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Two-symbol Pisot substitutions have pure discrete spectrum

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Abstract. We show that any Pisot substitution dynamical system on two symbols has pure discrete spectrum. This follows from an analysis of a combinatorial 'balanced pair algorithm', in view of recent progress on the Strong Coincidence Conjecture made by Barge and Diamond.

1. *Introduction*

There is considerable interest in determining the spectral type of a substitution dynamical system. The classical account of the spectral theory of substitution dynamical systems is that of Queffelec [**17**]. A very thorough survey of more recent results (up to 2000) can be found in [**22**]. Our goal is to prove that certain substitution systems have *pure discrete spectrum* or, equivalently, that they are metrically isomorphic to a translation on a compact Abelian group. Additional motivation for studying this question comes from the physics of quasi-crystals. There, substitution sequences are used as models of atomic configurations and pure discrete spectrum corresponds to the configuration being pure point diffractive, see [**2, 6, 7**]. Substitution dynamical systems are also closely related to tiling dynamical systems [**19, 28**] and adic transformations [**29**].

We study *Pisot substitutions*, which means that the matrix of the substitution has one eigenvalue greater than 1 and all other eigenvalues are strictly between 0 and 1 in modulus. There is a conjecture (we have not been able to identify its origins but it certainly dates back at least to the 1980s) that every Pisot substitution dynamical system has pure discrete spectrum. It is sometimes stated just for *unimodular* Pisot substitutions, i.e. for Pisot substitutions whose matrix has determinant ± 1.

There have been two main approaches to proving pure discrete spectrum. The first, geometric, pioneered by Rauzy [**20**], proceeds by constructing a semi-conjugacy from the substitution dynamical system onto a torus translation and then showing that it is almost

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everywhere one-to-one. Host [**11**], in a widely cited but unpublished manuscript, proved that a unimodular Pisot substitution system on two symbols is metrically conjugate to a circle rotation, provided the *strong coincidence condition* holds (see the next section for definitions). A similar condition was first introduced for substitutions of constant length by Dekking [**5**] who proved that it characterizes systems with pure discrete spectrum. Arnoux and Ito [**1**] introduced the strong coincidence condition in the non-constant length case. They used the geometric approach to establish that a unimodular Pisot substitution system on m symbols, satisfying the strong coincidence condition, is metrically conjugate to a *domain exchange* in $\mathbb{R}^{m-1}$. It was further extended in [**4**] and [**24**]. Recently, Siegel [**23**] has proved that in the case of non-unimodular Pisot substitutions there is an analogous realization on a product of $\mathbb{R}^{m-1}$ and p-adic spaces. However, being a domain exchange does not automatically imply that the dynamical system is pure discrete and additional conditions were needed to make such a claim [**1, 21**].

The second approach uses operator theory and/or harmonic analysis to show that every spectral measure for the dynamical system is pure discrete. It was used in [**12, 13, 25–27**] and independently, in a different form, in [**17, 18**]. The first author proved in his thesis [**8**] that a Pisot substitution system (not necessarily unimodular) on two symbols is pure discrete, under a condition which turned out to be equivalent to the strong coincidence condition. In this paper we present a revised and updated account of this result. Recently, Barge and Diamond [**3**] have proved the strong coincidence condition for all two-symbol Pisot substitutions and thus the discrete spectrum conjecture for such substitution systems has now been confirmed.

2. *Preliminaries and statement of main result*

Let $\mathcal{A} = \{0, 1, \ldots, d-1\}$, for some $d \geq 2$. Let $\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^i$ be the set of finite words on $\mathcal{A}$. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^*$. The substitution ζ is extended to maps (also denoted by ζ) $\mathcal{A}^* \to \mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by concatenation. We write $l_i(w)$ for the number of occurrences of the letter i in the word w and denote $\mathbf{l}(w) = [l_0(w), \ldots, l_{d-1}(w)]^T$. The matrix associated with the substitution ζ is defined as follows:

$$M_\zeta = (m_{i,j})_{d\times d}, \quad \text{where } m_{i,j} = l_i(\zeta(j)).$$

Note that $M_\zeta \mathbf{l}(w) = \mathbf{l}(\zeta(w))$ for all $w \in \mathcal{A}^*$. The substitution is *primitive* if there exists k such that all entries of M_ζ^k are strictly positive; equivalently, if for every $i, j \in \mathcal{A}$, the symbol j occurs in $\zeta^k(i)$. Suppose that $\zeta(0)$ begins with 0 and $|\zeta(0)| > 1$. The sequence $u = \lim_{n\to\infty} \zeta^n(0) \in \mathcal{A}^{\mathbb{N}}$, satisfying $\zeta(u) = u$, is called a *substitution sequence* for ζ.

Let $\mathcal{A}^{\mathbb{Z}}$ have the product topology and let σ be the shift transformation $(\sigma v)_n = v_{n+1}$ on $\mathcal{A}^{\mathbb{Z}}$. For $v \in \mathcal{A}^{\mathbb{Z}}$ we define $v_{[i,j]}$ as the finite word $v_i \ldots v_j$. Define X_ζ to be the set of all two-sided sequences $v \in \mathcal{A}^{\mathbb{Z}}$ such that $v_{[i,j]}$, for all $i < j$, occurs as a block in the substitution sequence u. Then (X_ζ, σ) is a topological dynamical system, called the *substitution dynamical system (for ζ)*. We have the following lemma.

LEMMA 2.1. (See [**17**, p. 89, pp. 95–98]) *Let ζ be a primitive substitution such that $\zeta(0)$ starts with* 0. *Then the following statements hold.*

(i) *The dynamical system (X_ζ, σ) is minimal, i.e. every orbit is dense.*

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(ii) *If a word w occurs in u, then there exists an integer L such that w is in $x_{[i,i+L]}$ for all $x \in X_\zeta$ and $i \in \mathbb{Z}$.*

(iii) *The dynamical system (X_ζ, σ) is uniquely ergodic, i.e. there is a unique σ-invariant Borel probability measure μ.*

Thus, (X_ζ, σ, μ) is a measure-preserving transformation and it is of interest to determine its spectral type. This is, by definition, the spectral type of the unitary operator $Uf(x) = f(\sigma x)$ on $L^2(X_\zeta, \mu)$.

We say that ζ is a *Pisot substitution* if the matrix M_ζ has one eigenvalue greater than 1 and all other eigenvalues are strictly between 0 and 1 in modulus. It is known that every Pisot substitution is primitive [**4**]. Since the property of being a Pisot substitution is preserved when passing from ζ to ζ^k, we can assume without loss of generality that $\zeta(0)$ starts with 0.

THEOREM 2.2. *Let ζ be a Pisot substitution on the alphabet $\mathcal{A} = \{0, 1\}$. Then the measure-preserving system (X_ζ, σ, μ) has pure discrete spectrum.*

Definition 2.3. A substitution ζ satisfies the *strong coincidence condition* (on prefixes) if there exist $n \in \mathbb{N}$, $i \in \mathcal{A}$, $p, q, t, r \in \mathcal{A}^*$, such that

$$\zeta^n(0) = pit \quad \text{and} \quad \zeta^n(1) = qir, \quad \text{with } \mathbf{l}(p) = \mathbf{l}(q).$$

Theorem 2.2 is immediate from the following two results.

THEOREM 2.4. *Let ζ be a Pisot substitution on the alphabet $\mathcal{A} = \{0, 1\}$. The measure-preserving system (X_ζ, σ, μ) has pure discrete spectrum if and only if ζ satisfies the strong coincidence condition.*

THEOREM 2.5. (Barge and Diamond [**3**]) *Every Pisot substitution on two symbols satisfies the strong coincidence condition.*

Theorem 2.4, which first appeared in [**8**], generalizes the unpublished work of Host [**11**] who obtained an analogous statement for unimodular Pisot substitutions.

3. *The balanced pair algorithm for pure discrete spectrum*

In this section we describe a procedure for deciding whether a given substitution dynamical system has pure discrete spectrum. We restrict ourselves to the case of two symbols, although there is a version of this procedure for any number of symbols.

A pair of words, $A = a_1 \dots a_n$, $B = b_1 \dots b_n$, is called a *balanced pair* if $\mathbf{l}(A) = \mathbf{l}(B)$. In this case, we write $[A, B]$ or $\left|\begin{smallmatrix} A \\ B \end{smallmatrix}\right|$. We say that the balanced pair *splits* at $k \in \{2, \dots, n\}$ if $[A_{[1,k-1]}, B_{[1,k-1]}]$ is a balanced pair. We also say that a pair of infinite words $[x, y]$ splits at k if the pair $[x_{[1,k-1]}, y_{[1,k-1]}]$ is balanced. If a balanced pair does not split non-trivially, we say that it is *irreducible*. We call the process of splitting a balanced pair into irreducible balanced subpairs *reduction*. An example of reduction follows this general discussion. A reduction is uniquely defined.

Recall that our alphabet is $\mathcal{A} = \{0, 1\}$ and u is the fixed point of ζ starting with 0. Define the set of *initial irreducible balanced pairs* I_1 as follows:

$$I_1 = \{[01^k, 1^k0] : 01^k0 \text{ is a word in } u\}.$$

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Inductively define, for $n > 1$,

$$I_n = \{[A, B] : [A, B] \text{ is an irreducible balanced subpair of } [\zeta(C), \zeta(D)], [C, D] \in I_{n-1}\}. \tag{3.1}$$

Then define the set I of *irreducible balanced pairs*:

$$I = \bigcup_{n=1}^{\infty} I_n.$$

Let $[A, B] \in I$ be a balanced pair. If there exists k such that the reduction of $[\zeta^k(A), \zeta^k(B)]$ contains the pair $[i, i]$ for some $i \in \mathcal{A}$, we say that the pair $[A, B]$ *leads to a coincidence*.

THEOREM 3.1. (Livshits [**13**]) *Let ζ be a primitive two-symbol substitution, with a non-periodic fixed point $u = 0 \ldots$. Assume that the set I of irreducible balanced pairs is finite. Then the following statements hold.*

(i) *If every $[A, B] \in I$ leads to a coincidence, then (X_ζ, σ, μ) has pure discrete spectrum.*

(ii) *If there exists a pair $[A, B] \in I$ which does not lead to a coincidence, then (X_ζ, σ, μ) has a continuous spectral component.*

Remark 1. If the substitution ζ satisfies the strong coincidence condition, then every irreducible balanced pair $[A, B]$ leads to a coincidence. Indeed, we have either $a_1 = b_1$, in which case the claim is obvious, or $\{a_1, b_1\} = \{0, 1\}$, and the reduction of $[\zeta^n(A), \zeta^n(B)]$ contains $[i, i]$, see Definition 2.3. Conversely, it is easy to see that if I is finite and every pair $[A, B] \in I$ leads to a coincidence, then the strong coincidence condition holds.

Remark 2. Livshits [**13**] also assumed the property of 'unique admissible decoding': for any $x \in X_\zeta$ there exists a unique $y \in X_\zeta$ such that $x = \sigma^k \zeta(y)$ for some $0 \le k < |\zeta(y_0)|$. From Mosse's results Mosse [**15, 16**] on bilateral recognizability, this property holds for every primitive substitution with a non-periodic fixed point (see also [**22**, Corollary 7.6.2]).

Remark 3. Although the theorem in its current formulation was found by Livshits [**12, 13**], many of its ideas appeared independently in other work. Michel [**14**] used the idea of coincidence density, which is integral to the proof of the theorem. In addition, in [**17, 18**] most of the ideas in this method are used in proving that a specific substitution dynamical system has pure discrete spectrum but no general formulation is implied. A proof of Theorem 3.1, based on ideas in [**17**], appeared in the first author's PhD dissertation [**8**]. The dissertation also includes an examination of substitutions whose alphabets have more than two symbols and includes an algorithm which can show whether the balanced pair algorithm will terminate.

Next we give an example of how the balanced pair algorithm is applied.

Example. Let ζ be the substitution defined by

$$0 \to 01100$$
$$1 \to 01.$$

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Then $u = \lim_{n\to\infty} \zeta^n(0) = 0110001010110001100\ldots$ and the words of the form 01^k0 are 010 and 0110. Thus the set I of irreducible balanced pairs begins:

$$I_1 = \left\{ \left| \begin{matrix} 0 \\ 0 \end{matrix} \right|, \left| \begin{matrix} 1 \\ 1 \end{matrix} \right|, \left| \begin{matrix} 01 \\ 10 \end{matrix} \right|, \left| \begin{matrix} 011 \\ 110 \end{matrix} \right| \right\}.$$

Those which are not coincidences break down as follows:

$$\left| \begin{matrix} 01 \\ 10 \end{matrix} \right| \to \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \left| \begin{matrix} 1 \\ 1 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 001 \\ 100 \end{matrix} \right|$$

$$\left| \begin{matrix} 011 \\ 110 \end{matrix} \right| \to \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \left| \begin{matrix} 1 \\ 1 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 0 \\ 1 \end{matrix} \right| \left| \begin{matrix} 0101 \\ 1100 \end{matrix} \right|.$$

Now we break down the new irreducible pairs:

$$\left| \begin{matrix} 001 \\ 100 \end{matrix} \right| \to \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \left| \begin{matrix} 1 \\ 1 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 001 \\ 100 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 001 \\ 100 \end{matrix} \right|$$

$$\left| \begin{matrix} 0101 \\ 1100 \end{matrix} \right| \to \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \left| \begin{matrix} 1 \\ 1 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 0 \\ 0 \end{matrix} \right| \left| \begin{matrix} 0101 \\ 1100 \end{matrix} \right| \left| \begin{matrix} 10 \\ 01 \end{matrix} \right| \left| \begin{matrix} 001 \\ 100 \end{matrix} \right|.$$

Since these are made up of pairs already considered, we are done. So

$$I = \left\{ \left| \begin{matrix} 0 \\ 0 \end{matrix} \right|, \left| \begin{matrix} 1 \\ 1 \end{matrix} \right|, \left| \begin{matrix} 01 \\ 10 \end{matrix} \right|, \left| \begin{matrix} 011 \\ 110 \end{matrix} \right|, \left| \begin{matrix} 100 \\ 001 \end{matrix} \right|, \left| \begin{matrix} 0101 \\ 1100 \end{matrix} \right| \right\}.$$

Note that each member of I has a 'descendant' which is a coincidence. Thus the spectrum is pure discrete by Theorem 3.1.

4. *Proof of Theorem 2.4*

LEMMA 4.1. (See [**17**, Proposition V.7]) *Let ζ be a primitive substitution on the alphabet $\mathcal{A} = \{0, 1\}$. Then for each $i \in \mathcal{A}$ the limit*

$$f_i = \lim_{k\to\infty} \frac{l_i(\zeta^k(0))}{|\zeta^k(0)|}$$

exists. Furthermore, $\mathbf{f} = [f_0, f_1]^T$ is a Perron–Frobenius eigenvector of M_ζ, such that $f_0 + f_1 = 1$.

We define the *discrepancy* d on a word w in the substitution sequence u by

$$d(w) = l_0(w) - f_0|w|. \tag{4.1}$$

LEMMA 4.2. *Let ζ be a Pisot substitution with a fixed point $u = 0\ldots$, and let γ be the non-Perron–Frobenius eigenvalue of M_ζ. Then for all words $w \in \mathcal{A}^*$,*

$$d(\zeta(w)) = \gamma d(w).$$

Proof. Let $\mathbf{g}$ be the eigenvector of M_ζ corresponding to γ and let $\mathbf{l}(w) = c_1\mathbf{f} + c_2\mathbf{g}$. A simple computation shows that $d(w) = c_2 \det[\mathbf{g}, \mathbf{f}]$, and it remains to note that $\mathbf{l}(\zeta(w)) = M_\zeta \mathbf{l}(w) = c_1\theta\mathbf{f} + c_2\gamma\mathbf{g}$. □

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LEMMA 4.3. *Let ζ be a Pisot substitution with a fixed point $u = 0\ldots$. Then there exists $C > 0$ such that for all words w in u, we have $|d(w)| \le C$.*

Proof. This is a standard result, see, for example, [**9**, Theorem A.2]. □

The following proposition is the key part of the proof. In view of Theorem 3.1, Theorem 2.4 will follow.

PROPOSITION 4.4. *Let ζ be a Pisot substitution on the alphabet $\mathcal{A} = \{0, 1\}$, with a fixed point $u = 0\ldots$. Then the set of irreducible balanced pairs I for ζ is finite.*

Proof. Recall (3.1)—the definition of I_n in the balanced pair algorithm. Consider the pairs of infinite words $P_k = [u, \zeta^k(\sigma u)]$, $k \ge 0$. Then I_1 equals the set of balanced pairs in the reduction of P_0. The set I_n is derived by applying ζ to I_{n-1}. By induction, for each n the set I_n is the set of balanced pairs in the reduction of P_{n-1}.

To prove the proposition we will find a word w in u and an integer L with the following property: for any $\ell \ge L$, wherever w occurs in u the pair P_ℓ splits somewhere between the beginning and the end of w. Because the word w occurs with bounded gaps in u (see Lemma 2.1(ii)), irreducible balanced pairs for ζ are bounded in length: this will ensure that I is finite.

Let C be the bound given by Lemma 4.3. Write $s_k = |\zeta^k(0)|$. Define for positive integers $n \ge 2$ and k:

$$d_n = d(u_{[1,n]});$$
$$e_{k,n} = d((\sigma^{s_k} u)_{[1,n]}) = d((\zeta^k(\sigma u))_{[1,n]}).$$

Then, we can state the following.

(1) $|d_n| \le C$ and $|e_{k,n}| \le C$; and

(2) $d_n - e_{k,n}$ is an integer by (4.1); in fact, it is the difference between the number of 0's in the first and the second word of the pair P_k curtailed at position n.

Therefore, also

(3) $|(d_n - e_{k,n}) - (d_{n+1} - e_{k,n+1})| \le 1$;

(4) $[u, \sigma^{s_k} u]$ splits at position n if and only if $d_n = e_{k,n}$.

The strategy is to find a word w in u with the following property: wherever w occurs in u (say $w = u_{[n+1,n+m]}$), the bounded function d_i takes values near both its minimum and maximum at some points between $i = n + 1$ and $i = n + m$. The pair P_k splits at position i if $d_i = e_{k,i}$. Because d_i and $e_{k,i}$ differ by an integer and, as we will show, have very similar upper and lower bounds, we will be guaranteed a split.

Define $D^+ = \sup_n d_n$ and $D^- = \inf_n d_n$. Choose $\epsilon \in (0, 1/4]$. Let L be so large that for $\ell > L$, by Lemma 4.2,

$$|d_{s_\ell}| = |d(\zeta^\ell(0))| = |\gamma|^\ell |d(0)| < \epsilon. \tag{4.2}$$

Since u begins with $\zeta^k(0)$, we can also find K large enough so that for $k > K$,

$$D^+ - \sup_n d((\zeta^k(0))_{[1,n]}) < \epsilon;$$
$$D^- - \inf_n d((\zeta^k(0))_{[1,n]}) > -\epsilon.$$

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Fix $k > K$, and let $w = \zeta^k(0)$. Let $i \le s_k$ and $j \le s_k$ be such that

$$D^+ - d_i < \epsilon; \tag{4.3}$$

$$D^- - d_j > -\epsilon. \tag{4.4}$$

Assume that w occurs at position $n+1$ in u. Then since w also occurs at position 1 in u,

$$d_{n+l} = d(u_{[1,n+l]}) = d(u_{[1,n]}) + d(u_{[n+1,n+l]}) = d_n + d(u_{[1,l]}) = d_n + d_l$$

for $1 \le l \le s_k$. We have

$$D^+ \ge d_{n+i} = d_n + d_i,$$

so $d_n < \epsilon$ by (4.3). Similarly,

$$D^- \le d_{n+j} = d_n + d_j,$$

so $d_n > -\epsilon$ by (4.4). Thus $|d_n| < \epsilon$.

We have for $\ell > L$,

$$\begin{aligned} d_{i+n} - e_{\ell,i+n} &= d_{i+n} - (d_{i+n+s_\ell} - d_{s_\ell}) \\ &> d_n + d_i - d_{i+n+s_\ell} - \epsilon \\ &> D^+ - d_{i+n+s_\ell} - 3\epsilon \\ &\ge -3\epsilon. \end{aligned}$$

Here we used (4.2) in the second line and (4.3), $d_n > -\epsilon$, in the third line. Similarly,

$$\begin{aligned} d_{j+n} - e_{\ell,j+n} &= d_{j+n} - (d_{j+n+s_\ell} - d_{s_\ell}) \\ &< d_n + d_j - d_{j+n+s_\ell} + \epsilon \\ &< D^- - d_{j+n+s_\ell} + 3\epsilon \\ &\le 3\epsilon. \end{aligned}$$

Since $\epsilon \le \frac{1}{4}$, for $\ell > L$, we have $d_{i+n} - e_{\ell,i+n} \ge 0$ and $d_{j+n} - e_{\ell,j+n} \le 0$. So for some m between i and j, the difference $d_{m+n} - e_{\ell,m+n}$ must be zero. This shows that P_ℓ splits wherever w occurs in u, which, by the opening remarks, shows that I is finite. □

We do not know whether Proposition 4.4 extends to the case of more than two symbols.

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