_e\{\phi D\nabla z_h\cdot\boldsymbol{n}_e\}[\xi_c]\Big|\\ \le& K\Big\{\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla z_h\cdot\boldsymbol{n}_e\}^2\Big\}\\ &+\varepsilon\big[J_T^\gamma(z_h,z_h)+J_T^\gamma(\xi_c,\xi_c)\big].\end{aligned}\tag{3.16}$$

Substituting these estimates into (3.10), we get the inequality (3.12). □

Lemma 3.4. *Under the condition of Lemma 3.3, there exists a following estimate*

$$\begin{aligned}&|||\xi_c(t)|||^2+\int_0^t\{|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\}d\tau\\ \le& K\int_0^t\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2\Big\}d\tau.\end{aligned}\tag{3.17}$$

Proof. Taking $z_h=\xi_c\in\mathcal{M}_h$ in (3.12), we have the estimate

$$\begin{aligned}&\Big(b(c_h)\frac{\partial\xi_c}{\partial t},\xi_c\Big)+\sum_{T\in\mathcal{T}_h}\int_T\phi D\nabla\xi_c\cdot\nabla\xi_c+J_T^\gamma(\xi_c,\xi_c)+(q\xi_c,\xi_c)\\ \le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2\\ &+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2\Big\}+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big]\end{aligned}\tag{3.18}$$

and note that

$$\Big(b(c_h)\frac{\partial\xi_c}{\partial t},\xi_c\Big)=\frac{1}{2}\frac{d}{dt}(b(c_h)\xi_c,\xi_c)-\frac{1}{2}\Big(\frac{\partial b(c_h)}{\partial c_h}\frac{\partial\xi_c}{\partial t}\xi_c,\xi_c\Big)-\frac{1}{2}\Big(\frac{\partial b(c_h)}{\partial c_h}\frac{\partial P_hc}{\partial t}\xi_c,\xi_c\Big),\tag{3.19}$$

we make another induction hypothesis: There exists a positive constant K_2 such that

$$\|\xi_c\|_{L^\infty}\le K_2.\tag{3.20}$$

And introduce a weighted L^2-projection operator $P_M:L^2(\Omega)\to\mathcal{M}_h$ such that

$$(b(c_h)(z-P_Mz),z_h)=0,\quad\forall z\in L^2(\Omega),\quad z_h\in\mathcal{M}_h.\tag{3.21}$$

Obviously, we have

$$\|z-P_Mz\|_{L^2(\Omega)}\le K\inf_{z_h\in\mathcal{M}_h}\|z-z_h\|_{L^2(\Omega)}.\tag{3.22}$$

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Then, using Lemma 3.3 we note that

$$\begin{aligned}&\Big(\frac{\partial b(c_h)}{\partial c_h}\frac{\partial \xi_c}{\partial t}\xi_c,\xi_c\Big)=\Big(b(c_h)\frac{\partial \xi_c}{\partial t},P_M\Big(\frac{\partial b(c_h)}{\partial c_h}\frac{\xi_c^2}{b(c_h)}\Big)\Big)\\ \le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial \zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\sum_{e\in \Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2\Big\}\\ &+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big].\end{aligned}\tag{3.23}$$

Hence, we can get

$$\begin{aligned}\Big(b(c_h)\frac{\partial \xi_c}{\partial t},\xi_c\Big)\ge&\frac{1}{2}\frac{d}{dt}(b(c_h)\xi_c,\xi_c)-K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial \zeta_c}{\partial t}\Big|\Big|\Big|^2\\ &+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\sum_{e\in \Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2\Big\}\\ &-\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big].\end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.18) and integrating (3.18) from 0 to t, for sufficiently small h_c and ε, we can obtain the estimate (3.17). □

Lemma 3.5. *Assume that $\alpha(c_h)$, $\alpha'(c_h)$, $\beta(c_h)$, $\beta'(c_h)$ and $\beta''(c_h)$ are bounded, there exists a following estimate*

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\alpha(c_h)\xi_\sigma),\boldsymbol{v}_h\Big)+(B\nabla\cdot\xi_\sigma,\nabla\cdot\boldsymbol{v}_h)\\ \le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial \zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\|\boldsymbol{v}_h\|^2\\ &+h_\sigma^2\|\nabla\cdot\zeta_\sigma\|^2+\sum_{e\in \Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\gamma^{-1}\|\nabla\cdot\boldsymbol{v}_h\|^2\Big\}\\ &+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)+\|\nabla\cdot\boldsymbol{v}_h\|^2\big].\end{aligned}\tag{3.25}$$

Proof. It is easily seen that

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\alpha(c_h)\xi_\sigma),\mathbf{v}_h\Big)+(B\nabla\cdot\xi_\sigma,\nabla\cdot\mathbf{v}_h)\\ =&\Big(\alpha'(c_h)\frac{\partial P_hc}{\partial t}\zeta_\sigma,\mathbf{v}_h\Big)+\Big(\alpha(c_h)\frac{\partial \zeta_\sigma}{\partial t},\mathbf{v}_h\Big)-\Big(\alpha'\frac{\partial \zeta_c}{\partial t}\boldsymbol{\sigma},\mathbf{v}_h\Big)\\ &+\Big(\alpha''\frac{\partial P_hc}{\partial t}\boldsymbol{\sigma}(c-c_h),\mathbf{v}_h\Big)+\Big((\alpha(c)-\alpha(c_h))\frac{\partial \boldsymbol{\sigma}}{\partial t},\mathbf{v}_h\Big)\\ &-\Big(B\phi\eta\frac{\partial \zeta_c}{\partial t},\nabla\cdot\mathbf{v}_h\Big)-\Big(\eta\mathbf{e}_3\frac{\partial \zeta_c}{\partial t}(c_h-c),\mathbf{v}_h\Big)+(Bq(\beta(c_h)-\beta(c)),\nabla\cdot\mathbf{v}_h)\\ &+(B\nabla\cdot\zeta_\sigma,\nabla\cdot\mathbf{v}_h)+\Big(\alpha'(c_h)\frac{\partial \xi_c}{\partial t}\zeta_\sigma,\mathbf{v}_h\Big)-\Big(\alpha'\frac{\partial \xi_c}{\partial t}\boldsymbol{\sigma},\mathbf{v}_h\Big)\\ &+\Big(\alpha''\frac{\partial \xi_c}{\partial t}\boldsymbol{\sigma}(c-c_h),\mathbf{v}_h\Big)-\Big(\eta\mathbf{e}_3\frac{\partial \xi_c}{\partial t}(c_h-c),\mathbf{v}_h\Big)-\Big(B\phi\eta\frac{\partial \xi_c}{\partial t},\nabla\cdot\mathbf{v}_h\Big)\\ =&I_1+I_2+\cdots+I_{14}.\end{aligned}\tag{3.26}$$

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Using Schwarz inequality and the inequality $ab \le 4a^2/\varepsilon + \varepsilon b^2$, we can get the estimate

$$\begin{aligned}
&|I_1|+|I_2|\cdots+|I_8| \\
\le& K\Big\{|||\xi_c|||^2+|||\zeta_c|||^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+\|\zeta_\sigma\|^2+\Big\|\frac{\partial\zeta_\sigma}{\partial t}\Big\|^2+\|\boldsymbol{v}_h\|^2\Big\}+\varepsilon\|\nabla\cdot\boldsymbol{v}_h\|^2, \qquad (3.27)
\end{aligned}$$

where we have used the induction hypothesis (3.20) in estimating the term I_7.

Using Lemma 3.2, we can get

$$|I_9| \le Kh_\sigma^2\|\nabla\cdot\zeta_\sigma\|^2+\varepsilon\|\nabla\cdot\boldsymbol{v}_h\|^2. \qquad (3.28)$$

From Lemma 3.3, we have

$$\begin{aligned}
|I_{10}| =& \Big|\Big(b(c_h)\frac{\partial\xi_c}{\partial t}, P_M(\alpha'(c_h)\zeta_\sigma\cdot\mathbf{v}_h/b(c_h))\Big)\Big| \qquad (3.29)\\
\le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+|||P_M(\alpha'(c_h)\zeta_\sigma\cdot\mathbf{v}_h/b(c_h))|||^2 \\
&+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla P_M(\alpha'(c_h)\zeta_\sigma\cdot\mathbf{v}_h/b(c_h))\cdot\boldsymbol{n}_e\}^2\Big\} \\
&+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big] \\
\le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\|\mathbf{v}_h\|^2 \\
&+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\gamma^{-1}\|\nabla\cdot\mathbf{v}_h\|^2\Big\}+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big]. \qquad (3.30)
\end{aligned}$$

Using the similar technique, we can obtain the estimate

$$\begin{aligned}
&|I_{11}|+|I_{12}|+|I_{13}|+|I_{14}| \\
\le& K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\|\mathbf{v}_h\|^2 \\
&+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2+\gamma^{-1}\|\nabla\cdot\mathbf{v}_h\|^2\Big\} \\
&+\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)+\|\nabla\cdot\mathbf{v}_h\|^2\big]. \qquad (3.31)
\end{aligned}$$

Substituting (3.27)-(3.31) into (3.26), we can easily get the inequality (3.25). □

Lemma 3.6. *Under the conditions of Lemma 3.5, we have the estimate*

$$\begin{aligned}
&\|\xi_\sigma(t)\|^2+\int_0^t\|\nabla\cdot\xi_\sigma\|^2d\tau \\
\le& K\int_0^t\Big[|||\xi_c|||^2+\|\xi_\sigma\|^2+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+\Big\|\frac{\partial\zeta_\sigma}{\partial t}\Big\|^2+h_\sigma^2\|\nabla\cdot\zeta_\sigma\|^2 \\
&+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2\Big]d\tau+\varepsilon\int_0^t\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big]d\tau. \qquad (3.32)
\end{aligned}$$

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Proof. Choose $v_h=\xi_\sigma$ in (3.25), and note that

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\alpha(c_h)\xi_\sigma),\xi_\sigma\Big)=\frac{1}{2}\Big[\frac{d}{dt}(\alpha(c_h)\xi_\sigma,\xi_\sigma)+\Big(\frac{\partial\alpha(c_h)}{\partial t}\xi_\sigma,\xi_\sigma\Big)\Big]\\&=\frac{1}{2}\Big[\frac{d}{dt}(\alpha(c_h)\xi_\sigma,\xi_\sigma)+\Big(\frac{\partial\alpha(c_h)}{\partial c_h}\frac{\partial\xi_c}{\partial t}\xi_\sigma,\xi_\sigma\Big)+\Big(\frac{\partial\alpha(c_h)}{\partial c_h}\frac{\partial P_hc}{\partial t}\xi_\sigma,\xi_\sigma\Big)\Big].\end{aligned}\tag{3.33}$$

Using the same technique to (3.29), we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t}(\alpha(c_h)\xi_\sigma),\xi_\sigma\Big)\geq&\frac{1}{2}\frac{d}{dt}(\alpha(c_h)\xi_\sigma,\xi_\sigma)-K\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2\\&+|||\zeta_c|||^2+\|\zeta_\sigma\|^2+\sum_{e\in\Upsilon_0}\int_e\gamma^{-1}h_e\{\phi D\nabla\xi_c\cdot\boldsymbol{n}_e\}^2\\&+\gamma^{-1}\|\nabla\cdot\mathbf{v}_h\|^2\Big\}-\varepsilon\big[|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)\big].\end{aligned}\tag{3.34}$$

Substituting this estimate into (3.25) and integrating it from 0 to t, and choosing γ large enough and ε, h_c small enough, we can get the inequality (3.32). □

Now, we complete our proof of Theorem 3.1.

Proof. Under the inductive hypotheses (3.14) and (3.20), using Lemma 3.4 and 3.6, we can obtain

$$\begin{aligned}&|||\xi_c(t)|||^2+\|\xi_\sigma(t)\|^2+\int_0^t\{|||\nabla\xi_c|||^2+J_T^\gamma(\xi_c,\xi_c)+\|\nabla\cdot\xi_\sigma\|^2\}d\tau\\&\leq K\int_0^t\Big\{|||\xi_c|||^2+\|\xi_\sigma\|^2+|||\zeta_c|||^2+h_\sigma^2\|\nabla\cdot\zeta_\sigma\|^2+\|\zeta_\sigma\|^2+\Big|\Big|\Big|\frac{\partial\zeta_c}{\partial t}\Big|\Big|\Big|^2+\Big\|\frac{\partial\zeta_\sigma}{\partial t}\Big\|^2\Big\}d\tau.\end{aligned}\tag{3.35}$$

Applying Gronwall's Lemma, we get

$$|||\xi_c(t)|||^2+\|\xi_\sigma(t)\|^2+\int_0^t[|||\nabla\xi_c(\tau)|||^2+\|\nabla\cdot\xi_\sigma(\tau)\|^2]d\tau\leq K\{h_c^{2(k+1)}+h_\sigma^{2(r+1)}\}.\tag{3.36}$$

It is obvious that we need to check the inductive hypotheses (3.14) and (3.20) to complete our proof. In fact, for $r,k\geq1$, we know that

$$\begin{aligned}&\|\xi_\sigma(t)\|_{L^\infty}\leq Kh_\sigma^{-\frac{3}{2}}\|\xi_\sigma(t)\|_{L^2}\leq Kh_\sigma^{-\frac{3}{2}}h_\sigma^{r+1}\leq K_1,\\&\|\xi_c(t)\|_{L^\infty}\leq Kh_c^{-\frac{3}{2}}|||\xi_c(t)|||\leq Kh_c^{-\frac{3}{2}}h_c^{k+1}\leq K_2.\end{aligned}$$

Hence, the inductive hypotheses (3.14) and (3.20) hold.

Finally, we consider the bound of ξ_H. Taking $w_h=\xi_H$ in (3.11) and applying (3.36), we can obtain the following estimate

$$\|\xi_H(t)\|^2\leq K\{h_p^{l+1}+h_c^{k+1}+h_\sigma^{r_1}\}.\tag{3.37}$$

This ends the proof of Theorem 3.1. □

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