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Solving Maxwell's Equation in Meta-Materials by a CG-DG Method

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Abstract. In this paper, an approach combining the DG method in space with CG method in time (CG-DG method) is developed to solve time-dependent Maxwell's equations when meta-materials are involved. Both the unconditional L^2-stability and error estimate of order $\mathcal{O}\left(\tau^{r+1}+h^{k+\frac{1}{2}}\right)$ are obtained when polynomials of degree at most r is used for the temporal discretization and at most k for the spatial discretization. Numerical results in 3D are given to validate the theoretical results.

AMS subject classifications: 35Q61, 65M12, 65M60

Key words: Maxwell's equations, meta-materials, full-discretization, CG-DG method, L^2-stability, L^2-error estimate.

1 Introduction

Meta-material is often referred as an artificially constructed electromagnetic material whose refraction index is negative. It was first theoretically proposed by Veselagi in 1967 [29], and practically constructed by Smith in 2000 [27]. Since then, the study of meta-materials has been a very hot topic due to its potential application in many areas, such as the design of invisibility and sub-wavelength imaging.

The numerical simulation of meta-materials plays an important role in the design of new meta-materials and the discovery of new phenomenon of meta-materials [8]. Among them, the widely used numerical methods for the simulation of meta-materials are the finite difference time domain methods (FDTD) [13, 28], the finite element methods (FEM) [21] and the commercial packages, such as HFSS and COMSOL, et al.. However, it is well known that the FDTD method has difficulties in solving problems with

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complex geometry. In the literature, there exist some works on finite element time domain (FETD) method for Maxwell's equation in dispersive media [2] and meta-materials [21, 22]. Their main advantage lies in their ability to handle arbitrary geometries via unstructured meshes of the domain of interest.

Since the development of the discontinuous Galerkin (DG) method in 1973 [26], it has become one of the most important methods in solving various kinds of differential equations. It can keep almost all the advantages of finite element method. Further, the DG method has many nice properties, e.g., applicability for non-conforming mesh, high-order accuracy, flexibility in handling material interface and high parallelizability. We refer to the survey paper [1, 7], the monographs [5, 9, 15] and their references therein for more details about it.

In recent years, DG methods have been used to solve Maxwell's equation in free space [4, 6, 10, 12, 14] and dispersive media [16, 17, 24, 30, 31], whose permittivity depends on the wave frequency. In [30] and [33], Xie et al. propose a semi-discrete DG method for dispersive media and space-time DG method for Maxwell's equation in free space. They prove that the convergence order is $\mathcal{O}(h^{k+\frac{1}{2}})$ for the semi-discrete DG method and $\mathcal{O}(\tau^{r+1}+h^{k+\frac{1}{2}})$ for the space-time DG method. It is worthwhile to point out that in the numerical examples of [30], the continuous Galerkin finite element method is used in the temporal discretization. Nevertheless, the convergence rate for the temporal variable is missing. To our knowledge this is the first effort to solve Maxwell's equation by the finite element methods (CG or DG) both for temporal and spatial discretization. Very recently, they extend this space-time DG method to dispersive media and provide the corresponding theoretical analysis and numerical results [31].

Recently, Li in [18] develops an approach using DG approach for spatial discretization and Runge-Kutta (finite difference) method in temporal discretization to solve the time-domain Maxwell's equation in meta-materials. Although several numerical examples are given to show that this method is efficient, the theoretical analysis is missing. In [23], Li et al. develop a leap-frog type DG method for solving the time-domain Maxwell's equation in meta-materials, and prove the stability and convergence of it. In [20], Li et al. develop a leap-frog type DG method for solving the time-domain Maxwell's equation in meta-materials based on auxiliary differential equation (ADE) method, and prove that, under some CFL condition, this method is stable and convergent. In [32], Wang et al. develop an implicit DG method for solving time-domain Maxwell's equations in meta-materials based on a system of integro-differential equations. They prove that the implicit DG scheme is unconditionally stable and the convergence rate is $\mathcal{O}(\tau^2+h^{k+\frac{1}{2}})$. However, the implicit DG scheme cannot make full use of the advantages of high order DG method.

Motivated by the works in [20, 30, 33], we introduce a method combining the DG method in space with the CG method in time (CG-DG method) for Maxwell's equation in meta-materials. Our work is based on a so-called auxiliary differential equation (ADE) method. The key point of it is to introduce another set of differential equations to include media interactions. We prove that this CG-DG scheme is unconditionally stable and con-

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vergent at a rate of $\mathcal{O}(\tau^{r+1}+h^{k+\frac{1}{2}})$ in the L^2-norm when polynomials of degree at most r is used for the temporal discretization and at most k for the spatial discretization. Compared with other numerical methods used for meta-materials, our scheme is a high-order scheme in temporal variable. To our knowledge, this is the first time to prove the error analysis of CGDG method for the Maxwell's equations. Theoretical results are validated by some numerical examples.

The rest of this paper is organized as follows. In Section 2, the governing equations for meta-materials is introduced. In Section 3, the numerical scheme is proposed. In Section 4, unconditionally stability for the numerical scheme is obtained. In Section 5, the convergence rate of $\mathcal{O}(\tau^{r+1}+h^{k+\frac{1}{2}})$ is obtained for the CG-DG method. In Section 6, numerical results in 3D are provided to support our theoretical prediction.

2 The governing equation

The electromagnetic wave propagation is governed by the general Maxwell's equation,

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, $\mathbf{D}$ and $\mathbf{B}$ are their corresponding electric and magnetic flux densities. The constitutive relations among them are given by

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}\equiv\epsilon\mathbf{E},\quad \mathbf{B}=\mu_0\mathbf{H}+\mathbf{M}\equiv\mu\mathbf{H}, \tag{2.2}$$

where ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability, $\mathbf{P}$ and $\mathbf{M}$ are the induced electric and magnetic polarizations, respectively.

Substituting (2.2) into (2.1), we get

$$\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{H}-\frac{\partial\mathbf{P}}{\partial t}\quad\text{in }\ \Omega\times(0,T), \tag{2.3}$$

$$\mu_0\frac{\partial\mathbf{H}}{\partial t}=-\nabla\times\mathbf{E}-\frac{\partial\mathbf{M}}{\partial t}\quad\text{in }\ \Omega\times(0,T). \tag{2.4}$$

We introduce the induced electric current $\mathbf{J}=\frac{\partial\mathbf{P}}{\partial t}$ and magnetic current $\mathbf{K}=\frac{\partial\mathbf{M}}{\partial t}$. Then (2.3) and (2.4) can be rewritten as

$$\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{H}-\mathbf{J}\quad\text{in }\ \Omega\times(0,T), \tag{2.5}$$

$$\mu_0\frac{\partial\mathbf{H}}{\partial t}=-\nabla\times\mathbf{E}-\mathbf{K}\quad\text{in }\ \Omega\times(0,T). \tag{2.6}$$

There exist three popular models to simulate the meta-materials, i.e., Lossy Drude polarization and magnetization model, Plasma-Lorentz model and Lorentz model. In the following, we will introduce their corresponding governing equations, respectively.

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For meta-materials, the permittivity ϵ and the permeability μ are not constants any more [21]. In frequency domain, the corresponding permittivity and permeability can be written as,

- **For the lossy Drude polarization and magnetization model** [34]:

$$\epsilon(\omega)=\epsilon_0\Big(1-\frac{\omega_{pe}^2}{\omega(\omega+i\Gamma_e)}\Big),\quad \mu(\omega)=\mu_0\Big(1-\frac{\omega_{pm}^2}{\omega(\omega+i\Gamma_m)}\Big); \tag{2.7}$$

- **For the Plasma-Lorentz model** [22,25,27]:

$$\epsilon(\omega)=\epsilon_0\Big(1-\frac{\omega_{pe}^2}{\omega(\omega+i\Gamma_e)}\Big),\quad \mu(\omega)=\mu_0\Big(1-\frac{F\omega_{pm}^2}{\omega^2-i\Gamma_m\omega-\omega_{m0}^2}\Big); \tag{2.8}$$

- **For the Lorentz model** [19]:

$$\epsilon(\omega)=\epsilon_0\Big(1-\frac{\omega_{pe}^2}{\omega^2-\omega_{e0}^2-i\Gamma_e\omega}\Big),\quad \mu(\omega)=\mu_0\Big(1-\frac{\omega_{pm}^2}{\omega^2-\omega_{m0}^2-i\Gamma_m\omega}\Big), \tag{2.9}$$

where ω_{pe}, ω_{pm} are the electric and magnetic plasma frequencies, Γ_e, Γ_m are the electric and magnetic damping frequencies, ω is the angle frequency, ω_{e0},ω_{m0} are the electric and magnetic resonance frequencies and $F\in(0,1)$ is a parameter depending on the geometry of the unit cell of meta-material.

Using a time-harmonic variation $\exp(i\omega t)$, substituting (2.7), (2.8) and (2.9) into (2.2), respectively, and using the definition of $\mathbf{J}$, $\mathbf{K}$, we get the following equations for the three different meta-materials,

- **For the lossy Drude polarization and magnetization model**:

$$\begin{aligned}&\frac{\partial\mathbf{J}}{\partial t}+\Gamma_e\mathbf{J}=\epsilon_0\omega_{pe}^2\mathbf{E},\\&\frac{\partial\mathbf{K}}{\partial t}+\Gamma_m\mathbf{K}=\mu_0\omega_{pm}^2\mathbf{H};\end{aligned} \tag{2.10}$$

- **For the Plasma-Lorentz model**:

$$\begin{aligned}&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\partial\mathbf{J}}{\partial t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\mathbf{J}=\mathbf{E},\\&\frac{1}{\mu_0\omega_{m0}^2F}\frac{\partial\mathbf{K}}{\partial t}+\frac{\Gamma_m}{\mu_0\omega_{m0}^2F}\mathbf{K}+\frac{1}{\mu_0F}\mathbf{M}=\mathbf{H},\\&\mathbf{K}=\frac{\partial\mathbf{M}}{\partial t};\end{aligned} \tag{2.11}$$

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- **For the Lorentz model**:

$$\begin{aligned}
&\frac{1}{\epsilon_0\omega_{pe}^2}\frac{\partial \mathbf{J}}{\partial t}+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\mathbf{J}+\frac{\omega_{e0}^2}{\epsilon_0\omega_{pe}^2}\mathbf{P}-\mathbf{E}=0,\\
&\frac{1}{\mu_0\omega_{pm}^2}\frac{\partial \mathbf{K}}{\partial t}+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\mathbf{K}+\frac{\omega_{m0}^2}{\mu_0\omega_{pm}^2}\mathbf{M}-\mathbf{H}=0,\\
&\mathbf{J}=\frac{\partial \mathbf{P}}{\partial t},\\
&\mathbf{K}=\frac{\partial \mathbf{M}}{\partial t}.
\end{aligned} \tag{2.12}$$

It is worthwhile to point out that the governing equations for the three meta-materials above share a common feature, i.e., all of them can be viewed as (2.5) and (2.6) along with $2-4$ differential equations. It is observed that both the numerical method and its corresponding theoretical analysis for them are very similar. For simplicity, we just focus on the Lossy Drude polarization and magnetization model in this paper.

Assume that Ω is a bounded convex domain with its boundary $\partial\Omega$ perfect conducting, i.e.,

$$\mathbf{n}\times\mathbf{E}=0 \quad \text{on } \partial\Omega\times(0,T),$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. The initial conditions for the system (2.5), (2.6) and (2.10) are

$$\mathbf{E}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x}),\quad \mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x}),\quad \mathbf{J}(\mathbf{x},0)=\mathbf{J}_0(\mathbf{x}),\quad \mathbf{K}(\mathbf{x},0)=\mathbf{K}_0(\mathbf{x}). \tag{2.13}$$

Denote $H^k(\Omega)$ and $H^k(I)$ the standard Sobolev space with the standard norms $\|\cdot\|_{k,\Omega}$ and $\|\cdot\|_{k,I}$, respectively. Then define

$$L^2(I,H^k(\Omega))=\left\{u|u(\cdot,t)\in H^k(\Omega),\forall t\in I \text{ and } \int_I ||u(\cdot,t)||_{k,\Omega}^2 dt<\infty\right\},$$

with the norm $|||u|||_{k,0}=\left(\int_I \|u(\cdot,t)\|_{k,\Omega}^2 dt\right)^{\frac{1}{2}}$ and

$$H^k(I,L^2(\Omega))=\left\{u|u(\mathbf{x},\cdot)\in H^k(I),\forall \mathbf{x}\in\Omega \text{ and } \int_\Omega ||u(\mathbf{x},\cdot)||_{k,I}^2 d\mathbf{x}<\infty\right\},$$

with norm $|||u|||_{0,k}=\left(\int_\Omega \|u(\mathbf{x},\cdot)\|_{k,I}^2 d\mathbf{x}\right)^{\frac{1}{2}}$. The corresponding vector functional spaces are denoted by $\left(H^k(\Omega)\right)^3$, $\left(L^2(I,H^k(\Omega))\right)^3$ and $\left(H^k(I,L^2(\Omega))\right)^3$. For $\mathbf{Q}=\begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix}$, $\mathbf{U}=(\mathbf{E},\mathbf{H})^T$ and $\mathbf{E}(\cdot,t),\mathbf{H}(\cdot,t)\in\left(H^k(\Omega)\right)^3$, $\forall t\in I$, define

$$\begin{aligned}
&\|\mathbf{U}(\cdot,t)\|_{k,\Omega}=\left(\|\mathbf{E}(\cdot,t)\|_{k,\Omega}^2+\|\mathbf{H}(\cdot,t)\|_{k,\Omega}^2\right)^{1/2},\\
&\|\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t)\|_{k,\Omega}=\left(\epsilon_0\|\mathbf{E}(\cdot,t)\|_{k,\Omega}^2+\mu_0\|\mathbf{H}(\cdot,t)\|_{k,\Omega}^2\right)^{1/2}.
\end{aligned}$$

We denote C to be a constant which is independent of the mesh size h and Δt, but with different value in different inequality.

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3 Numerical scheme

We assume that Ω is decomposed by a regular mesh $\mathcal{T}_h$ with maximum diameter h. An element is denoted by K, a face by e, and the outward unit normal by $\mathbf{n}_K$. We also denote the union of all interior faces of $\mathcal{T}_h$ by $\mathcal{E}_\mathcal{I}$, the union of all boundary faces of $\mathcal{T}_h$ by $\mathcal{E}_\mathcal{D}$, and the union of all faces of $\mathcal{T}_h$ by $\mathcal{E}=\mathcal{E}_\mathcal{I}\bigcup\mathcal{E}_\mathcal{D}$. Moreover, let $0=t_0<t_1<\cdots<t_n=T$ be a triangulation of the interval $I=[0,T]$ with elements denoted by $I_j=[t_{j-1},t_j], j=1,2,\cdots,n$. Set $\Delta t_j=t_j-t_{j-1}$ and $\tau=\max\{\Delta t_j, j=1,2,\cdots n\}$.

Denote $P^k(K)$ (or $P^r(I)$) the spaces of polynomials with degree at most k (or r) in K (or I_j). Then the DG finite element space for spatial discretization is given by

$$\mathbf{S}^k_{h,\Omega}=\{v\in L^2(\Omega): v|_K\in P^k(K),\ K\in\mathcal{T}_h\}.$$

The continuous finite element space for temporal discretization is

$$\mathbf{S}^r_{\tau,I}=\{\mathbf{v}\in L^2(I): \mathbf{v}|_{I_j}\in P^r(I_j),\ j=1,2,\cdots,n\}.$$

Thus our CG-DG space is defined by

$$\mathbf{V}^{r,k}_h=\bar{\mathbf{V}}^{r,k}_h\oplus\bar{\mathbf{V}}^{r,k}_h,$$

where $\bar{\mathbf{V}}^{r,k}_h=(\mathbf{S}^r_{\tau,I}\otimes\mathbf{S}^k_{h,\Omega})^3$.

It is known that the choice of the numerical flux is very important in the definition of DG scheme. For this purpose, we need to introduce some notations first. Let e be an interior face belonging to the element K. Set

$$\mathbf{v}^{int(K)}(\mathbf{x})=\lim_{\delta\to0-}\mathbf{v}(\mathbf{x}+\delta\mathbf{n}_K),\qquad \mathbf{v}^{ext(K)}(\mathbf{x})=\lim_{\delta\to0+}\mathbf{v}(\mathbf{x}+\delta\mathbf{n}_K),\quad\forall\mathbf{x}\in e.$$

Then the average and tangential jump of $\mathbf{v}$ on any interior face e are defined as follows:

$$\{\{\mathbf{v}\}\}=\frac{\mathbf{v}^{int(K)}+\mathbf{v}^{ext(K)}}{2},\quad [\![\mathbf{v}]\!]_T=\mathbf{n}_K\times\mathbf{v}^{int(K)}-\mathbf{n}_K\times\mathbf{v}^{ext(K)}.$$

For a boundary face $e\subset\mathcal{E}_\mathcal{D}$ which is belong to the element K, we define

$$\mathbf{v}^{int}(\mathbf{x})=\mathbf{v}^{int(K)}(\mathbf{x}),\qquad\forall\mathbf{x}\in e.$$

To introduce the DG formulation, we rewrite (2.5), (2.6) and (2.10) with the initial condition (2.13) into the form

$$\mathbf{Q}\mathbf{U}_t+\nabla\cdot\mathbf{f}(\mathbf{U})+\mathbf{L}=0\quad\text{in }\Omega\times(0,T_0],\tag{3.1}$$

$$\mathbf{S}\mathbf{L}_t+\mathbf{N}\mathbf{L}=\mathbf{U}\quad\text{in }\Omega\times(0,T_0],\tag{3.2}$$

$$\mathbf{U}(\mathbf{x},0)=\mathbf{U}_0\quad\text{in }\Omega,\tag{3.3}$$

$$\mathbf{L}(\mathbf{x},0)=\mathbf{L}_0\quad\text{in }\Omega,\tag{3.4}$$

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where

$$\mathbf{U}=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix},\qquad \mathbf{U}_0=\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix},\qquad \mathbf{Q}=\begin{pmatrix}\epsilon_0 I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & \mu_0 I_{3\times3}\end{pmatrix},$$

$$\mathbf{f}(\mathbf{U})=[\mathbf{f}_1(\mathbf{U}),\mathbf{f}_2(\mathbf{U}),\mathbf{f}_3(\mathbf{U})]^T,\qquad \mathbf{f}_i(\mathbf{U})=\begin{pmatrix}-\mathbf{e}_i\times\mathbf{H}\\ \mathbf{e}_i\times\mathbf{E}\end{pmatrix},\qquad \mathbf{L}=\begin{pmatrix}\mathbf{J}\\ \mathbf{K}\end{pmatrix},$$

$$\mathbf{L}_0=\begin{pmatrix}\mathbf{J}_0\\ \mathbf{K}_0\end{pmatrix},\quad \mathbf{S}=\begin{pmatrix}\frac{1}{\epsilon_0\omega_{pe}^2}I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & \frac{1}{\mu_0\omega_{pm}^2}I_{3\times3}\end{pmatrix},\quad \mathbf{N}=\begin{pmatrix}\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & \frac{\Gamma_m}{\mu_0\omega_{pm}^2}I_{3\times3}\end{pmatrix}.$$

Here $\mathbf{e}_i$, $i=1,2,3$ denote the three Cartesian unit vectors.

Multiplying (3.1) by a test function $\mathbf{v}=(v_{\mathbf{E}},\, v_{\mathbf{H}})^T$ and (3.2) by a test function $\mathbf{w}=(w_{\mathbf{E}},w_{\mathbf{H}})^T$ and then integrating by parts over $K\times I_j$, we obtain the following weak formulation

$$\int_{I_j}\int_K \mathbf{Q}\mathbf{U}_t\cdot\mathbf{v}d\mathbf{x}dt+\int_{I_j}\int_{\partial K}(\mathbf{f}(\mathbf{U})\cdot\mathbf{n}_K)\cdot\mathbf{v}dsdt-\int_{I_j}\int_K \mathbf{f}(\mathbf{U})\cdot\nabla\mathbf{v}d\mathbf{x}dt+\int_{I_j}\int_K \mathbf{L}\cdot\mathbf{v}d\mathbf{x}dt=0, \quad (3.5)$$

$$\int_{I_j}\int_K(\mathbf{S}\frac{\partial}{\partial t}\mathbf{L}+\mathbf{N}\mathbf{L})\cdot\mathbf{w}d\mathbf{x}dt=\int_{I_j}\int_K \mathbf{U}\cdot\mathbf{w}d\mathbf{x}dt. \quad (3.6)$$

Based on it, our CG-DG Scheme is to find $\mathbf{U}_h$, $\mathbf{L}_h\in\mathbf{V}_h^{r,k}$ such that

$$\int_{I_j}\int_K \mathbf{Q}\frac{\partial}{\partial t}\mathbf{U}_h\cdot\mathbf{v}_h d\mathbf{x}dt+\int_{I_j}\int_{\partial K}\widehat{(\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}_K)}\cdot\mathbf{v}_h dsdt$$
$$-\int_{I_j}\int_K \mathbf{f}(\mathbf{U}_h)\cdot\nabla\mathbf{v}_h d\mathbf{x}dt+\int_{I_j}\int_K \mathbf{L}_h\cdot\mathbf{v}_h d\mathbf{x}dt=0, \quad (3.7)$$

$$\int_{I_j}\int_K(\mathbf{S}\frac{\partial}{\partial t}\mathbf{L}_h+\mathbf{N}\mathbf{L}_h)\cdot\mathbf{w}_h d\mathbf{x}dt=\int_{I_j}\int_K \mathbf{U}_h\cdot\mathbf{w}_h d\mathbf{x}dt, \quad (3.8)$$

for all $\mathbf{v}_h,\mathbf{w}_h\in(P^{r-1}(I_j)\otimes\mathbf{S}_{h,\Omega}^k)^3\oplus(P^{r-1}(I_j)\otimes\mathbf{S}_{h,\Omega}^k)^3$ and all elements $K\times I_j$, where $\widehat{(\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}_K)}$ is the numerical flux on the face $e\subset\mathcal{E}$. We take

$$\widehat{\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}}_K=\begin{pmatrix}-\mathbf{n}_K\times(\{\{\mathbf{H}_h\}\}+\frac{1}{2Z}[\![\mathbf{E}_h]\!]_T)\\ \mathbf{n}_K\times(\{\{\mathbf{E}_h\}\}-\frac{Z}{2}[\![\mathbf{H}_h]\!]_T)\end{pmatrix} \quad (3.9)$$

on an interior face $e=\partial K\bigcap\mathcal{E}_I$ and

$$\widehat{\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}}_K=\begin{pmatrix}-\mathbf{n}\times\left(\mathbf{H}_h^{int}+\frac{1}{Z}\mathbf{n}\times\mathbf{E}_h^{int}\right)\\ \mathbf{0}_{3\times1}\end{pmatrix} \quad (3.10)$$

on a boundary face $e=\partial K\bigcap\mathcal{E}_D$, where $Z=\sqrt{\frac{\mu_0}{\epsilon_0}}$ denotes the impedance of the medium. Obviously this numerical flux is consistent with $\mathbf{f}(\mathbf{U})\cdot\mathbf{n}_K$.

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Moreover, set

$$\mathbf{U}_h(\mathbf{x},0)=\mathbb{P}_h\mathbf{U}(\mathbf{x},0),\quad \mathbf{L}_h(\mathbf{x},0)=\mathbb{P}_h\mathbf{L}(\mathbf{x},0), \tag{3.11}$$

with $\mathbb{P}_h$ the L^2 projection operator which will be defined later.

The algorithm of the CGDG method:

Firstly, let $\Phi_j, j=0,1,\cdots,Nt$ be the vector basis of $(\mathbf{S}^k_{h,\Omega})^3\oplus(\mathbf{S}^k_{h,\Omega})^3$. We introduce the space discretization: Find vector coefficient $\mathbf{U}_h(t),\mathbf{L}_h(t)\in(\mathbf{S}^k_{h,\Omega})^3\oplus(\mathbf{S}^k_{h,\Omega})^3$, such that

$$\begin{aligned}\int_K \mathbf{Q}\frac{\partial}{\partial t}\mathbf{U}_h\cdot\mathbf{v}_h d\mathbf{x}+\int_{\partial K}(\mathbf{f}(\widehat{\mathbf{U}_h)\cdot\mathbf{n}_K})\cdot\mathbf{v}_h ds\\ -\int_K \mathbf{f}(\mathbf{U}_h)\cdot\nabla\mathbf{v}_h d\mathbf{x}+\int_K \mathbf{L}_h\cdot\mathbf{v}_h d\mathbf{x}=0,\end{aligned} \tag{3.12}$$

$$\int_K(\mathbf{S}\frac{\partial}{\partial t}\mathbf{L}_h+\mathbf{N}\mathbf{L}_h)\cdot\mathbf{w}_h d\mathbf{x}=\int_K \mathbf{U}_h\cdot\mathbf{w}_h d\mathbf{x}, \tag{3.13}$$

for all $\mathbf{v}_h,\mathbf{w}_h\in(\mathbf{S}^k_{h,\Omega})^3\oplus(\mathbf{S}^k_{h,\Omega})^3$ and all elements K.

Then we can obtain an ordinary system about $\mathbf{U}_h(t),\mathbf{L}_h(t)$, written as

$$\begin{aligned}&\mathbb{A}\frac{\partial\mathbf{U}_h(t)}{\partial t}+\mathbb{B}\mathbf{U}_h(t)+\mathbb{C}\mathbf{L}_h(t)=0,\\ &\mathbb{C}_1\frac{\partial\mathbf{L}_h(t)}{\partial t}+\mathbb{C}_2\mathbf{L}_h(t)=\mathbb{C}\mathbf{U}_h(t),\end{aligned} \tag{3.14}$$

where $\mathbb{A},\mathbb{B},\mathbb{C},\mathbb{C}_1,\mathbb{C}_2$ denote the matrix generalized by the space DG discretization (3.12)-(3.13).

Finally, we will discretization (3.15). Let $\phi_j(t)$ $(j=0,1,\cdots,r)$ be the basis of $P^r(I)$. Our fully discretization scheme is: Find $\mathbf{U}^j_h,\mathbf{L}^j_h\in(\mathbf{S}^k_{h,\Omega})^3\oplus(\mathbf{S}^k_{h,\Omega})^3$ $(j=0,1,2,\cdots,r)$, such that

$$\begin{aligned}&\left(\mathbb{A}\sum_{j=0}^r\mathbf{U}^j_h\frac{\partial\phi_j(t)}{\partial t},\psi(t)\right)_I+\left(\mathbb{B}\sum_{j=0}^r\mathbf{U}^j_h\phi_j,\psi(t)\right)_I+\left(\mathbb{C}\sum_{j=0}^r\mathbf{L}^j_h\phi_j,\psi(t)\right)_I=0,\\ &\left(\mathbb{C}_1\sum_{j=0}^r\partial\mathbf{L}^j_h\frac{\partial\phi_j(t)}{\partial t},\psi(t)\right)+\left(\mathbb{C}_2\sum_{j=0}^r\mathbf{L}_h\phi(t),\psi(t)\right)=\left(\mathbb{C}\sum_{j=0}^r\mathbf{U}^j_h\phi_j(t),\psi(t)\right),\end{aligned} \tag{3.15}$$

for all $\psi(t)\in P^{r-1}(I)$ and all elements I.

In fact, we obtain a system as

$$\mathbb{D}_1\tilde{\mathbf{U}}_h+\mathbb{D}_2\tilde{\mathbf{L}}_h=0; \tag{3.16}$$

$$\mathbb{D}_3\tilde{\mathbf{L}}_h=\mathbb{D}_4\tilde{\mathbf{U}}_h, \tag{3.17}$$

where the matrices are generalized by both the DG in space and CG in time, $\tilde{\mathbf{U}}_h=(\mathbf{U}^0_h,\mathbf{U}^1_h,\cdots,\mathbf{U}^r_h)^T$, $\tilde{\mathbf{L}}_h=(\mathbf{L}^0_h,\mathbf{L}^1_h,\cdots,\mathbf{L}^r_h)^T$. In our numerical implement, we first insert (3.17) into (3.16) and solve it to get $\tilde{\mathbf{U}}_h$, then insert $\tilde{\mathbf{U}}_h$ into (3.17) to get $\tilde{\mathbf{L}}_h$. Because (3.13) is a ordinary equation, we conclude that it is easy to get $\mathbb{D}_3^{-1}\mathbb{D}_4$.

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4 L^2 stability analysis

In this section, we focus on the L^2-stability of our numerical scheme. For this purpose, we introduce an projection operator and its corresponding properties first.

For time-dependent functions, define an L^2-projection operator $\mathcal{P}^t: L^2(0,T) \to P^{r-1}(I_j)$, which satisfies

$$\int_{I_j} (\mathcal{P}^t \varphi - \varphi) v dt = 0, \quad \text{for } \forall\, v \in P^{r-1}(I_j). \tag{4.1}$$

Actually the projection operator $\mathcal{P}^t$ has the following properties:

Lemma 4.1. *Assume* $\mathbf{U}_h$, $\mathbf{L}_h \in \mathbf{V}_h^{r,k}$, $\mathbf{v}_h \in (P^{r-1}(I_j) \otimes \mathbf{S}_{h,\Omega}^k)^3 \oplus (P^{r-1}(I_j) \otimes \mathbf{S}_{h,\Omega}^k)^3$. *Further suppose that* ζ *is a linear function of* $\mathbf{U}_h$. *Then*

$$\int_{I_j} \zeta\Big(\mathbf{U}_h\Big) \mathbf{v}_h dt = \int_{I_j} \zeta\Big(\mathcal{P}^t \mathbf{U}_h\Big) \mathbf{v}_h dt, \tag{4.2}$$

$$\int_{I_j} \int_\Omega \frac{\partial}{\partial t} \mathbf{U}_h \cdot \mathcal{P}^t \mathbf{U}_h d\mathbf{x} dt = \frac{1}{2} ||\mathbf{U}_h(t_j, \cdot))||^2 - \frac{1}{2} ||\mathbf{U}_h(t_{j-1}, \cdot))||^2, \tag{4.3}$$

$$\int_{I_j} \int_\Omega \mathbf{U}_h \cdot \mathcal{P}^t \mathbf{L}_h d\mathbf{x} dt = \int_{I_j} \int_\Omega \mathbf{L}_h \cdot \mathcal{P}^t \mathbf{U}_h d\mathbf{x} dt, \tag{4.4}$$

$$||\mathcal{P}^t \mathbf{U}_h(\mathbf{x}, \cdot)|| \le ||\mathbf{U}_h(\mathbf{x}, \cdot)||. \tag{4.5}$$

Proof. The definition of $\mathcal{P}^t$ and the linearity of $\mathcal{P}^t$ and ζ yields

$$\int_{I_j} \zeta\Big(\mathbf{U}_h\Big) \mathbf{v}_h dt = \int_{I_j} \mathcal{P}^t \zeta\Big(\mathbf{U}_h\Big) \mathbf{v}_h dt = \int_{I_j} \zeta\Big(\mathcal{P}^t \mathbf{U}_h\Big) \mathbf{v}_h dt,$$

which is exactly (4.2).

By the definition of $\mathcal{P}^t$,

$$\int_{I_j} \int_\Omega \frac{\partial}{\partial t} \mathbf{U}_h \cdot \mathcal{P}^t \mathbf{U}_h dt d\mathbf{x} = \int_\Omega \int_{I_j} \frac{\partial}{\partial t} \mathbf{U}_h \cdot \mathbf{U}_h d\mathbf{x} dt = \frac{1}{2} ||\mathbf{U}_h(\cdot, t_j)||^2 - \frac{1}{2} ||\mathbf{U}_h(\cdot, t_{j-1})||^2.$$

Hence, (4.3)holds.

Direct calculation shows that,

$$\begin{aligned} &\int_{I_j} \int_\Omega \mathbf{U}_h \cdot \mathcal{P}^t \mathbf{L}_h d\mathbf{x} dt - \int_{I_j} \int_\Omega \mathbf{L}_h \cdot \mathcal{P}^t \mathbf{U}_h d\mathbf{x} dt \\ =& \int_{I_j} \int_\Omega (\mathbf{U}_h - \mathcal{P}^t \mathbf{U}_h) \cdot \mathcal{P}^t \mathbf{L}_h d\mathbf{x} dt + \int_{I_j} \int_\Omega (\mathcal{P}^t \mathbf{L}_h - \mathbf{L}_h) \cdot \mathcal{P}^t \mathbf{U}_h d\mathbf{x} dt \\ =& 0, \end{aligned}$$

which leads to (4.4).

(4.5) follows by a straightforward conclusion of (4.2) and the Cauchy-Schwartz inequality. □

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It is worthwhile to point out that the projection operator $\mathcal{P}^t$ plays a significant role in both stability and error analysis later. In our CG-DG method the test function is one degree lower than the shape function, it makes us to take the test function as the projection of the shape function.

Denote

$$\mathbf{B}_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h}\Big(\int_{I_j}\int_K \mathbf{Q}\frac{\partial}{\partial t}\mathbf{U}_h\cdot\mathbf{v}_h d\mathbf{x}dt+\int_{I_j}\int_K \mathbf{L}_h\cdot\mathbf{v}_h d\mathbf{x}dt$$
$$+\int_{I_j}\sum_{e\in\partial K}\int_e(\widehat{\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}_K})\cdot\mathbf{v}_h dsdt-\int_{I_j}\int_K \mathbf{f}(\mathbf{U}_h)\cdot\nabla\mathbf{v}_h d\mathbf{x}dt\Big). \tag{4.6}$$

According to (3.7), we have

$$\mathbf{B}_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathbf{v}_h)=0,\quad \text{for all } \mathbf{v}_h\in(P^{r-1}(I_j)\otimes\mathbf{S}^k_{h,\Omega})^3\oplus P^{r-1}(I_j)\otimes\mathbf{S}^k_{h,\Omega})^3. \tag{4.7}$$

Lemma 4.2. *For each* $\mathbf{U}_h$, $\mathbf{L}_h\in\mathbf{V}_h^{r,k}$, *we have*

$$B_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathcal{P}^t\mathbf{U}_h)=\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t_j)||^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t_{j-1})||^2+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)$$
$$+\int_{I_j}\int_\Omega \mathbf{L}_h\cdot\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt,$$

where

$$\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)=2\int_{I_j}\sum_{e\subset\mathcal{E}_\mathcal{D}}\int_e\frac{1}{Z}\Big|\mathbf{n}\times\mathcal{P}^t\mathbf{E}_h^{int}\Big|^2 dsdt$$
$$+\int_{I_j}\sum_{e\subset\mathcal{E}_\mathcal{I}}\int_e Z\Big|[\![\mathcal{P}^t\mathbf{H}_h(\mathbf{x},t)]\!]_T\Big|^2+\frac{1}{Z}\Big|[\![\mathcal{P}^t\mathbf{E}_h(\mathbf{x},t)]\!]_T\Big|^2 dsdt. \tag{4.8}$$

Proof. Taking $\mathbf{v}_h=\mathcal{P}^t\mathbf{u}_h$ in (4.6), and implementing (4.3) in Lemma 4.1, we obtain

$$\begin{aligned}&B_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathcal{P}^t\mathbf{U}_h)\\&=\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t_j)||^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t_{j-1})||^2+\int_{I_j}\int_\Omega \mathbf{L}_h\cdot\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt\\&+\int_{I_j}\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\int_e(\widehat{\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\mathbf{U}_h dsdt-\int_{I_j}\int_\Omega \mathbf{f}(\mathbf{U}_h)\cdot\nabla\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt.\end{aligned} \tag{4.9}$$

By (4.2) in Lemma 4.1,

$$\begin{aligned}&\int_{I_j}\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\int_e(\widehat{\mathbf{f}(\mathbf{U}_h)\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\mathbf{U}_h dsdt-\int_{I_j}\int_\Omega \mathbf{f}(\mathbf{U}_h)\cdot\nabla\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt\\&=\int_{I_j}\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\int_e(\widehat{\mathbf{f}(\mathcal{P}^t\mathbf{U}_h)\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\mathbf{U}_h dsdt-\int_{I_j}\int_\Omega \mathbf{f}(\mathcal{P}^t\mathbf{U}_h)\cdot\nabla\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt.\end{aligned} \tag{4.10}$$

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According to Lemma 3.1 in [30,33],

$$\int_{I_j}\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\int_e(\mathbf{f}(\widehat{\mathcal{P}^t\mathbf{U}_h})\cdot\mathbf{n}_K)\cdot\mathcal{P}^t\mathbf{U}_h ds dt-\int_{I_j}\int_\Omega\mathbf{f}(\mathcal{P}^t\mathbf{U}_h)\cdot\nabla\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt$$
$$=\frac{1}{2}\Theta_{I_j,T_h}(\mathcal{P}^t\mathbf{U}_h). \tag{4.11}$$

The combination of (4.9)-(4.11) leads to the conclusion. □

Based on the discussion above, we get the following stability result for our CG-DG approach:

Theorem 4.1. *Let* $\mathbf{U}_h=(\mathbf{E}_h,\mathbf{H}_h)^T$ *and* $\mathbf{L}_h=(\mathbf{J}_h,\mathbf{K}_h)^T$ *be the solutions of* (3.7) *and* (3.8). *Then,*

$$\begin{aligned}&\epsilon_0||\mathbf{E}_h(\mathbf{x},T)||^2+\mu_0||\mathbf{H}_h(\mathbf{x},T)||^2+\frac{1}{\epsilon_0\omega_{pe}^2}||\mathbf{J}_h(\mathbf{x},T)||^2+\frac{1}{\mu_0\omega_{pm}^2}||\mathbf{K}_h(\mathbf{x},T)||^2\\&\quad+\Theta_{T,\mathcal{T}_h}(\mathcal{P}^t\mathbf{u}_h)+2\int_0^T||\mathbf{N}^{\frac{1}{2}}\mathcal{P}^t\mathbf{L}(\mathbf{x},t)||^2dt\\=&\epsilon_0||\mathbf{E}_h(\mathbf{x},0)||^2+\mu_0||\mathbf{H}_h(\mathbf{x},0)||^2+\frac{1}{\epsilon_0\omega_{pe}^2}||\mathbf{J}_h(\mathbf{x},0)||^2+\frac{1}{\mu_0\omega_{pm}^2}||\mathbf{K}_h(\mathbf{x},0)||^2,\end{aligned}$$

where

$$\Theta_{T,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)=\sum_{j=0}^{n-1}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h),$$

with $\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{u}_h)$ *depicted in* (4.8).

Proof. Taking $\mathbf{v}_h=\mathcal{P}^t\mathbf{U}_h$ in (4.7)and implementing Lemma 4.2 leads to

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathcal{P}^t\mathbf{U}_h)=&\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_j)||_0^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_{j-1})||_0^2+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)\\&+\int_{I_j}\int_\Omega\mathbf{L}_h\cdot\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt\\=&0.\end{aligned}\tag{4.12}$$

Setting $\mathbf{w}_h=\mathcal{P}^t\mathbf{L}_h$ in (3.8) and implementing (4.4), we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K(\mathbf{S}\frac{\partial}{\partial t}\mathbf{L}_h+\mathbf{N}\mathbf{L}_h)\cdot\mathcal{P}^t\mathbf{L}_h d\mathbf{x}dt=&\int_{I_j}\int_\Omega\mathbf{U}_h\cdot\mathcal{P}^t\mathbf{L}_h d\mathbf{x}dt\\=&\int_{I_j}\int_\Omega\mathbf{L}_h\cdot\mathcal{P}^t\mathbf{U}_h d\mathbf{x}dt.\end{aligned}\tag{4.13}$$

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Hence, plugging (4.13) into (4.12) and using (4.3) and (4.4), we have

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{U}_h,\mathbf{L}_h;\mathcal{P}^t\mathbf{U}_h)=&\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_j)||_0^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_{j-1})||_0^2+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)\\&+\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K(\mathbf{S}\frac{\partial}{\partial t}\mathbf{L}_h+\mathbf{N}\mathbf{L}_h)\cdot\mathcal{P}^t\mathbf{L}_h d\mathbf{x}dt\\=&\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_j)||_0^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,t_{j-1})||_0^2+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)\\&+\frac{1}{2}||\mathbf{S}^{\frac{1}{2}}\mathbf{L}_h(\cdot,t_j)||_0^2-\frac{1}{2}||\mathbf{S}^{\frac{1}{2}}\mathbf{L}_h(\cdot,t_{j-1})||_0^2+\int_{I_j}||\mathbf{N}^{\frac{1}{2}}\mathcal{P}^t\mathbf{L}(\mathbf{x},t)||^2dt\\=&0.\end{aligned}$$

Summing up the above equations over j from 1 to n, we get

$$\begin{aligned}&\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,T)||_0^2-\frac{1}{2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}_h(\cdot,0)||_0^2+\frac{1}{2}||\mathbf{S}^{\frac{1}{2}}\mathbf{L}_h(\cdot,T)||_0^2-\frac{1}{2}||\mathbf{S}^{\frac{1}{2}}\mathbf{L}_h(\cdot,0)||_0^2\\&+\frac{1}{2}\Theta_{T,\mathcal{T}_h}(\mathcal{P}^t\mathbf{U}_h)+\int_T||\mathbf{N}^{\frac{1}{2}}\mathcal{P}^t\mathbf{L}(\mathbf{x},t)||^2dt\\=&0.\end{aligned}$$

This completes the proof of Theorem 4.1. □

5 Error estimates

Now we turn to the error estimates of our CG-DG approach. For this propose, we introduce the projection operators $\mathcal{R}^t$ and $\mathbb{P}_h$ in the following.

Let us introduce the Gauss-Lobatto projection operator $\mathcal{R}^t$ for time dependent functions, as $\mathcal{R}^t: H^{r+1}(0,T)\to\mathbf{S}^r_{\tau,I_j}$, satisfying

$$\begin{aligned}&\int_{I_j}(\mathcal{R}^tu-u)vdt=0\quad\text{for } v\in\mathcal{P}^{r-2}(I_j),\\&(\mathcal{R}^tu)(t_j)=u(t_j),\qquad(\mathcal{R}^t\varphi)(t_{j-1})=u(t_{j-1}).\end{aligned}$$

The projection operator $\mathcal{R}^t$ has the following properties:

Lemma 5.1 ([3]). *If $u\in H^{r+1}(I_j)$ and $\mathcal{R}^tu$ is its Gauss-Lobatto projection operator $\mathcal{R}^t$, then*

$$(u-\mathcal{R}^tu)\perp P^{r-2}\quad\textit{and}\quad||u-\mathcal{R}^tu||_{L^2(I_j)}\le C\tau^{r+1}||u||_{H^{r+1}(I_j)}.$$

Moreover, we also need to introduce the classical L^2-projection operator $\mathbb{P}_h: H^{k+1}(\Omega)\to S^k_{h,\Omega}$ satisfying

$$\int_K\mathbb{P}_huvd\mathbf{x}=\int_K uvd\mathbf{x},\quad\forall v\in P^k(K),\quad\forall K\in\mathcal{T}_h.\tag{5.1}$$

In terms of it, we have the following approximation lemma.

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Lemma 5.2. *Assume that $u\in H^{k+1}(K)$ and $\mathbb{P}_h u$ is its L^2-projection into $P^k(K)$. Then*

$$||u-\mathbb{P}_h u||_{0,K}\leq Ch^{k+1}|u|_{k+1,K},\quad ||u-\mathbb{P}_h u||_{0,\partial K}\leq Ch^{k+1/2}|u|_{k+1,K}.$$

It is known that the error analysis of numerical methods for time-dependent problems is more difficult than those for the time-independent ones. In this paper, we will implement an approach based on the operator-decomposition technique introduced in [31, 33]. To our knowledge, this technique was first proposed by Douglas, Dupont and Wheeler [11], and further used for parabolic and hyperbolic problems by Chen [3].

According to [31, 33], we have the following operation decomposition

$$I-\mathcal{R}^t\otimes\mathbb{P}_h=I-\mathcal{R}^t+I-\mathbb{P}_h-(I-\mathcal{R}^t)\otimes(I-\mathbb{P}_h). \tag{5.2}$$

Actually, the two operators $I-\mathcal{R}^t$ and $I-\mathbb{P}_h$ are independent of each other. By the properties of $\mathcal{R}^t u$ and $\mathcal{P}_h u$, we have the following orthogonality relations:

$$\int_{I_j}\left[(I-\mathcal{R}^t)_t\otimes(I-\mathbb{P}_h)u\right]vdt=0,\quad \forall v\in P^{r-1}(I_j)\times S^k_{h,K},\quad j=1,2,\cdots,n,\quad r\geq 1, \tag{5.3}$$

$$\int_{K}\left[(I-\mathcal{R}^t)\otimes(I-\mathbb{P}_h)u\right]vd\mathbf{x}=0,\quad \forall v\in S^k_{h,K},\quad K\in\mathcal{T}_h. \tag{5.4}$$

It is easy to see that the numerical fluxes defined in (3.9), (3.10) are consistent. Set $\mathbf{e_U}=\mathbf{U}-\mathbf{U}_h$, $\mathbf{e}_L=\mathbf{L}-\mathbf{L}_h$. By (3.7) and (4.6), we have

$$\mathbf{B}_{I_j}(\mathbf{e_U},\mathbf{e_L};\mathbf{v}_h)=0\quad\forall\,\mathbf{v}_h\in(P^{r-1}(I_j)\otimes\mathbf{S}^k_{h,\Omega})^3\oplus P^{r-1}(I_j)\otimes\mathbf{S}^k_{h,\Omega})^3. \tag{5.5}$$

The error $\mathbf{e_U}$, $\mathbf{e}_L$ can be decomposed into

$$\begin{aligned}\mathbf{e}_{U}&=\mathbf{U}-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{U}-(\mathbf{U}_h-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{U})\triangleq\mathbf{R_U}-\theta_\mathbf{U},\\ \mathbf{e}_L&=\mathbf{L}-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{L}-(\mathbf{L}_h-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{L})\triangleq\mathbf{R_L}-\theta_\mathbf{L},\end{aligned} \tag{5.6}$$

where $\mathbf{R_U}=\mathbf{U}-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{U}$, $\theta_\mathbf{U}=\mathbf{U}_h-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{U}$, $\mathbf{R_L}=\mathbf{L}-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{L}$ and $\theta_\mathbf{L}=\mathbf{L}_h-\mathcal{R}^t\otimes\mathbb{P}_h\mathbf{L}$, respectively.

Plugging $e_\mathbf{U}=\mathbf{R_U}-\theta_\mathbf{U}$ and $e_\mathbf{L}=\mathbf{R_L}-\theta_\mathbf{L}$ into (5.5) and taking $v_h=\mathcal{P}^t\theta_\mathbf{U}$, we have

$$\mathbf{B}_{I_j}(\theta_\mathbf{U},\theta_\mathbf{L};\mathcal{P}^t\theta_\mathbf{U})=\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_\mathbf{U}). \tag{5.7}$$

Lemma 5.3. *In terms of $\mathbf{B}_{I_j}(\theta_\mathbf{U},\theta_\mathbf{L};\mathcal{P}^t\theta_\mathbf{U})$, we have the following estimate*

$$\begin{aligned}\mathbf{B}_{I_j}(\theta_\mathbf{U},\theta_\mathbf{L};\mathcal{P}^t\theta_\mathbf{U})=&\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_\mathbf{U}(\cdot,t_j)\|^2_{0,\Omega}-\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_\mathbf{U}(\cdot,t_{j-1})\|^2_{0,\Omega}+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\theta_\mathbf{U})\\&+\int_{I_j}\int_\Omega\theta_\mathbf{L}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt\end{aligned} \tag{5.8}$$

for $j=1,2,\cdots,n$.

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Proof. By the definition of $\mathbf{B}_{I_j}(\mathbf{e_U},\mathbf{e_L};\mathbf{v}_h)$, we have

$$\begin{aligned}\mathbf{B}_{I_j}(\theta_\mathbf{U},\theta_\mathbf{L};\mathcal{P}^t\theta_\mathbf{U}) &= \sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K \mathbf{Q}\frac{\partial}{\partial t}\theta_\mathbf{U}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt+\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_{\partial K}(\widehat{\mathbf{f}(\theta_\mathbf{U})\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\theta_\mathbf{U}dsdt\\ &\quad-\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K\mathbf{f}(\theta_\mathbf{U})\cdot\nabla\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt+\int_{I_j}\int_\Omega\theta_\mathbf{L}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt. \end{aligned}\tag{5.9}$$

Using (4.3), we get

$$\sum_{K\in T_h}\int_{I_j}\int_K\mathbf{Q}\frac{\partial}{\partial t}\theta_\mathbf{U}\cdot\mathcal{P}^t(\theta_\mathbf{U})d\mathbf{x}dt=\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_\mathbf{U}(\cdot,t_j)\|^2_{0,\Omega}-\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_\mathbf{U}(\cdot,t_{j-1})\|^2_{0,\Omega}.\tag{5.10}$$

Following the same strategy as that in the proof of Lemma 4.2, it is easy to show that

$$\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_{\partial K}(\widehat{\mathbf{f}(\theta_\mathbf{U})\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\theta_\mathbf{U}dsdt-\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K\mathbf{f}(\theta_\mathbf{U})\cdot\nabla\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt=\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\theta_\mathbf{U}).\tag{5.11}$$

The combination of (5.9), (5.10) and (5.11) yields (5.8). □

We aim to estimate $\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_\mathbf{U})$ for $j=1,2,\cdots,n$. Firstly,

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_\mathbf{U}) &= \int_{I_j}\int_\Omega\mathbf{Q}\frac{\partial}{\partial t}\mathbf{R_U}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt+\int_{I_j}\int_\Omega\mathbf{R_L}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt\\ &\quad+\sum_{K\in\mathcal{T}_h}\int_{I_j}\sum_{e\in\partial K}\int_e(\widehat{\mathbf{f}(\mathbf{R_U})\cdot\mathbf{n}_K})\cdot\mathcal{P}^t\theta_\mathbf{U}dsdt-\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K\mathbf{f}(\mathbf{R_U})\cdot\nabla\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt\end{aligned}\tag{5.12}$$

for $j=1,2,\cdots n$.

By (5.2), the operator $\mathbf{R_U}$ is decomposed into three parts, i.e.,

$$\mathbf{R_U}=(I-\mathcal{R}^t\otimes\mathbb{P}_h)\mathbf{U}\triangleq\xi_\mathbf{U}+\eta_\mathbf{U}+\rho_\mathbf{U},$$

where $\xi_\mathbf{U}=(I-\mathcal{R}^t)\mathbf{U}$, $\eta_\mathbf{U}=(I-\mathbb{P}_h)\mathbf{U}$, $\rho_\mathbf{U}=-((I-\mathcal{R}^t)\otimes(I-\mathbb{P}_h))\mathbf{U}$.

According to the definitions of the projection operators $\mathcal{P}^t$, $\mathbb{P}_h$, $\mathcal{R}^t$, we have the following orthogonality relations:

$$\begin{aligned}&\int_{I_j}\frac{\partial}{\partial t}\xi_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)dt=0,\quad \int_{I_j}\frac{\partial}{\partial t}\rho_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)dt=0,\quad\forall\,\mathbf{x}\in\Omega,\\ &\int_K\eta_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)d\mathbf{x}=0,\quad \int_K\frac{\partial}{\partial t}\eta_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)d\mathbf{x}=0,\quad\forall t\in(0,T],\\ &\int_K\rho_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)d\mathbf{x}=0,\quad \int_K\frac{\partial}{\partial t}\rho_\mathbf{U}(\mathbf{x},t)\cdot\mathcal{P}^t\theta_\mathbf{U}(\mathbf{x},t)d\mathbf{x}=0,\quad\forall t\in(0,T].\end{aligned}\tag{5.13}$$

Meanwhile, $\mathbf{R_L}$ can be decomposed in the same way and has the same orthogonality relations as introduced in (5.13).

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As a consequence of (5.13),

$$\int_{I_j}\int_\Omega \mathbf{Q}\frac{\partial}{\partial t}\mathbf{R_u}\cdot\mathcal{P}^t\theta_{\mathbf{u}}d\mathbf{x}dt=\int_{I_j}\int_\Omega \mathbf{Q}\Big(\frac{\partial}{\partial t}\xi_{\mathbf{u}}+\frac{\partial}{\partial t}\eta_{\mathbf{u}}+\frac{\partial}{\partial t}\rho_{\mathbf{u}}\Big)\cdot\mathcal{P}^t\theta_{\mathbf{u}}d\mathbf{x}dt=0. \tag{5.14}$$

Lemma 5.4. *In terms of* $\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_{\mathbf{U}})$, *we have*

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_{\mathbf{U}})\leq& Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||^2_{k+1,\Omega}dt+\frac{1}{2}\int_{I_j}\|\mathbf{Q}^{\frac{1}{2}}\mathcal{P}^t\theta_{\mathbf{U}}(\cdot,t)\|^2_{0,\Omega}dt\\&+C\tau^{2r+2}\int_\Omega\Big(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||^2_{r+1,I_j}+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||^2_{r+1,I_j}\Big)d\mathbf{x}\\&+C\tau^{2r+2}\int_\Omega(||\mathbf{J}(\mathbf{x},t)||^2_{r+1,I_j}+||\mathbf{K}(\mathbf{x},t)||^2_{r+1,I_j})d\mathbf{x}+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\theta_{\mathbf{U}}).\end{aligned}$$

Proof. Implementing (5.14) and the decomposition of $\mathbf{R_U},\mathbf{R_L}$, (5.12) can be rewritten as

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{R_u},\mathbf{R_L};\mathcal{P}^t\theta_{\mathbf{U}})=&\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_{\partial K}\Big(\widehat{\mathbf{f}(\mathbf{R_U})\cdot\mathbf{n}}_K\Big)\cdot\mathcal{P}^t\theta_{\mathbf{U}}dsdt-\sum_{K\in\mathcal{T}_h}\int_{I_j}\int_K\mathbf{f}(\mathbf{R_U})\cdot\nabla\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\\&+\int_{I_j}\int_\Omega\mathbf{R_L}\cdot\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\\=&\sum_{K\in\mathcal{T}_h}\Big(\int_{I_j}\int_{\partial K}\Big(\widehat{\mathbf{f}(\xi_{\mathbf{U}})\cdot\mathbf{n}}_K\Big)\cdot\mathcal{P}^t\theta_{\mathbf{U}}dsdt-\int_{I_j}\int_K\mathbf{f}(\xi_{\mathbf{U}})\cdot\nabla\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\Big)\\&+\sum_{K\in\mathcal{T}_h}\Big(\int_{I_j}\int_{\partial K}\Big(\widehat{\mathbf{f}(\eta_{\mathbf{U}})\cdot\mathbf{n}}_K\Big)\cdot\mathcal{P}^t\theta_{\mathbf{U}}dsdt-\int_{I_j}\int_K\mathbf{f}(\eta_{\mathbf{U}})\cdot\nabla\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\Big)\\&+\sum_{K\in\mathcal{T}_h}\Big(\int_{I_j}\int_{\partial K}\Big(\widehat{\mathbf{f}(\rho_{\mathbf{U}})\cdot\mathbf{n}}_K\Big)\cdot\mathcal{P}^t\theta_{\mathbf{U}}dsdt-\int_{I_j}\int_K\mathbf{f}(\rho_{\mathbf{U}})\cdot\nabla\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\Big)\\&+\int_{I_j}\int_\Omega\mathbf{R_L}\cdot\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\\\triangleq&\mathcal{M}_j^1+\mathcal{M}_j^2+\mathcal{M}_j^3+\mathcal{M}_j^4.\end{aligned}\tag{5.15}$$

In the following, we will estimate $\mathcal{M}_j^1,\mathcal{M}_j^2,\mathcal{M}_j^3,\mathcal{M}_j^4$, respectively.

Following almost the same line as that of in [30,33],

$$\begin{aligned}M_j^1\leq&\int_\Omega\int_{I_j}\Big(\frac{1}{\mu_0}|\nabla\times\xi_{\mathbf{E}}|^2+\frac{1}{\epsilon_0}|\nabla\times\xi_{\mathbf{H}}|^2+\frac{\mu_0}{4}|\mathcal{P}^t\theta_{\mathbf{H}}|^2+\frac{\epsilon_0}{4}|\mathcal{P}^t\theta_{\mathbf{E}}|^2\Big)dtd\mathbf{x}\\\leq&C\tau^{2r+2}\int_\Omega\Big(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||^2_{r+1,I_j}+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||^2_{r+1,I_j}\Big)d\mathbf{x}+\frac{1}{4}\int_{I_j}\|\mathbf{Q}^{\frac{1}{2}}\mathcal{P}^t\theta_{\mathbf{U}}(\cdot,t)\|^2_{0,\Omega}dt,\end{aligned}\tag{5.16}$$

$$\begin{aligned}M_j^2\leq&Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||^2_{k+1,\Omega}dt+\frac{1}{4}\int_{I_j}\sum_{e\subset\mathcal{E}_{\mathcal{I}}}\int_e\Big(\frac{1}{Z}\Big|[\![\mathcal{P}^t\theta_{\mathbf{E}}]\!]_T\Big|^2+Z\Big|[\![\mathcal{P}^t\theta_{\mathbf{H}}]\!]_T\Big|^2\Big)dsdt\\&+\frac{1}{2Z}\int_{I_j}\sum_{e\subset\mathcal{E}_{\mathcal{D}}}\int_e\Big|\mathbf{n}\times\mathcal{P}^t\theta^{int}_{\mathbf{E}}\Big|^2dsdt,\end{aligned}\tag{5.17}$$

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and

$$M_j^3 \leq Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||_{k+1,\Omega}^2 dt+\frac{1}{4}\int_{I_j}\sum_{e\subset\mathcal{E}_\mathcal{I}}\int_e\left(\frac{1}{Z}\left|[\![\mathcal{P}^t\theta_\mathbf{E}]\!]_T\right|^2+Z\left|[\![\mathcal{P}^t\theta_\mathbf{H}]\!]_T\right|^2\right)dsdt$$
$$+\frac{1}{2Z}\int_{I_j}\sum_{e\subset\mathcal{E}_\mathcal{D}}\int_e\left|\mathbf{n}\times\mathcal{P}^t\theta_\mathbf{E}^{int}\right|^2 dsdt. \tag{5.18}$$

Finally using the operator decomposition of $\mathbf{R}_L$ and (5.13), we obtain

$$\begin{aligned}M_j^4&=\int_{I_j}\int_\Omega \mathbf{R_L}\cdot\mathcal{P}^t\theta_\mathbf{U}d\mathbf{x}dt\\&=\int_{I_j}\int_\Omega(\xi_\mathbf{J}\cdot\mathcal{P}^t\theta_\mathbf{E}+\xi_\mathbf{K}\cdot\mathcal{P}^t\theta_\mathbf{H})d\mathbf{x}dt\\&\leq\int_\Omega\int_{I_j}\left(\frac{1}{\epsilon_0}|\xi_\mathbf{J}|^2+\frac{1}{\mu_0}|\xi_\mathbf{K}|^2+\frac{\epsilon_0}{4}|\mathcal{P}^t\theta_\mathbf{E}|^2+\frac{\mu_0}{4}|\mathcal{P}^t\theta_\mathbf{H}|^2\right)d\mathbf{x}dt\\&\leq C\tau^{2r+2}\int_\Omega(||\mathbf{J}(\mathbf{x},t)||_{r+1,I_j}^2+||\mathbf{K}(\mathbf{x},t)||_{r+1,I_j}^2)d\mathbf{x}+\frac{1}{4}\int_{I_j}\|\mathbf{Q}^{\frac{1}{2}}\mathcal{P}^t\theta_\mathbf{U}(\cdot,t)\|_{0,\Omega}^2dt.\end{aligned}\tag{5.19}$$

Hence, the combination of (5.15)-(5.19) yields

$$\begin{aligned}\mathbf{B}_{I_j}(\mathbf{R_U},\mathbf{R_L};\mathcal{P}^t\theta_\mathbf{U})\leq{}&Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||_{k+1,\Omega}^2dt+\frac{1}{2}\int_{I_j}\|\mathbf{Q}^{\frac{1}{2}}\mathcal{P}^t\theta_\mathbf{U}(\cdot,t)\|_{0,\Omega}^2dt\\&+C\tau^{2r+2}\int_\Omega\left(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||_{r+1,I_j}^2+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||_{r+1,I_j}^2\right)d\mathbf{x}\\&+C\tau^{2r+2}\int_\Omega(||\mathbf{J}(\mathbf{x},t)||_{r+1,I_j}^2+||\mathbf{K}(\mathbf{x},t)||_{r+1,I_j}^2)d\mathbf{x}+\frac{1}{2}\Theta_{I_j,\mathcal{T}_h}(\mathcal{P}^t\theta_\mathbf{U}).\end{aligned}$$

This completes the proof of Lemma 5.4. □

Theorem 5.1. *Let* $\mathbf{U}_h=(\mathbf{E}_h,\mathbf{H}_h)^T$ *and* $\mathbf{L}_h=(\mathbf{J}_h,\mathbf{K}_h)^T$ *be the solutions of* (3.7) *and* (3.8), *and let* $\mathbf{U}=(\mathbf{E},\mathbf{H})^T$ *and* $\mathbf{L}=(\mathbf{J},\mathbf{K})^T$ *be the exactly solutions of* (3.1) *and* (3.2). *Assume that*

$$\mathbf{E},\mathbf{H},\mathbf{J},\mathbf{K}\in L^2([0,T],(H^{k+1}(\Omega))^3),\quad\nabla\times\mathbf{E},\nabla\times\mathbf{H}\in H^{r+1}([0,T],(L^2(\Omega))^3).$$

Then there exists a constant C *independent of the mesh size* h *and the time step* τ *such that*

$$\begin{aligned}&\epsilon_0||\mathbf{E}(\mathbf{x},T)-\mathbf{E}_h(\mathbf{x},T)||_0+\mu_0||\mathbf{H}(\mathbf{x},T)-\mathbf{H}_h(\mathbf{x},T)||_0\\&\quad+\frac{1}{\epsilon_0\omega_{pe}^2}||\mathbf{J}(\mathbf{x},T)-\mathbf{J}_h(\mathbf{x},T)||_0+\frac{1}{\mu_0\omega_{pm}^2}||\mathbf{K}(\mathbf{x},T)-\mathbf{K}_h(\mathbf{x},T)||_0\\\leq{}&Ch^{k+\frac{1}{2}}|||\mathbf{U}(\cdot,t)|||_{k+1,0}+C\tau^{r+1}(|||\nabla\times\mathbf{E}(\mathbf{x},\cdot)|||_{0,r+1}+|||\nabla\times\mathbf{H}(\mathbf{x},\cdot)|||_{0,r+1})\\&+C\tau^{r+1}(|||\mathbf{J}(\mathbf{x},\cdot)|||_{0,r+1}+|||\mathbf{K}(\mathbf{x},\cdot)|||_{0,r+1})+C\tau^{r+1}|||\mathbf{U}(\mathbf{x},\cdot)|||_{0,r+1}\\&+Ch^{k+1}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,T)||_{0,\Omega}+Ch^{k+1}\|\mathbf{S}^{\frac{1}{2}}\mathbf{L}(\cdot,T)\|_{0,\Omega}.\end{aligned}$$

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Proof. In term of Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_j)\|_{0,\Omega}^2-\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_{j-1})\|_{0,\Omega}^2+\int_{I_j}\int_{\Omega}\theta_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt\\
\leq& Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||_{k+1,\Omega}^2dt+\frac{1}{2}\int_{I_j}\|\mathbf{Q}^{1/2}\mathcal{P}^t\theta_{\mathbf{U}}(\cdot,t)\|_{0,\Omega}^2dt \qquad (5.20)\\
&+C\tau^{2r+2}\int_{\Omega}\left(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||_{r+1,I_j}^2+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||_{r+1,I_j}^2\right)d\mathbf{x}+C\tau^{2r+2}\int_{\Omega}||\mathbf{L}(\mathbf{x},t)||_{r+1,I_j}^2d\mathbf{x}.
\end{aligned}$$

It remains essential to estimate the term $\int_{I_j}\int_{\Omega}\theta_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{U}}d\mathbf{x}dt$.

Using (3.6) and (3.8), we have

$$\int_{I_j}\int_{K}\left(\mathbf{S}\frac{\partial}{\partial t}e_{\mathbf{L}}+\mathbf{N}e_{\mathbf{L}}\right)\cdot\mathbf{w}_hd\mathbf{x}dt=\int_{I_j}\int_{K}e_{\mathbf{U}}\cdot\mathbf{w}_hd\mathbf{x}dt. \qquad (5.21)$$

Summing over all $K\in\mathcal{T}_h$, inserting $e_{\mathbf{U}}=\mathbf{R}_{\mathbf{U}}-\theta_{\mathbf{U}}$ and $e_{\mathbf{L}}=\mathbf{R}_{\mathbf{L}}-\theta_{\mathbf{L}}$ into (5.21) and taking $w_h=\mathcal{P}^t\theta_{\mathbf{L}}$ leads to

$$\begin{aligned}
\int_{I_j}\int_{\Omega}\theta_{\mathbf{U}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt=&\int_{I_j}\int_{\Omega}\mathbf{R}_{\mathbf{U}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt\\
&-\int_{I_j}\int_{K}\left(\mathbf{S}\frac{\partial}{\partial t}(\mathbf{R}_{\mathbf{L}}-\theta_{\mathbf{L}})+\mathbf{N}(\mathbf{R}_{\mathbf{L}}-\theta_{\mathbf{L}})\right)\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt. \qquad (5.22)
\end{aligned}$$

Implementing (4.2) and (4.3),

$$\begin{aligned}
&\int_{I_j}\int_{\Omega}\mathbf{N}\theta_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt=\int_{I_j}||\mathbf{N}^{1/2}\mathcal{P}^t\theta_{\mathbf{L}}||_{0,\Omega}^2dt,\\
&\int_{I_j}\int_{\Omega}\mathbf{S}\frac{\partial}{\partial t}\theta_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt=\frac{1}{2}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_j)\|_{0,\Omega}^2-\frac{1}{2}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_{j-1})\|_{0,\Omega}^2. \qquad (5.23)
\end{aligned}$$

Similar to (5.14), by (5.13), we have

$$\int_{I_j}\int_{\Omega}\mathbf{S}\frac{\partial}{\partial t}\mathbf{R}_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt=0. \qquad (5.24)$$

Similar to the estimate of $\mathcal{M}_j^4$ in Lemma 5.4, a direct calculation yields

$$\begin{aligned}
&\int_{I_j}\int_{\Omega}\mathbf{R}_{\mathbf{U}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt\leq C\tau^{2r+2}\int_{\Omega}||\mathbf{U}(\mathbf{x},t)||_{r+1,I_j}^2d\mathbf{x}+\int_{I_j}\|\mathbf{S}^{\frac{1}{2}}\mathcal{P}^t\theta_{\mathbf{L}}(\cdot,t)\|_{0,\Omega}^2dt,\\
&\int_{I_j}\int_{\Omega}\mathbf{R}_{\mathbf{L}}\cdot\mathcal{P}^t\theta_{\mathbf{L}}d\mathbf{x}dt\leq C\tau^{2r+2}\int_{\Omega}||\mathbf{L}(\mathbf{x},t)||_{r+1,I_j}^2d\mathbf{x}+\int_{I_j}\|\mathbf{S}^{\frac{1}{2}}\mathcal{P}^t\theta_{\mathbf{L}}(\cdot,t)\|_{0,\Omega}^2dt. \qquad (5.25)
\end{aligned}$$

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Substituting (5.23)-(5.25) into (5.20), and using (4.5) in Lemma 4.1, we get

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_j)\|_{0,\Omega}^2-\frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_{j-1})\|_{0,\Omega}^2\\
&\quad+\frac{1}{2}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_j)\|_{0,\Omega}^2-\frac{1}{2}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_{j-1})\|_{0,\Omega}^2+\int_{I_j}||\mathbf{N}^{1/2}\mathcal{P}^t\theta_{\mathbf{L}}||_{0,\Omega}^2dt\\
\le&Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||_{k+1,\Omega}^2dt+\int_{I_j}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t)\|_{0,\Omega}^2dt+\int_{I_j}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t)\|_{0,\Omega}^2dt\\
&+C\tau^{2r+2}\int_{\Omega}\left(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||_{r+1,I_j}^2+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||_{r+1,I_j}^2\right)d\mathbf{x}\\
&+C\tau^{2r+2}\int_{\Omega}(||\mathbf{J}(\mathbf{x},t)||_{r+1,I_j}^2+||\mathbf{K}(\mathbf{x},t)||_{r+1,I_j}^2)d\mathbf{x}+C\tau^{2r+2}\int_{\Omega}\|\mathbf{U}(\mathbf{x},\cdot)||_{r+1,I_j}d\mathbf{x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_j)\|_{0,\Omega}^2+\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_j)\|_{0,\Omega}^2\\
\le&e^{\tau}\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,t_{j-1})\|_{0,\Omega}+e^{\tau}\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,t_{j-1})\|_{0,\Omega}^2+e^{\tau}Ch^{2k+1}\int_{I_j}||\mathbf{U}(\cdot,t)||_{k+1,\Omega}^2dt\\
&+Ce^{\tau}\tau^{2r+2}\int_{\Omega}\left(||\nabla\times\mathbf{E}(\mathbf{x},\cdot)||_{r+1,I_j}^2+||\nabla\times\mathbf{H}(\mathbf{x},\cdot)||_{r+1,I_j}^2\right)d\mathbf{x}\\
&+Ce^{\tau}\tau^{2r+2}\int_{\Omega}(||\mathbf{J}(\mathbf{x},t)||_{r+1,I_j}^2+||\mathbf{K}(\mathbf{x},t)||_{r+1,I_j}^2)d\mathbf{x}+Ce^{\tau}\tau^{2r+2}\int_{\Omega}\|\mathbf{U}(\mathbf{x},\cdot)\|_{r+1,I_j}^2d\mathbf{x}
\end{aligned}$$

by using the Gronwall's inequality.

Summing over $j=1$ to $j=n$,

$$\begin{aligned}
&\|\mathbf{Q}^{\frac{1}{2}}\theta_{\mathbf{U}}(\cdot,T)\|_{0,\Omega}^2+\|\mathbf{S}^{\frac{1}{2}}\theta_{\mathbf{L}}(\cdot,T)\|_{0,\Omega}^2\\
\le&Ch^{2k+1}|||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t)|||_{k+1,0}^2+C\tau^{2r+2}\left(|||\nabla\times\mathbf{E}(\mathbf{x},\cdot)|||_{0,r+1}^2+|||\nabla\times\mathbf{H}(\mathbf{x},\cdot)|||_{0,r+1}^2\right)\\
&+\tau^{2r+2}(|||\mathbf{J}(\mathbf{x},\cdot)|||_{0,r+1}^2+|||\mathbf{K}(\mathbf{x},\cdot)|||_{0,r+1}^2)+C\tau^{2r+2}|||\mathbf{U}(\mathbf{x},\cdot)|||_{0,r+1}^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
||\mathbf{Q}^{\frac{1}{2}}\mathbf{R}_{\mathbf{U}}(\cdot,T)||_{0,\Omega}^2=||\mathbf{Q}^{\frac{1}{2}}\eta_{\mathbf{U}}(\cdot,T)||_{0,\Omega}^2\le Ch^{2k+2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,T)||_{0,\Omega}^2,\\
||\mathbf{S}^{\frac{1}{2}}\mathbf{R}_{\mathbf{L}}(\cdot,T)||_{0,\Omega}^2=||\mathbf{S}^{\frac{1}{2}}\eta_{\mathbf{L}}(\cdot,T)||_{0,\Omega}^2\le Ch^{2k+2}\|\mathbf{S}^{\frac{1}{2}}\mathbf{L}(\cdot,T)\|_{0,\Omega}^2.
\end{aligned}$$

Hence, by the triangular inequality, we have

$$\begin{aligned}
&\|\mathbf{Q}^{\frac{1}{2}}e_{\mathbf{U}}(\cdot,T)\|_{0,\Omega}^2+\|\mathbf{S}^{\frac{1}{2}}e_{\mathbf{L}}(\cdot,T)\|_{0,\Omega}^2\\
\le&Ch^{2k+1}|||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,t)|||_{k+1,0}^2+C\tau^{2r+2}\left(|||\nabla\times\mathbf{E}(\mathbf{x},\cdot)|||_{0,r+1}^2+|||\nabla\times\mathbf{H}(\mathbf{x},\cdot)|||_{0,r+1}^2\right)\\
&+C\tau^{2r+2}(|||\mathbf{J}(\mathbf{x},\cdot)|||_{0,r+1}^2+|||\mathbf{K}(\mathbf{x},\cdot)|||_{0,r+1}^2)+C\tau^{2r+2}|||\mathbf{U}(\mathbf{x},\cdot)|||_{0,r+1}^2\\
&+Ch^{2k+2}||\mathbf{Q}^{\frac{1}{2}}\mathbf{U}(\cdot,T)||_{0,\Omega}^2+Ch^{2k+2}\|\mathbf{S}^{\frac{1}{2}}\mathbf{L}(\cdot,T)\|_{0,\Omega}^2,
\end{aligned}$$

which completes the proof. □

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6 Numerical results

In this section, 3D numerical results are given to justify our theoretical prediction. According to the theoretical analysis above, we know that our numerical scheme is stable without any restriction on the time step size τ. Actually we obtain accurate numerical solutions when τ is equal to h. In our numerical example, we use hexahedral element for simplicity. Moreover, in this section, the L^2-errors are computed in the following way:

$$||\mathbf{u}-\mathbf{u}_h||_0=\Big(\sum_{K\in\mathcal{T}_h}\int_K|\mathbf{u}-\mathbf{u}_h|^2d\mathbf{x}\Big)^{\frac{1}{2}}.$$

We consider the following 3-D problem,

$$\begin{aligned}
&\epsilon_0\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\mathbf{J}(\mathbf{E})=\mathbf{R}_1,\\
&\mu_0\frac{\partial\mathbf{H}}{\partial t}+\nabla\times\mathbf{E}+\mathbf{K}(\mathbf{H})=\mathbf{R}_2,\\
&\frac{\partial\mathbf{J}}{\partial t}+\Gamma_e\mathbf{J}=\epsilon_0\omega_{pe}^2\mathbf{E},\\
&\frac{\partial\mathbf{K}}{\partial t}+\Gamma_m\mathbf{K}=\mu_0\omega_{pm}^2\mathbf{H},
\end{aligned}$$

in $\Omega=[0,1]^3$. In our numerical test, we take $\epsilon_0=\mu_0=\Gamma_e=\Gamma_m=\omega_{pe}=\omega_{pm}=1$ and $\mathbf{R}_1$,$\mathbf{R}_2$ such that the exact solution of this problem is

$$\mathbf{E}=\begin{pmatrix}(y-y^2)(z-z^2)\\(x-x^2)(z-z^2)\\(x-x^2)(y-y^2)\end{pmatrix}te^{-(t+x+y+z)},\qquad \mathbf{H}=\begin{pmatrix}y-z\\z-x\\x-y\end{pmatrix}te^{-(t+x+y+z)},$$

$$\mathbf{J}=\frac{1}{2}\begin{pmatrix}(y-y^2)(z-z^2)\\(x-x^2)(z-z^2)\\(x-x^2)(y-y^2)\end{pmatrix}t^2e^{-(t+x+y+z)},\qquad \mathbf{K}=\frac{1}{2}\begin{pmatrix}y-z\\z-x\\x-y\end{pmatrix}t^2e^{-(t+x+y+z)}.$$

We choose the time step size τ equal to the spatial mesh size h and the same order ($r=k$) polynomials for temporal and spatial discretization. The L^2-errors and their corresponding convergence order are shown in Table 1, Table 2 and Table 3, Table 4 for $r=k=1$ and $r=k=2$, respectively. It is observed that the convergence rate of $\mathbf{E}_h$, $\mathbf{H}_h$, $\mathbf{J}_h$, $\mathbf{K}_h$ in L^2-norm are $\mathcal{O}(h^{k+1})$, which is better than the theoretical prediction.

7 Conclusions

In this paper, we developed an CG-DG method for time-dependent Maxwell's equation when metamaterials are involved. Both the L^2-stability and the convergence rate

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Table 1: The convergence rate of L^2 error for $r=k=1$.

time	time step	h	$\|\|\mathbf{E}(\cdot,T)-\mathbf{E}_h(\cdot,T)\|\|_0$	order	$\|\|\mathbf{H}(\cdot,T)-\mathbf{H}_h(\cdot,T)\|\|_0$	order
T=0.25	N=1	$\frac{1}{4}$	1.4824e-3		1.2234e-3	
	N=2	$\frac{1}{8}$	3.9219e-4	1.9184	3.5858e-4	1.7705
	N=4	$\frac{1}{16}$	9.9438e-5	1.9797	9.4230e-5	1.9280
	N=8	$\frac{1}{32}$	2.5061e-5	1.9884	2.4059e-5	1.9696
T=0.5	N=2	$\frac{1}{4}$	1.6944e-3		2.3683e-3	
	N=4	$\frac{1}{8}$	4.2899e-4	1.9818	6.3735e-4	1.8937
	N=8	$\frac{1}{16}$	1.0738e-4	1.9982	1.6975e-4	1.9087
	N=16	$\frac{1}{32}$	2.6875e-5	1.9984	4.3438e-5	1.9664
T=1	N=4	$\frac{1}{4}$	2.4795e-3		2.2242e-3	
	N=8	$\frac{1}{8}$	6.3574e-4	1.9635	6.0414e-4	1.8803
	N=16	$\frac{1}{16}$	1.5977e-4	1.9924	1.7520e-4	1.7859
	N=32	$\frac{1}{32}$	4.0052e-5	1.9961	4.7093e-5	1.8954

Table 2: The convergence rate of L^2 error for $r=k=1$.

time	time step	h	$\|\|\mathbf{J}(\cdot,T)-\mathbf{J}_h(\cdot,T)\|\|_0$	order	$\|\|\mathbf{K}(\cdot,T)-\mathbf{K}_h(\cdot,T)\|\|_0$	order
T=0.25	N=1	$\frac{1}{4}$	1.3649e-4		4.5465e-4	
	N=2	$\frac{1}{8}$	4.2885e-5	1.6703	1.1386e-4	1.9975
	N=4	$\frac{1}{16}$	1.1552e-5	1.8923	2.8470e-5	1.9997
	N=8	$\frac{1}{32}$	2.9505e-6	1.9691	7.1278e-6	1.9979
T=0.5	N=2	$\frac{1}{4}$	4.2071e-4		5.7778e-4	
	N=4	$\frac{1}{8}$	1.1591e-4	1.8598	1.4355e-4	2.0090
	N=8	$\frac{1}{16}$	2.9912e-5	1.9542	3.6533e-5	1.9743
	N=16	$\frac{1}{32}$	7.6905e-6	1.9596	9.3509e-6	1.9660
T=1	N=4	$\frac{1}{4}$	8.8159e-4		1.0535e-3	
	N=8	$\frac{1}{8}$	2.2838e-4	1.9487	2.8654e-4	1.8784
	N=16	$\frac{1}{16}$	5.7359e-5	1.9933	7.8929e-5	1.8601
	N=32	$\frac{1}{32}$	1.4358e-5	1.9982	2.0583e-5	1.9391

of $\mathcal{O}(\tau^{r+1}+h^{k+\frac{1}{2}})$ are obtained. Numerical examples are given to validate the theoretical predictions. The practical application of this model will be investigated in our future work.

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Table 3: The convergence rate of L^2 error for $r=k=2$.

time	time step	h	$\|\|\mathbf{E}(\cdot,T)-\mathbf{E}_h(\cdot,T)\|\|_0$	order	$\|\|\mathbf{H}(\cdot,T)-\mathbf{H}_h(\cdot,T)\|\|_0$	order
T=0.5	N=1	$\frac{1}{2}$	9.2853e-4		7.8794e-4	
	N=2	$\frac{1}{4}$	1.1371e-4	3.0295	1.0985e-4	2.8425
	N=4	$\frac{1}{8}$	1.4304e-5	2.9908	1.4417e-5	2.9297
	N=8	$\frac{1}{16}$	1.7751e-6	3.0104	1.8731e-6	2.9443
	N=16	$\frac{1}{32}$	2.2307e-7	2.9923	2.3960e-7	2.9667
T=1	N=2	$\frac{1}{2}$	1.0426e-3		1.0095e-3	
	N=4	$\frac{1}{4}$	1.4800e-4	2.8165	1.3985e-4	2.8516
	N=8	$\frac{1}{8}$	1.8554e-5	2.9957	1.8325e-5	2.9320
	N=16	$\frac{1}{16}$	2.2735e-6	3.0287	2.2570e-6	3.0213
	N=32	$\frac{1}{32}$	2.8593e-7	2.9912	2.8103e-7	3.0056

Table 4: The convergence rate of L^2 error for $r=k=2$.

time	time step	h	$\|\|\mathbf{J}(\cdot,T)-\mathbf{J}_h(\cdot,T)\|\|_0$	order	$\|\|\mathbf{K}(\cdot,T)-\mathbf{K}_h(\cdot,T)\|\|_0$	order
T=0.5	N=1	$\frac{1}{2}$	1.8245e-4		1.9126e-4	
	N=2	$\frac{1}{4}$	2.6560e-5	2.7801	2.6611e-5	2.8454
	N=4	$\frac{1}{8}$	3.4432e-6	2.9474	3.5234e-6	2.9170
	N=8	$\frac{1}{16}$	4.3340e-7	2.9900	4.5743e-7	2.9453
	N=16	$\frac{1}{32}$	5.4249e-8	2.9980	5.5743e-8	2.9611
T=1	N=2	$\frac{1}{2}$	4.9254e-4		4.7190e-4	
	N=4	$\frac{1}{4}$	6.9282e-5	2.8296	6.6806e-5	2.8204
	N=8	$\frac{1}{8}$	8.7956e-6	2.9776	8.8239e-6	2.9204
	N=16	$\frac{1}{16}$	1.0907e-6	3.0115	1.1552e-6	2.9333
	N=32	$\frac{1}{32}$	1.3597e-7	3.0039	1.4478e-7	2.9962

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