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Kneading theory for tree maps

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Abstract. Using the techniques developed in Alves and Sousa Ramos (*Comm. Math. Phys.* **204**(1) (1999), 89–114), we generalize the Milnor and Thurston results about the zeta function, semiconjugacy and topological entropy of interval maps to tree maps.

1. *Introduction*

Motivated by some applications concerning the study of two-dimensional real and one-dimensional complex dynamics, continuous maps from a tree into itself have been studied in recent years by several authors (see [**12**]). It is well-known that important problems such as how to obtain estimates for the topological entropy of a tree map (see [**4, 5**]), and how to characterize the set of all periods of a tree map (see [**11**]), are closely related with the study of the Artin–Mazur zeta function of a tree map.

Including the results which appeared in [**2**] and in the PhD thesis of the first author, the aim of the present paper is to extend the kneading theory developed by Milnor and Thurston in [**18**] to tree maps.

The main difficulty in generalizing the kneading theory consists in obtaining a good replacement for the kneading matrix introduced by Milnor and Thurston for interval maps. To circumvent this difficulty, we introduce the pair of linear endomorphisms $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$.

Section 2 is devoted to the construction of this pair which is a natural extension of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ introduced in [**3**] for interval maps. Also here this pair of linear endomorphisms presents two fundamental properties which play an important role. As an immediate consequence of the functoriality of $\epsilon_{\#0}$ and $\epsilon_{\#1}$ we obtain

$$\epsilon_{\#0}(f)^n = \epsilon_{\#0}(f^n) \quad \text{and} \quad \epsilon_{\#1}(f)^n = \epsilon_{\#1}(f^n),$$

which show that these endomorphisms behave well under iteration. The nature of the second property is exclusively algebraic. We prove that the pair of linear endomorphisms $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ has finite rank, which will enable us to define the trace and the determinant of this pair (see Appendix A). This determinant is the characteristic

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polynomial of a finite matrix $M(t)$ with entries in the ring $\mathbb{Q}[[t]]$, and will play an analogous role to the kneading determinant. Observe also that, for the special case of Markov maps, this determinant coincides with the characteristic polynomial of the Markov transition matrix of the map.

Section 3 is devoted to the computation of the Artin–Mazur [**7**] zeta function $\zeta(t)$ of a tree map. Another approach to computing $\zeta(t)$ can be found in [**8**] (see also [**16, 22, 10, 15**]). The main result of this section shows that the determinant of $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ plays a relevant role in the computation of $\zeta(t)$, and extends the Milnor and Thurston relation for tree maps

$$\exp \sum_{n\geq 1} (2N^{-}_{f^n} - 1)\frac{t^n}{n} = D_f(t)^{-1}, \tag{1}$$

where $D_f(t)$ denotes the kneading determinant of the interval map $f : I \to I$, and $N^{-}_{f^n}$ denotes the number of fixed points of f^n of negative type (a fixed point of f^n, lying in the interior of I, is of negative type if f^n is decreasing in a neighborhood of that point).

In §4 we study the existence of piecewise linear models for tree maps. Recently, Baillif and de Carvalho [**9**] generalized the theorems of Parry [**23**] and Milnor–Thurston for tree maps, which state that every piecewise monotone interval map with positive topological entropy $\log(s)$ is semiconjugated to a piecewise linear interval map with slope $\pm s$ everywhere. Using total variations, we give in this section an alternative way to construct these semiconjugacies. The main theorem in this section generalizes the Parry and Milnor–Thurston theorems, and furthermore shows an unexpected role played by total variations on the construction of other piecewise linear models.

Finally, in §5 we extend for tree maps the main relation between the topological entropy $h_t(f)$ and the growth number of fixed points of negative type (see [**19**, Theorem 4.11]) which states that

$$h_t(f) = \log\left(\max\left\{1, \overline{\lim_n} \sqrt[n]{N^{-}_{f^n}}\right\}\right).$$

2. *The functors $\epsilon_{\#0}$ and $\epsilon_{\#1}$*

A simply connected topological space T is a tree if there exist subspaces $I_1, \ldots, I_m$ of T such that $T = I_1 \cup I_2 \cup \cdots \cup I_m$, and each I_i is homeomorphic to $[0, 1]$. As usual, if T is a tree and $x \in T$, we define the valence of x to be the number of connected components of $T\backslash\{x\}$. The valence of x will be denoted by $\mathrm{val}(x)$ and a point $x \in T$ is a vertex of T if $\mathrm{val}(x) \neq 2$. The set of all vertices of T is denoted by V_T and the closures of the connected components of $T\backslash V_T$ are called the edges of T. A connected (and non-empty) set I contained in some edge of T is called an interval of T. Given $x, y \in T$ we denote by $\langle x, y\rangle$ the convex hull of $\{x, y\}$, in other words $\langle x, y\rangle$ is the smallest connected set containing $\{x, y\}$. Note that, in general, $\langle x, y\rangle$ is not an interval of T.

Let $g : T \to T'$ be a continuous map between trees and let I be an interval of T such that $\mathrm{int}(I) \neq \emptyset$ and $g(I)$ is an interval of T'. Under these conditions, we say that g is monotone on I if either g is constant on I or g is injective on I. We say that g is piecewise monotone if it is possible to cover T by finitely many intervals of monotonicity of g. If, in addition to this, the map g has no intervals of constancy, then it will be called piecewise strictly monotone.

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By a pointed tree we mean a pair (T, K) where T is a tree and K is a finite subset of T containing all vertices of T. If (T, K) is a pointed tree, then the elements of K are the vertices of (T, K) and the closures of the connected components of $T\backslash K$ are called the edges of (T, K). A connected and non-empty set $I \subseteq T$ is called an interval of (T, K) if there exists an edge T_i of (T, K) such that $I \subseteq T_i$. Given pointed trees (T, K) and (T', K'), a map $g : T \to T'$ will be called vertex preserving if $g(K) \subseteq K'$.

To define the functors $\epsilon_{\#0}$ and $\epsilon_{\#1}$ we have to consider an additional structure on the pointed tree (T, K). By an oriented tree we mean a pointed tree (T, K) where each edge T_i is equipped with a linear ordering $\geq_i$ satisfying the following: there exists a homeomorphism φ_i from T_i into $[0, 1] \subset \mathbb{R}$ that satisfies:

$$y \geq_i x \text{ if and only if } \varphi_i(y) \geq \varphi_i(x), \quad \text{for all } x, y \in T_i.$$

Let x and y be points of T, lying in the same edge T_i of the oriented tree (T, K). If $y \geq_i x$ we define the closed interval

$$[x, y] \stackrel{\text{def}}{=} \{z \in T_i : y \geq_i z \geq_i x\}.$$

Observe that every closed interval of (T, K) has this form, and more generally, any interval of (T, K) has one of the following forms: $]x, y] \stackrel{\text{def}}{=} [x, y]\backslash\{x\}$, $[x, y[\stackrel{\text{def}}{=} [x, y]\backslash\{y\}$ and $]x, y[\stackrel{\text{def}}{=} [x, y]\backslash\{x, y\}$.

Let $g : T \to T'$ be a piecewise monotone map. If (T, K) and (T', K') are oriented trees and $I = [x, y]$ is a closed interval of (T, K) on which g is monotone, we know that there exists an edge T'_j of (T', K') such that $g([x, y]) \subseteq T'_j$, and therefore we can define the sign

$$\epsilon_{g,1}([x, y]) = \begin{cases} 0 & \text{if } g(x) = g(y) \\ -1 & \text{if } g(x) >_j g(y) \\ 1 & \text{if } g(y) >_j g(x). \end{cases} \tag{2}$$

Let T be a tree and let $S_0(T, \mathbb{R})$ be the $\mathbb{R}$-vector space whose basis is the set of formal symbols x, with $x \in T$†. Denote by $S_1(T, \mathbb{R})$ the subspace of $S_0(T, \mathbb{R})$ which is generated by

$$\{y - x \in S_0(T, \mathbb{R}) : x, y \in T\}.$$

If (T, K) is an oriented tree and $I = [x, y]$ is a closed interval of (T, K), we use the symbol $\overrightarrow{I}$ to denote the vector $y - x \in S_1(T, \mathbb{R})$. More generally, if $X \neq \emptyset$ is a closed and connected subset of T and $T_1, \ldots, T_k$ are the edges of (T, K) which intersect X, we define the vector

$$\overrightarrow{X} = \overrightarrow{I_1} + \cdots + \overrightarrow{I_k} \in S_1(T, \mathbb{R}) \tag{3}$$

with $I_i = X \cap T_i$. Observe that, with this notation, we can say that $S_1(T, \mathbb{R})$ is the subspace of $S_0(T, \mathbb{R})$ which is generated by

$$\mathcal{I} = \{\overrightarrow{I} : I \text{ is a closed interval of } (T, K)\}. \tag{4}$$

† More generally, for an arbitrary set X, we denote by $S_0(X, \mathbb{R})$ the $\mathbb{R}$-vector space whose basis is the set of formal symbols x, with $x \in X$. The subspace of $S_0(X, \mathbb{R})$ which is generated by $\{y - x \in S_0(X, \mathbb{R}) : x, y \in X\}$ will be denoted by $S_1(X, \mathbb{R})$.

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Now we can define the functor $\epsilon_{\#1}$. Any tree map $g : T \to T'$ induces the linear maps

$$g_{\#0} : S_0(T, \mathbb{R}) \to S_0(T', \mathbb{R}) \quad \text{and} \quad g_{\#1} : S_1(T, \mathbb{R}) \to S_1(T', \mathbb{R}),$$

defined by: $g_{\#0}(x) = g(x)$, for all $x \in T$, and $g_{\#1} = g_{\#0}|_{S_1(T,\mathbb{R})}$. In other words, $g_{\#1}$ is the unique linear map from $S_1(T, \mathbb{R})$ into $S_1(T', \mathbb{R})$ that satisfies

$$g_{\#1}(y - x) = g(y) - g(x), \quad \text{for all } x, y \in T. \tag{5}$$

Assume now that (T, K) and (T', K') are oriented trees. If $g : T \to T'$ is a piecewise monotone map, we know that there exists a finite family $\{I_k\}$ of intervals of (T, K) on which g is monotone and such that $T = \cup I_k$. In algebraic terms, this means that the subset of $\mathcal{I}$

$$\mathcal{I}_g = \{\overrightarrow{I} \in \mathcal{I} : g \text{ is monotone on } I\} \tag{6}$$

generates $S_1(T, \mathbb{R})$, and from (5) we conclude that $g_{\#1}$ is the unique linear map from $S_1(T, \mathbb{R})$ into $S_1(T', \mathbb{R})$ that satisfies

$$g_{\#1}(\overrightarrow{I}) = \epsilon_{g,1}(I) \cdot \overrightarrow{g(I)}, \quad \text{for all } \overrightarrow{I} \in \mathcal{I}_g. \tag{7}$$

Using $\epsilon_{g,1}$ and $g_{\#1}$ we define $\epsilon_{\#1}(g)$.

Definition 1. Let (T, K) and (T', K') be oriented trees and let $g : T \to T'$ be a piecewise monotone map. Define $\epsilon_{\#1}(g)$ to be the unique linear map from $S_1(T, \mathbb{R})$ into $S_1(T', \mathbb{R})$ that satisfies

$$\epsilon_{\#1}(g)(\overrightarrow{I}) = \epsilon_{g,1}(I) \cdot g_{\#1}(\overrightarrow{I}), \quad \text{for all } \overrightarrow{I} \in \mathcal{I}_g.$$

Observe that, from (7), $\epsilon_{\#1}(g)$ can be regarded as the unique linear map from $S_1(T, \mathbb{R})$ into $S_1(T', \mathbb{R})$ that satisfies

$$\epsilon_{\#1}(g)(\overrightarrow{I}) = \overrightarrow{g(I)}, \quad \text{for all } \overrightarrow{I} \in \mathcal{I}_g, \tag{8}$$

and as an immediate consequence of this we have the following.

PROPOSITION 2. *Let £ be the category whose objects are the oriented trees and whose morphisms are piecewise monotone maps between oriented trees. Then the association*

$$(T, K) \to S_1(T, \mathbb{R})$$
$$g \to \epsilon_{\#1}(g)$$

is a covariant functor from £ to the category of vector spaces over $\mathbb{R}$.

Now we define the linear map $\epsilon_{\#0}(g)$. In order to obtain a functor we define the linear map $\epsilon_{\#0}(g)$ on $S_0(T\backslash K, \mathbb{R})$. Let (T, K) and (T', K') be oriented trees and let $g : T \to T'$ be a piecewise monotone map. Define the map $\epsilon_{g,0} : T \to \{-1, 0, 1\}$ as follows: for any $x \in T\backslash K$ lying in the interior of some interval I on which g is monotone, define $\epsilon_{g,0}(x) = \epsilon_{g,1}(I)$; for the remaining points of T, define $\epsilon_{g,0}(x) = 0$.

It is clear that, in general, the linear map $g_{\#0} : S_0(T, \mathbb{R}) \to S_0(T', \mathbb{R})$ does not map $S_0(T\backslash K, \mathbb{R})$ into $S_0(T'\backslash K', \mathbb{R})$. However, if $g(x) \in K'$, then $\epsilon_{g,0}(x) = 0$, and therefore

$$\epsilon_{g,0}(x) \cdot g_{\#0}(x) \in S_0(T'\backslash K', \mathbb{R}), \quad \text{for all } x \in T\backslash K.$$

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Definition 3. Let (T, K) and (T', K') be oriented trees and let $g : T \to T'$ be a piecewise monotone map. Define $\epsilon_{\#0}(g)$ to be the unique linear map from $S_0(T\backslash K, \mathbb{R})$ into $S_0(T'\backslash K', \mathbb{R})$ that satisfies

$$\epsilon_{\#0}(g)(x) = \epsilon_{g,0}(x) \cdot g_{\#0}(x), \quad \text{for all } x \in T\backslash K.$$

Let (T, K), (T', K') and (T'', K'') be oriented trees, and let $g : T \to T'$ and $h : T' \to T''$ be piecewise monotone maps. In general, we have

$$\epsilon_{\#0}(h) \circ \epsilon_{\#0}(g) \neq \epsilon_{\#0}(h \circ g),$$

however, if $g^{-1}(K') \subseteq K$ or $h(K') \subseteq K''$, then

$$\epsilon_{\#0}(h) \circ \epsilon_{\#0}(g) = \epsilon_{\#0}(h \circ g). \tag{9}$$

Thus we can write the following.

PROPOSITION 4. *Let $\pounds'$ be the category whose objects are oriented trees (T, K) and whose morphisms are vertex preserving piecewise monotone tree maps. Then the association*

$$(T, K) \to S_0(T\backslash K, \mathbb{R})$$
$$g \to \epsilon_{\#0}(g)$$

is a covariant functor from $\pounds'$ to the category of vector spaces over $\mathbb{R}$.

3. *The determinant of* $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$

Our goal is to study the endomorphisms of an oriented tree (T, K). From now on, an endomorphism of $\pounds$ (see Proposition 2) will be called a K-piecewise monotone map. An endomorphism of $\pounds'$ (see Proposition 4) will be called a K-preserving piecewise monotone map. In other words, a K-piecewise monotone map is simply a piecewise monotone map $f : T \to T$, regarded as an endomorphism of (T, K), and a K-preserving piecewise monotone map is a K-piecewise monotone map that verifies $f(K) \subseteq K$. So, for each K-piecewise monotone map f, we obtain two linear endomorphism, $\epsilon_{\#0}(f)$ and $\epsilon_{\#1}(f)$, defined in different subspaces of $S_0(T, \mathbb{R})$: the first is defined on $S_0(T\backslash K, \mathbb{R})$, and the second is defined on $S_1(T, \mathbb{R})$.

Note that, by Proposition 2, if f is a K-piecewise monotone map, then we have

$$\epsilon_{\#1}(f^n) = \epsilon_{\#1}(f)^n, \tag{10}$$

for all $n \geq 0$. However, as we have mentioned before, the equation $\epsilon_{\#0}(f^n) = \epsilon_{\#0}(f)^n$ may not hold for a map which is only K-piecewise monotone. Nevertheless, as a consequence of Proposition 4, if f is a K-preserving piecewise monotone map, then we obtain

$$\epsilon_{\#0}(f^n) = \epsilon_{\#0}(f)^n, \tag{11}$$

for all $n \geq 0$. Indeed, this is the reason why we consider K-preserving maps.

These relations will play an essential role in the sequel, as will the following theorem which states that the pair of linear endomorphisms $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ has finite rank (see Definition 36). This fact enables us to define the trace and the determinant of this pair.

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THEOREM 5. *Let (T, K) be an oriented tree and let f be a K-piecewise monotone map. Then the pair of linear endomorphisms $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ has finite rank.*

Since the linear endomorphisms $\epsilon_{\#0}(f)$ and $\epsilon_{\#1}(f)$ are defined in different subspaces of $S_0(T, \mathbb{R})$, to prove Theorem 5 we introduce the endomorphisms θ_0 and θ_1 that are extensions of $\epsilon_{\#0}(f)$ and $\epsilon_{\#1}(f)$, respectively, to

$$S_0(T, \mathbb{R}) = S_0(T\backslash K, \mathbb{R}) + S_1(T, \mathbb{R}).$$

Definition 6. Let (T, K) be an oriented tree, and let f be a K-piecewise monotone map. Define θ_0 to be the unique linear endomorphism of $S_0(T, \mathbb{R})$ such that the diagram

$$\begin{array}{ccc} S_0(T\backslash K, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \\ \downarrow{\scriptstyle \epsilon_{\#0}(f)} & & \downarrow{\scriptstyle \theta_0} \\ S_0(T\backslash K, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \end{array}$$

is commutative, and $\theta_0(x) = 0$, for all $x \in K$.

Definition 7. Let (T, K) be an oriented tree, and let f be a K-piecewise monotone map. Define θ_1 to be the unique linear endomorphism of $S_0(T, \mathbb{R})$ such that the diagram

$$\begin{array}{ccc} S_1(T, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \\ \downarrow{\scriptstyle \epsilon_{\#1}(f)} & & \downarrow{\scriptstyle \theta_1} \\ S_1(T, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \end{array}$$

is commutative, and $\theta_1(\sum_{x\in K} x) = 0$.

We have then

$$\theta_0(y) = \begin{cases} \epsilon_{\#0}(f)(y) & \text{if } y \in T\backslash K \\ 0 & \text{if } y \in K \end{cases} \in S_0(T\backslash K, \mathbb{R}),$$

and

$$\theta_1(y) = \frac{1}{\#K} \sum_{x\in K} \epsilon_{\#1}(f)(y - x) \in S_1(T, \mathbb{R})$$

for all $y \in T$, and consequently $\mathrm{Im}(\theta_0) \subseteq S_0(T\backslash K, \mathbb{R})$ and $\mathrm{Im}(\theta_1) \subseteq S_1(T, \mathbb{R})$. Thus, from Proposition 37, to prove Theorem 5 it suffices to show that $\theta_1 - \theta_0$ has finite rank. Let $T_1, \ldots, T_m$ be the edges of the oriented tree (T, K). For each $x \in T$ define the linear form ω_x on $S_0(T, \mathbb{R})$ as follows: set $X_0 = \{x\}$ and denote by $X_1, \ldots, X_{\mathrm{val}(x)}$ the connected components of $T\backslash\{x\}$. For each $j = 0, 1, \ldots, \mathrm{val}(x)$, define

$$\nu(X_j) = \#(X_j \cap K) \quad \text{and} \quad \epsilon(X_j) = \lim_{\substack{y\to x \\ y\in X_j}} \epsilon_{f,0}(y).$$

Define ω_x to be the unique linear form on $S_0(T, \mathbb{R})$ that satisfies

$$\omega_x(y) = -\epsilon(X_l) + (m + 1)^{-1} \sum_{j=0}^{\mathrm{val}(x)} \epsilon(X_j)\nu(X_j), \tag{12}$$

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for all $y \in X_l$ and $l = 0, 1, \ldots, \mathrm{val}(x)$. Observe that the linear form ω_x measures the discontinuity of the map $\epsilon_{f,0}$ at x. Indeed, if we define

$$C_f \stackrel{\text{def}}{=} \{x \in T : \epsilon_{f,0} \text{ is discontinuous at } x\}, \tag{13}$$

then we obtain

$$C_f = \{x \in T : \omega_x \neq 0\}. \tag{14}$$

From the definition of $\epsilon_{f,0}$, we also have that C_f is a finite set. Thus, as a consequence of the following proposition we see that $\theta_1 - \theta_0$ has finite rank. The proof of the next proposition is a simple verification.

PROPOSITION 8. *Let (T, K) be an oriented tree, and let f be a K-piecewise monotone map. Then we have $\theta_0(y) = \epsilon_{f,0}(y) \cdot f_{\#0}(y)$, for all $y \in T$, and*

$$\theta_1 - \theta_0 = \sum_{x \in C_f} \omega_x \otimes f_{\#0}(x).$$

As a consequence of Theorem 5 we know that, for each K-piecewise monotone map f, the trace of $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ and the corresponding determinant (Definition 39)

$$D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) \stackrel{\text{def}}{=} \exp \sum_{n \geq 1} -\frac{t^n}{n} \mathrm{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n)$$

are defined. From Propositions 8 and 40, we can use the same pair (θ_0, θ_1) to compute the formal power series $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$: considering the matrix with entries in $\mathbb{Q}[[t]]$

$$M(t) = \begin{bmatrix} \sum_{n \geq 0} \omega_{x_1}(\theta_0^n(f_{\#0}(x_1)))t^n & \ldots & \sum_{n \geq 0} \omega_{x_1}(\theta_0^n(f_{\#0}(x_l)))t^n \\ \vdots & & \vdots \\ \sum_{n \geq 0} \omega_{x_l}(\theta_0^n(f_{\#0}(x_1)))t^n & \ldots & \sum_{n \geq 0} \omega_{x_l}(\theta_0^n(f_{\#0}(x_l)))t^n \end{bmatrix}, \tag{15}$$

with $\{x_1, \ldots, x_l\} = C_f$, we have the following.

PROPOSITION 9. *Let (T, K) be an oriented tree and let f be a K-piecewise monotone map. Then we have*

$$D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = \mathrm{Det}(\mathrm{id} - tM(t)).$$

Example 10. The matrix $M(t)$ contains all the symbolic information on the orbits of the points of C_f. If, in particular, we know the symbolic orbits of these points, then the matrix $M(t)$ can be computed explicitly. As an example, let (T, K) be the oriented tree defined by $K = \{1, 0, i, -i\}$ and $T = \{x \in \mathbb{C} : |x| \leq 1, \arg(x) = -\pi/2, 0, \pi/2\} = [0, 1] \cup [0, i] \cup [-i, 0]$. Let $f : T \to T$ be the K-preserving piecewise monotone map defined by

$$f(x) = \begin{cases} i(1 + 2\sqrt{2})x(1 - x) & \text{if } x \in [0, 1] \\ -x & \text{if } x \in [0, i] \\ ix & \text{if } x \in [-i, 0]. \end{cases}$$

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We have $C_f = \{x_1, x_2, x_3, x_4, x_5\}$ with $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$, $x_4 = i$, and $x_5 = -i$. From the construction of f, we also have $f(x_1) = x_1$, $f(x_3) = x_1$, $f(x_4) = x_5$ and $f(x_5) = x_1$. Note that the orbit of x_2 is not finite, nevertheless it can be shown that

$$f^{3k-2}(x_2) \in]x_1, x_4[, \quad f^{3k-1}(x_2) \in]x_5, x_1[, \quad f^{9k-6}(x_2) \in]x_2, x_3[,$$
$$f^{9k}(x_2) \in]x_2, x_3[\quad \text{and} \quad f^{9k-3}(x_2) \in]x_1, x_2[$$

for all $k \geq 1$. With this, we obtain

$$M(t) = \begin{bmatrix} -\frac{1}{4} & \frac{(3-3t-5t^2)(1-t^3-t^6)}{4(1-t^9)} & -\frac{1}{4} & \frac{3}{4} & -\frac{5}{4} \\ -\frac{1}{2} & \frac{(-1+t+3t^2)(1-t^3-t^6)+4t^5}{2(1-t^9)} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{4} & \frac{(1-t+t^2)(1-t^3-t^6)}{4(1-t^9)} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{(1-t+t^2)(1-t^3-t^6)}{4(1-t^9)} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{(1-t+t^2)(1-t^3-t^6)}{4(1-t^9)} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix},$$

and from Proposition 9,

$$D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = \frac{1-t^3-t^6}{1+t^3+t^6}.$$

3.1. *Markov maps.* Now, we are going to see that, for a very special class of piecewise monotone maps, there is an alternative way of computing $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$. Let (T, K) be an oriented tree and let f be a K-preserving piecewise monotone map. Denoting the edges of (T, K) by $T_1, \ldots, T_m$, we say that f is Markov if either f is constant on T_i or f is injective on T_i, for all $i = 1, \ldots, m$. It is easy to see that, for these maps, the determinant $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ is always a polynomial. In fact, if f is Markov, then we have $f(C_f) \subseteq K$, and therefore the entries of the matrix $M(t)$, defined in (15), are rational numbers. More precisely this matrix has the form

$$M = \begin{bmatrix} \omega_{x_1}(f_{\#0}(x_1)) & \ldots & \omega_{x_1}(f_{\#0}(x_l)) \\ \vdots & & \vdots \\ \omega_{x_l}(f_{\#0}(x_1)) & \ldots & \omega_{x_l}(f_{\#0}(x_l)) \end{bmatrix}, \tag{16}$$

with $\{x_1, \ldots, x_l\} = C_f$, and from Proposition 9 we obtain

$$D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = \mathrm{Det}(\mathrm{id} - tM) \quad \text{and} \quad \mathrm{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n) = \mathrm{Tr}(M^n), \tag{17}$$

for all $n \geq 1$. Observe also that, if f is Markov, then $S_1(K, \mathbb{R})$ is an $\epsilon_{\#1}(f)$-invariant subspace of $S_1(T, \mathbb{R})$, so, denoting by P the matrix that represents $\epsilon_{\#1}(f)|_{S_1(K,\mathbb{R})}$ with respect to the basis $\overrightarrow{T_1}, \overrightarrow{T_2}, \ldots, \overrightarrow{T_m}$ (see (3)), we obtain

$$\mathrm{Tr}(P^n) = \mathrm{Tr}(\epsilon_{\#1}(f)^n|_{S_1(K,\mathbb{R})}) \tag{18}$$

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for all $n \geq 0$. Note that the matrix P coincides with what is usually called the Markov transition matrix of f. In fact, from the definition of $\epsilon_{\#1}(f)$, we see that P is an $m \times m$ matrix with coefficients in $\{0, 1\}$ and such that

$$P(i, j) = \begin{cases} 1 & \text{if } T_i \subseteq f(T_j) \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Using (18) and (17) we now prove that, for Markov maps, $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ coincides with the characteristic polynomial of P.

THEOREM 11. *Let (T, K) be an oriented tree and let $f : T \to T$ be a K-preserving piecewise monotone map. If f is Markov, then we have*

$$\mathrm{Tr}(P^n) = \mathrm{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n)$$

for all $n \geq 1$, and

$$D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = \mathrm{Det}(\mathrm{id} - tP).$$

Proof. From (17) and (18) it suffices to show that

$$\mathrm{Tr}(\epsilon_{\#1}(f)^n|_{S_1(K,\mathbb{R})}) = \mathrm{Tr}(M^n), \quad \text{for all } n \geq 1,$$

where M is the matrix defined in (16). From Definition 7 we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_1(K,\mathbb{R}) & \longrightarrow & S_0(K,\mathbb{R}) & \longrightarrow & \dfrac{S_0(K,\mathbb{R})}{S_1(K,\mathbb{R})} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon_{\#1}(f)|_{S_1(K,\mathbb{R})}} & & \downarrow{\scriptstyle \theta_1|_{S_0(K,\mathbb{R})}} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & S_1(K,\mathbb{R}) & \longrightarrow & S_0(K,\mathbb{R}) & \longrightarrow & \dfrac{S_0(K,\mathbb{R})}{S_1(K,\mathbb{R})} & \longrightarrow & 0 \end{array}$$

with exact rows, and therefore

$$\mathrm{Tr}(\epsilon_{\#1}(f)^n|_{S_1(K,\mathbb{R})}) = \mathrm{Tr}(\theta_1^n|_{S_0(K,\mathbb{R})}), \quad \text{for all } n \geq 1.$$

On the other hand, from Proposition 8, we also conclude that $\theta_0|_{S_0(K,\mathbb{R})} = 0$ and

$$\theta_1|_{S_0(K,\mathbb{R})} = \sum_{x \in C_f} (\omega_x|_{S_0(K,\mathbb{R})}) \otimes f_{\#0}(x),$$

and consequently

$$\mathrm{Tr}(\theta_1^n|_{S_0(K,\mathbb{R})}) = \mathrm{Tr}(M^n),$$

for all $n \geq 1$, as desired. □

Theorem 11 shows in particular that, for Markov maps, $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ defines a holomorphic function on $\mathbb{C}$. For piecewise monotone maps in general this is not true (see Example 10), however, from (12), we see easily that

$$|\omega_x(\theta_0^n(y))| = |\epsilon_{f,0}(y) \cdots \epsilon_{f,0}(f^{n-1}(y))\omega_x(f_{\#0}^n(y))| \leq 2,$$

for all $x, y \in T$, and this shows that the entries of the matrix $M(t)$ of (15) are holomorphic on the unit disk $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$. Thus, as a consequence of Proposition 9, we obtain the following.

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PROPOSITION 12. *Let (T, K) be an oriented tree and let f be a K-piecewise monotone map. Then the function $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ is holomorphic on $\mathbb{D}$.*

4. *The Artin–Mazur zeta function*

Recall that if $f : X \to X$ is a function defined in an arbitrary set X, and each iterate f^n has only finitely many fixed points, then the Artin–Mazur zeta function of f is defined to be the formal power series

$$\zeta(t) = \exp \sum_{n\geq 1} N_{f^n} t^n/n,$$

where N_{f^n} denotes the total number of fixed points of the iterate f^n. The aim of this section is to show that the determinant of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ plays a fundamental role in the computation of $\zeta(t)$. We begin by introducing some notation. Let $g : T \to T$ be a piecewise strictly monotone map and denote by $\mathbb{T}$ the set whose elements are the pairs (x, C) such that $x \in T$ and C is a connected component of $T\backslash\{x\}$. Consider the map

$$G : \mathbb{T} \to \mathbb{T} \tag{20}$$

induced by g, defined by $G(x, C) = (x', C')$ if and only if $g(x) = x'$ and there exists a neighborhood V of x such that $g(V \cap C) \subset C'$. Note that a pair $(x, C) \in \mathbb{T}$ is a fixed point of G if and only if x is a fixed point of g and there exists a neighborhood V of x such that $g(V \cap C) \subset C$.

Definition 13. Let T be a tree and let $g : T \to T$ be a piecewise strictly monotone map with isolated fixed points. If (x, C) is a fixed point of G and there exists a neighborhood V of x such that $g(y)$ lies between x and y, for all $y \in V \cap C$, then (x, C) will be called a formally stable pair of g. A fixed point of G which is not formally stable will be called a formally unstable pair of g. The number of all formally stable pairs of g is denoted by S_g.

So, if $f : T \to T$ is a strictly piecewise monotone map, and each iterate f^n has only finitely many fixed points, we can define the formal power series

$$S(t) \stackrel{\text{def}}{=} \exp \sum_{n\geq 1} \frac{S_{f^n}}{n} t^n.$$

Consider now a pointed tree (T, K) with m edges and assume that f is a K-piecewise monotone map. Under these conditions we define the transition matrices Q and R as follows. Denoting by $x_1, \ldots, x_{m+1}$ the elements of K, Q is a $(m + 1) \times (m + 1)$ matrix defined by

$$Q(i, j) = \begin{cases} 1 & \text{if } f(x_j) = x_i \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Denoting by $F : \mathbb{T} \to \mathbb{T}$ the map induced by f and by $X_1, \ldots, X_{2m}$ the elements of $\{(x, C) \in \mathbb{T} : x \in K\}$, R is a $2m \times 2m$ matrix defined by

$$R(i, j) = \begin{cases} 1 & \text{if } F(X_j) = X_i \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Finally we are in a position to state the main theorem of this section. An analogous result can be provided for K-piecewise strictly monotone maps (see [**1**]).

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THEOREM 14. *Let (T, K) be an oriented tree and let f be a K-preserving piecewise strictly monotone map. If each iterate f^n has only finitely many fixed points, then*

$$\zeta(t) = \frac{D_R(t)S(t)}{D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)\ D_Q(t)},$$

where $D_Q(t)$ (respectively $D_R(t)$) is the characteristic polynomial of Q (respectively R).

Remark 15. In the particular case of the oriented tree $([a, b], \{a, b\})$ it is easy to see that, for a given $\{a, b\}$-preserving piecewise strictly monotone, the matrices P and Q and coincide, and from Theorem 14 we obtain

$$\zeta(t) = \frac{S(t)}{D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)}.$$

Observe also that a theorem of Julia and Fatou (see [**17, 14**]), valid for polynomials of degree $n \geq 2$, and a theorem of Singer (see [**25**]), valid for piecewise monotone interval maps with negative Schwarzian derivative, enable us to conclude that, for these maps, $S(t)^{-1}$ is a cyclotomic polynomial, i.e.

$$S(t)^{-1} = (1 - t^{n_1})(1 - t^{n_2}) \cdots (1 - t^{n_k}).$$

For an arbitrary piecewise strictly monotone tree map $f : T \to T$ the computation of $S(t)$ may be very complicated. Nevertheless, defining a formally stable p_o-periodic orbit of f as a p_o-periodic orbit of $F : \mathbb{T} \to \mathbb{T}$ containing a stable pair of f^{p_o}, we have

$$S(t)^{-1} = \prod_{o \in \mathbb{O}} (1 - t^{p_o}),$$

where $\mathbb{O}$ denotes the set (possibly not finite) of all formally stable periodic orbits of f. As an example, let f be the map described in Example 10. It can be shown that there exists one and only one formally stable periodic orbit of f. Since the period of this orbit is 9, and, on the other hand, $D_Q(t) = 1 - t$ and $D_R(t) = 1 - t^3$, we obtain

$$\zeta(t) = \frac{(1 + t^3 + t^6)(1 - t^3)}{(1 - t^3 - t^6)(1 - t)(1 - t^9)}.$$

4.1. *Traces and fixed points (Proof of Theorem 14).* Let (T, K) be an oriented tree and let g be a K-piecewise monotone map. We know from Proposition 8 that the trace of the pair $(\epsilon_{\#0}(g), \epsilon_{\#1}(g))$ is defined. Now we show that there exists an important relationship between this trace and the number of fixed points of g. First we introduce some more notation. Denote by Fix_g the set of all fixed points of g. If $x \in \mathrm{Fix}_g$, then we say that x is a fixed point of g of positive (respectively negative) type if $\epsilon_{g,0}(x) > 0$ (respectively $\epsilon_{g,0}(x) < 0$). If $x \in \mathrm{Fix}_g$ and $\epsilon_{g,0}(x) = 0$, then we say that x is a fixed point of g of critical type. We denote by Fix_g^+ (respectively Fix_g^-) the set of all fixed points of g of positive (respectively negative) type, and by Fix_g^0 the set of all fixed points of g of critical type. In other words, we have

$$\mathrm{Fix}_g^+ = \{x \in \mathrm{Fix}_g : \epsilon_{g,0}(x) > 0\}, \quad \mathrm{Fix}_g^- = \{x \in \mathrm{Fix}_g : \epsilon_{g,0}(x) < 0\}$$

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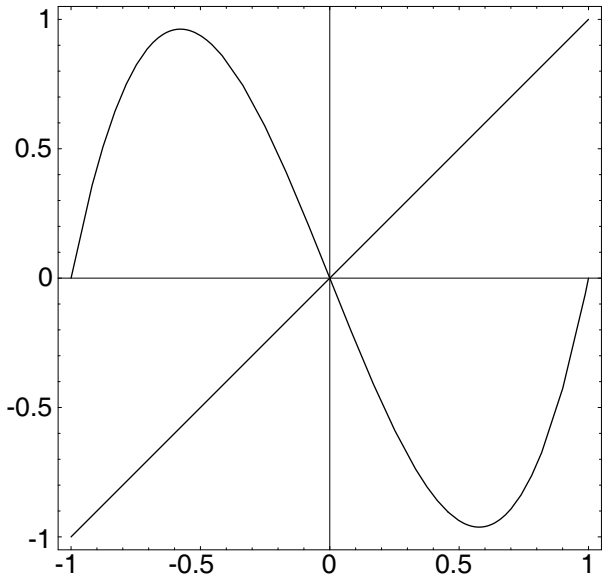


FIGURE 1. A piecewise monotone map $g : [-1, 1] \to [-1, 1]$.

and

$$\mathrm{Fix}_g^0 = \{x \in \mathrm{Fix}_g : \epsilon_{g,0}(x) = 0\}.$$

Note that Fix_g^+, Fix_g^- and Fix_g^0 are independent of the order of the pointed tree (T, K); nevertheless these sets depend on K. Figure 1 shows the graph of a piecewise monotone map $g : [-1, 1] \to [-1, 1]$. Regarding g as a $\{-1, 1\}$-piecewise monotone map, we obtain $\mathrm{Fix}_g^- = \{0\}$ and $\mathrm{Fix}_g^0 = \emptyset$. However, for the same map regarded as a $\{-1, 0, 1\}$-piecewise monotone map, we obtain $\mathrm{Fix}_g^- = \emptyset$ and $\mathrm{Fix}_g^0 = \{0\}$.

It is clear that the cardinals

$$N_g \stackrel{\mathrm{def}}{=} \#\,\mathrm{Fix}_g \quad \text{and} \quad N_g^+ \stackrel{\mathrm{def}}{=} \#\,\mathrm{Fix}_g^+,$$

may not be finite; however, since g is piecewise monotone, we see at once that

$$N_g^- \stackrel{\mathrm{def}}{=} \#\,\mathrm{Fix}_g^- \quad \text{and} \quad N_g^0 \stackrel{\mathrm{def}}{=} \#\,\mathrm{Fix}_g^0$$

are both natural numbers.

Observe that, if (T, K) is an oriented tree and g is a K-piecewise monotone map, then the pair of linear endomorphisms $(\epsilon_{\#0}(g), \epsilon_{\#1}(g))$ depends on the orientation of (T, K). However, as an immediate consequence of the next result, we see that the trace and the determinant of this pair is independent of the orientation of (T, K).

THEOREM 16. *Let (T, K) be an oriented tree and let g be a piecewise K-monotone map. Then we have*

$$\mathrm{Tr}(\epsilon_{\#0}(g), \epsilon_{\#1}(g)) = 2N_g^- - 1 + N_g^0.$$

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Let $T_1, \ldots, T_m$ be the edges of the oriented tree (T, K) and consider the oriented trees $(T_1, K_1), \ldots, (T_m, K_m)$, with $K_j = K \cap T_j$. For each $j = 1, \ldots, m$, let $i_j : T_j \to T$ be the injection of T_j in T, and $p_j : T \to T_j$ the projection of T into T_j (p_j is the unique continuous map that verifies $p_j(x) = x$, for all $x \in T_j$, and p_j is constant on each connected component of $T \backslash T_j$). Clearly i_j and p_j are piecewise monotone maps, so if $g : T \to T$ is a piecewise monotone map, then the interval map

$$g_j = p_j \circ g \circ i_j : T_j \to T_j \tag{23}$$

can be regarded as a K_j-piecewise monotone map, and by Lemma 2.23 of [**3**]†, we know that

$$\mathrm{Tr}(\epsilon_{\#0}(g_j), \epsilon_{\#1}(g_j)) = 2N_{g_j}^- - 1 + N_{g_j}^0.$$

Thus, Theorem 16 is an immediate consequence of the two following lemmas.

LEMMA 17. *Let (T, K) be an oriented tree with m edges, and let g be a K-piecewise monotone map. Then we have*

$$\mathrm{Tr}(\epsilon_{\#0}(g), \epsilon_{\#1}(g)) = \sum_{j=1}^{m} \mathrm{Tr}(\epsilon_{\#0}(g_j), \epsilon_{\#1}(g_j)).$$

Proof. Setting $\varphi = \epsilon_{\#0}(g)$, $\psi = \epsilon_{\#1}(g)$, $U = S_0(T, \mathbb{R})$, $V = S_0(T \backslash K, \mathbb{R})$, $W = S_1(T, \mathbb{R})$, $V_j = S_0(T_j \backslash K_j, \mathbb{R})$, and $W_j = S_1(T_j, \mathbb{R})$, it is easy to check that we are under the conditions of Proposition 38, and therefore

$$\mathrm{Tr}(\epsilon_{\#0}(g), \epsilon_{\#1}(g)) = \sum_{j=1}^{m} \mathrm{Tr}(p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j}, p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j}).$$

So it suffices to show that

$$p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j} = \epsilon_{\#0}(g_j) \quad \text{and} \quad p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j} = \epsilon_{\#1}(g_j).$$

From the definition of $\epsilon_{\#0}$, we verify easily that $p_{V_j} = \epsilon_{\#0}(p_j)$ and $i_{V_j} = \epsilon_{\#0}(i_j)$, where $i_j : T_j \to T$ and $p_j : T \to T_j$ are the piecewise monotone maps of (23). Since $i_j^{-1}(V) = V_j$ and $p_j(V) = V_j$, it follows from (9) that

$$p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j} = \epsilon_{\#0}(p_j) \circ \epsilon_{\#0}(g) \circ \epsilon_{\#0}(i_j) = \epsilon_{\#0}(p_j \circ g \circ i_j) = \epsilon_{\#0}(g_j).$$

By definition of $\epsilon_{\#1}$, we have $p_{W_j} = \epsilon_{\#1}(p_j)$ and $i_{W_j} = \epsilon_{\#1}(i_j)$, and from Proposition 2

$$p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j} = \epsilon_{\#1}(p_j) \circ \epsilon_{\#1}(g) \circ \epsilon_{\#1}(i_j) = \epsilon_{\#1}(p_j \circ g \circ i_j) = \epsilon_{\#1}(g_j),$$

as desired. □

LEMMA 18. *Let (T, K) be an oriented tree with m edges, and let g be a K-piecewise monotone map. Then we have*

$$2N_g^- - 1 + N_g^0 = \sum_{j=1}^{m} (2N_{g_j}^- - 1 + N_{g_j}^0).$$

† Actually this lemma is proved for piecewise strictly monotone interval maps, but with small changes it can be shown for the wider class of piecewise monotone interval maps. This remark is important, because even for a piecewise strictly monotone tree map g the maps g_j may not be strictly monotone.

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Proof. Let $x \in K$ be one of the $m+1$ vertices of the oriented tree (T, K), and let $T_{x_1}, \ldots, T_{x_{\mathrm{val}(x)}}$ be the edges of (T, K) that contain $\{x\}$. By definition of g_j, we have

$$\sum_{j=1}^{\mathrm{val}(x)} \#(\mathrm{Fix}^0_{g_{x_j}} \cap \{x\}) = \#(\mathrm{Fix}^0_g \cap \{x\}) - 1 + \mathrm{val}(x),$$

for all $x \in K$. Thus we can write

$$\begin{aligned}
\sum_{j=1}^{m} \#(\mathrm{Fix}^0_{g_j} \cap K) &= \sum_{x \in K} \sum_{j=1}^{\mathrm{val}(x)} \#(\mathrm{Fix}^0_{g_{x_j}} \cap \{x\}) \\
&= \sum_{x \in K} [\#(\mathrm{Fix}^0_g \cap \{x\}) - 1 + \mathrm{val}(x)] \\
&= \#(\mathrm{Fix}^0_g \cap K) - (m+1) + \sum_{x \in K} \mathrm{val}(x) \\
&= \#(\mathrm{Fix}^0_g \cap K) - (m+1) + 2m \\
&= \#(\mathrm{Fix}^0_g \cap K) + m - 1
\end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{j=1}^{m} (2N^-_{g_j} - 1 + N^0_{g_j}) &= 2N^-_g - m + \#(\mathrm{Fix}^0_g \cap (T \backslash K)) + \sum_{j=1}^{m} \#(\mathrm{Fix}^0_{g_j} \cap K) \\
&= 2N^-_g - 1 + \#(\mathrm{Fix}^0_g \cap (T \backslash K)) + \#(\mathrm{Fix}^0_g \cap K) \\
&= 2N^-_g - 1 + N^0_g,
\end{aligned}$$

as desired. □

Observe that Theorem 16 holds for any K-piecewise monotone map, even for maps with an infinite number of fixed points. Assuming that g is a piecewise strictly monotone map with isolated fixed points, it is possible to establish a precise relationship between the trace of the pair $(\epsilon_{\#0}(g), \epsilon_{\#1}(g))$ and the total number of fixed points of g.

Once again it is convenient to consider first the case of the oriented tree $([a, b], \{a, b\})$. Let g be an $\{a, b\}$-piecewise monotone map with isolated fixed points. In order to relate the numbers N_g and $2N^-_g - 1$, Milnor and Thurston proved in [**18**] that the difference $N_g - (2N^-_g - 1)$ is equal to the number of formally left stable fixed points of g plus the number of formally right stable fixed points of g†. In other words

$$N_g - (2N^-_g - 1 + N^0_g) = \sum_{x \in I} l_g(x) + \sum_{x \in I} r_g(x) - \sum_{x \in I} n^0_g(x),$$

where the numbers $l_g(x)$, $r_g(x)$ and $n^0_g(x)$ are defined by: $l_g(x) = 1$ if x is a formally left stable fixed point of g, and $l_g(x) = 0$ for the remaining points of I; $r_g(x) = 1$ if x is a formally right stable fixed point of g, and $r_g(x) = 0$ for the remaining points of I;

† Recall that, following Milnor and Thurston, an isolated fixed point x_0 of g, not of negative type, is formally left stable if $g(x) > x$ throughout some interval $]x_0 - \epsilon, x_0[$, or also if x_0 is the leftmost point of the entire interval I. Similarly, an isolated fixed point x_0 of g, not of negative type, is formally right stable if $g(x) < x$ throughout some interval $]x_0, x_0 + \epsilon[$, or also if x_0 is the rightmost point of the entire interval I.

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$n_g^0(x) = 1$ if x is a fixed point of g of zero type, and $n_g^0(x) = 0$ for the remaining points of I. Since $l_g(a) = 1$ if and only if a is a fixed point of g, and $r_g(b) = 1$ if and only if b is a fixed point of g, we arrive at

$$\begin{aligned}&\#(\mathrm{Fix}_g \cap]a, b[) - (2N_g^- - 1 + N_g^0)\\ &\quad = \sum_{x \in\,]a,b[} (l_g(x) + r_g(x) - n_g^0(x)) + r_g(a) - n_g^0(a) + l_g(b) - n_g^0(b). \qquad (24)\end{aligned}$$

Using (24) we prove now the following general result.

THEOREM 19. *Let (T, K) be an oriented tree, let g be a K-piecewise strictly monotone map, with isolated fixed points, and $G : \mathbb{T} \to \mathbb{T}$ be the map induced by g. Define*

$$A_g = \#(\mathrm{Fix}_g \cap K) \quad \textit{and} \quad B_g = \#\{(x, C) \in \mathbb{T} : x \in K \textit{ and } (x, C) \in \mathrm{Fix}_G\}.$$

Then we have

$$N_g - (2N_g^- - 1 + N_g^0) = A_g - B_g + S_g.$$

Proof. Let $T_1 = [a_1, b_1], \ldots, T_m = [a_m, b_m]$ be the edges of (T, K). For each $j = 1, \ldots, m$, let C_j (respectively D_j) be the unique connected component of $T\backslash\{a_j\}$ (respectively $T\backslash\{b_j\}$) that intersects $[a_j, b_j]$, and let g_j be the $\{a_j, b_j\}$-piecewise monotone map defined in (23). From Lemma 18 and (24) we have

$$\begin{aligned}N_g - (2N_g^- - 1 + N_g^0) &= A_g + \sum_{j=1}^{m} (\#(\mathrm{Fix}_{g_j} \cap]a_j, b_j[) - (2N_{g_j}^- - 1 + N_{g_j}^0))\\ &= A_g + \sum_{j=1}^{m} \sum_{x \in\,]a_j, b_j[} (l_{g_j}(x) + r_{g_j}(x) - n_{g_j}^0(x))\\ &\quad + \sum_{j=1}^{m} (r_{g_j}(a_j) - n_{g_j}^0(a_j)) + \sum_{j=1}^{m} (l_{g_j}(b_j) - n_{g_j}^0(b_j)).\end{aligned}$$

Finally the proof follows from the following equalities:

$$r_{g_j}(a_j) - n_{g_j}^0(a_j) = \begin{cases} -1 & \text{if } (a_j, C_j) \text{ is an unstable pair of } g\\ 0 & \text{otherwise,}\end{cases} \qquad \text{for all } j = 1, \ldots, m;$$

$$l_{g_j}(b_j) - n_{g_j}^0(b_j) = \begin{cases} -1 & \text{if } (b_j, D_j) \text{ is an unstable pair of } g\\ 0 & \text{otherwise,}\end{cases} \qquad \text{for all } j = 1, \ldots, m;$$

and

$$l_{g_j}(x) + r_{g_j}(x) - n_{g_j}^0(x) = \#\{(x, C) \in \mathbb{T} : (x, C) \text{ is a stable pair of } g\},$$

for all $x \in]a_j, b_j[$ and $j = 1, \ldots, m$. □

We are now in position to state a fundamental relationship between the determinant

$$D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) \stackrel{\text{def}}{=} \exp \sum_{n \geq 1} - \mathrm{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n) t^n / n,$$

and a modified zeta function of f defined by

$$\hat{\zeta}(t) \stackrel{\text{def}}{=} \exp \sum_{n \geq 1} (2N_{f^n}^- - 1 + N_{f^n}^0) t^n / n.$$

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Since, by (10) and Theorem 16, we have

$$\mathrm{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n) = \mathrm{Tr}(\epsilon_{\#0}(f^n), \epsilon_{\#1}(f^n)) = 2N^-_{f^n} - 1 + N^0_{f^n},$$

for all $n \geq 1$, we can write (compare with (1)) the following.

THEOREM 20. *Let (T, K) be an oriented tree and let f be a K-preserving piecewise monotone map. Then we have*

$$\hat{\zeta}(t) = D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)^{-1}.$$

Finally, we can prove Theorem 14. From Theorem 20 it suffices to show that

$$\zeta(t)\hat{\zeta}_f(t)^{-1} = \frac{D_R(t)S(t)}{D_Q(t)}.$$

In other words we have to prove that

$$N_{f^n} - (2N^-_{f^n} - 1 + N^0_{f^n}) = \mathrm{Tr}(Q^n) - \mathrm{Tr}(R^n) + S_{f^n},$$

for all $n \geq 1$, and this follows from Theorem 19, because

$$\mathrm{Tr}(Q^n) = A_{f^n} \quad \text{and} \quad \mathrm{Tr}(R^n) = B_{f^n}, \quad \text{for all } n \geq 1. \qquad \square$$

Turning our attention to the specific case of Markov maps, the following corollary is an immediate consequence of Theorems 14 and 20.

COROLLARY 21. *Let (T, K) be an oriented tree and let $f : T \to T$ be a K-preserving monotone map. Assume that f is Markov. Then we have*

$$\mathrm{Tr}(P^n) = 2N^-_{f^n} - 1 + N^0_{f^n},$$

for all $n \geq 1$, and therefore

$$\hat{\zeta}(t) = \frac{1}{D_P(t)},$$

where P is the Markov transition matrix of f defined in (19). If, in addition to this, f is piecewise strictly monotone, and each of its iterates f^n has only finitely many fixed points, then we have

$$\mathrm{Tr}(P^n) + \mathrm{Tr}(Q^n) - \mathrm{Tr}(R^n) = N_{f^n} - S_{f^n},$$

for all $n \geq 1$, and therefore

$$\zeta(t) = \frac{D_R(t)S(t)}{D_P(t)D_Q(t)},$$

where Q and R are the transition matrices of f defined in (21) and (22).

Example 22. As we can see through the following examples, even for interval maps, the matrices P, Q and R play an important role in the computation of $\zeta(t)$. Let $f : [-1, 1] \to [-1, 1]$ be the tent map defined by $f(x) = b(|x| - 1) + 1$, with $b \in]1, 2]$. Since $b > 1$, we have $S(t) = 1$, and from Remark 15 we can regard f as a $\{-1, 1\}$-preserving piecewise monotone map to obtain

$$\zeta(t) = \frac{1}{D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)}.$$

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If, in particular, the orbit of 0 is finite, Corollary 21 gives an alternative way of computing $\zeta(t)$. For $b = (\sqrt{5}+1)/2$, the set $K = \{f^n(x) : x = -1, 0, 1 \text{ and } n \geq 0\}$ is finite because $f^3(0) = 0$. So we can consider the oriented tree $([-1, 1], K)$. Regarding f as a K-piecewise monotone map, we see that f is Markov with $D_P(t) = 1 - 2t + t^3$ and $D_Q(t) = D_R(t)$, and from Corollary 21

$$\zeta(t) = \frac{1}{D_P(t)} = \frac{1}{1 - 2t + t^3}.$$

For $b = \sqrt{2}$, the set K is also finite (we have $f^3(0) = f^2(0)$), but in this case the orbit of 0 is not periodic and therefore $D_Q(t) \neq D_R(t)$. More precisely we have $D_P(t) = 1 - t - 2t^2 + 2t^3$, $D_Q(t) = (1 - t)^2$ and $D_R(t) = (1 - t^2)(1 - t)$, and from Corollary 21,

$$\zeta(t) = \frac{D_R(t)}{D_P(t)\, D_Q(t)} = \frac{(1 - t^2)(1 - t)}{(1 - t - 2t^2 + 2t^3)(1 - t)^2}.$$

To end this section, we give another consequence of Theorem 20 which will play a relevant role in the study of the topological entropy. Define

$$\rho \stackrel{\text{def}}{=} \max(\{|t|^{-1} : D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) = 0\} \cup \{1\}). \tag{25}$$

COROLLARY 23. *Let (T, K) be an oriented tree and let f be K-preserving piecewise monotone map. Then we have*

$$\rho = \max\left\{\overline{\lim_n} \sqrt[n]{N^-_{f^n}}, 1\right\}.$$

5. *Piecewise linear models*

In this section we study the existence of piecewise linear models for a given piecewise monotone tree map. Baillif and de Carvalho [**9**] showed how to construct a piecewise linear model for a given piecewise strictly monotone tree map with positive topological entropy. Our goal here is to show how to obtain not only one, but all piecewise linear models of a given piecewise monotone tree map.

For that purpose, letting T be a tree, a continuous semimetric $d : T \times T \to \mathbb{R}^+$ is called linear if

$$d(x, z) = d(x, y) + d(y, z),$$

for all $x, y \in T$ and $z \in \langle x, y\rangle$, where $\langle x, y\rangle$ denotes the convex hull of $\{x, y\}$. If d is a linear semimetric on T, a piecewise monotone map $f : T \to T$ is called d-piecewise linear with slope $s(d) \in \mathbb{R}^+$ if

$$d(f(x), f(y)) = s(d)d(x, y),$$

for all x and y lying in a same interval on which f is injective or constant.

Note that, for a given piecewise monotone map $f : I = [a, b] \to I$, a continuous and increasing (not necessarily strictly increasing) map $\lambda : I \to [0, 1]$ is a semiconjugacy from f to a piecewise linear map $F : [0, 1] \to [0, 1]$, with slope $\pm s$ everywhere, if and only

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if f is a d-piecewise linear map with slope s, where d is the continuous linear semimetric on I, defined by

$$\begin{aligned} d : I \times I &\to \mathbb{R}^+ \\ (x, y) &\to |\lambda(y) - \lambda(x)|. \end{aligned}$$

So, with this terminology, the theorems of Parry and Milnor–Thurston can be restated as follows: if $f : I \to I$ is a piecewise strictly monotone interval map with positive topological entropy $\log(s)$, then there exists a continuous and linear semimetric d on I such that f is a d-piecewise linear map with slope s. Our goal in this section is to extend this result for tree maps.

Let T be a tree and $d : T \times T \to \mathbb{R}^+$ a linear semimetric on T. If $g : T \to T$ is a piecewise monotone map, we define the total variation of g, with respect to d, by

$$\mathrm{Var}_d(g) = \sup_{\mathcal{P}} \left\{ \sum_{i=1}^{k} d(g(x_i), g(y_i)) \right\},$$

where $P = \{[x_1, y_1], \ldots, [x_k, y_k]\}$ denotes an arbitrary partition of T. So for each piecewise monotone map $f : T \to T$, and each compact and connected set $X \subseteq T$ we can define the formal power series

$$V(d; X; t) \stackrel{\text{def}}{=} \sum_{n \geq 0} \mathrm{Var}_d(f^n|_X) t^{n+1}.$$

These formal power series are holomorphic on $\{t \in \mathbb{C} : |t| < s(d)^{-1}\}$, with

$$s(d) = \overline{\lim_n} \sqrt[n]{\mathrm{Var}_d(f^n)},$$

and the next result shows that they are meromorphic on the unit disk $\mathbb{D}$ (compare with [**24**]).

THEOREM 24. *Let (T, K) be an oriented tree, let f be a K-piecewise monotone tree map, and let d be a linear semimetric on T. Then, for any compact and connected set $X \subseteq T$, the function $V(d; X; t)$ has a meromorphic extension to the unit disk whose poles are contained in the set of zeros of $D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)$. Furthermore, if $s(d) > 1$, then $V(d; T; t)$ has a pole at $s(d)^{-1}$.*

Note that, if $s(d) > 1$, Theorem 24 shows that the function

$$\frac{V(d; X; t)}{V(d; T; t)}$$

is meromorphic on $\mathbb{D}$, and (since $V(d; X; t) \leq V(d; T; t)$, for all $t \in [0, s(d)^{-1}]$) has a removable singularity at $s(d)^{-1}$. Thus we can construct a new linear semimetric $\Lambda(d) : T \times T \to [0, 1]$, defined by

$$\Lambda(d)(x, y) = \lim_{t \to 1/s(d)} \frac{V(d; \langle x, y \rangle; t)}{V(d; T; t)}, \quad \text{for all } (x, y) \in T \times T.$$

where $\langle x, y \rangle$ denotes the convex hull of $\{x, y\}$. We are now in a position to prove the main theorem of this section.

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THEOREM 25. *Let $f : T \to T$ be a piecewise monotone tree map and let d be a linear semimetric d on T such that $s(d) > 1$. Then $\Lambda(d)$ is a continuous and linear semimetric on T such that f is $\Lambda(d)$-piecewise linear with slope $s(d)$.*

Proof. Assume that d is a linear semimetric on T such that $s(d) > 1$. We have

$$\begin{aligned} V(d; \langle x, y\rangle; t) &= \sum_{n\geq 0} \mathrm{Var}_d(f^n|_{\langle x,y\rangle})t^{n+1} \\ &= d(x, y)t + t\sum_{n\geq 1} \mathrm{Var}_d(f^{n-1}|_{\langle f(x),f(y)\rangle})t^n \\ &= d(x, y)t + tV(d; \langle f(x), f(y)\rangle; t), \end{aligned}$$

for all x and y lying in a same interval on which f is injective or constant. But from Theorem 24 we know that $V(d; T; t)$ has a pole at $s(d)^{-1}$ and therefore

$$\Lambda(d)(x, y) = \lim_{t\to 1/s(d)} \frac{V(d; \langle x, y\rangle; t)}{V(d; T; t)} = s(d)^{-1}\Lambda(d)(f(x), f(y)).$$

This shows that f is a $\Lambda(d)$-piecewise linear map, with slope $s(d)$, since f is piecewise monotone map and $s(d) > 1$, it follows that $\Lambda(d)$ is continuous on $T \times T$. □

Remark 26. Recall that Misiurewicz and Szlenk proved in [**20**] that the topological entropy of a piecewise strictly monotone interval map f coincides with the logarithm of

$$\max\left\{\overline{\lim_n}\sqrt[n]{\mathrm{Var}(f^n)}, 1\right\}.$$

Assume now that $f : T \to T$ is a piecewise strictly monotone tree map with positive topological entropy $\log(s)$. If d is an arbitrary linear metric on T, the same arguments can be used to show that

$$s = \overline{\lim_n}\sqrt[n]{\mathrm{Var}_d(f^n)},$$

and from Theorem 25 we can conclude that f is $\Lambda(d)$-piecewise linear map with slope s.

However, Theorem 25 also shows that, for the same map f, other semiconjugacies may exist. As an example, if f is Markov, we can use the eigenvalues of the transition matrix P (19) to construct other semiconjugacies. Let $T_1 = [a_1, b_1], \ldots, T_m = [a_m, b_m]$ be the edges of the oriented tree (T, K), and suppose that $\overrightarrow{\alpha} = (\alpha_1 \ldots \alpha_m)^T$ (with $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$) is an eigenvector of P^T, associated to an eigenvalue $r > 1$. If $d_{\overrightarrow{\alpha}} : T \times T \to \mathbb{R}^+$ is a linear semimetric such that $d(a_i, b_i) = \alpha_i$, for $i = 1, \ldots, m$, then we have

$$\mathrm{Var}_{d_{\overrightarrow{\alpha}}}(f^n) = (\alpha_1 \ldots \alpha_m)P^n(1 \ldots 1)^T = r^n(\alpha_1 + \cdots + \alpha_m)$$

and $s(d_{\overrightarrow{\alpha}}) = r > 1$, and from Theorem 25 we conclude that f is a $\Lambda(d_{\overrightarrow{\alpha}})$-piecewise linear map with slope r (compare with [**6**]).

Finally, note that any piecewise linear model of a tree map with $s(d) > 1$ can be obtained (less a multiplicative constant) through Λ. Indeed if $f : T \to T$ is a d-piecewise linear map with slope $s(d) > 1$, then we check easily $\Lambda(d) = \mathrm{Var}_d(\mathrm{id}_T)^{-1} \cdot d$.

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5.1. *Total variations (Proof of Theorem 24).* Let d be a linear semimetric on T and let $f : T \to T$ be a piecewise monotone map. We have seen in the previous section the importance of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ in the computation of the number of fixed points of f^n. To prove Theorem 24 we show that the same pair of linear endomorphisms plays an equally important role in the computation of the total variation of f^n with respect to d. We begin by introducing the linear form ξ_d. Since $\mathcal{I}$ (see (4)) generates $S_1(T, \mathbb{R})$, we can define this linear form as follows.

Definition 27. Let (T, K) be an oriented tree and let d be a linear semimetric on T. Define ξ_d to be the unique linear form on $S_1(T, \mathbb{R})$ that verifies

$$\xi_d(y - x) = d(x, y), \quad \text{for all } y - x \in \mathcal{I}.$$

Let (T, K) be an oriented tree. If $g : T \to T$ is a piecewise monotone map and d is a linear semimetric d on T, we can construct another linear semimetric d_g as follows:

$$d_g(x, y) \stackrel{\text{def}}{=} \text{Var}_d(g|_{\langle x,y\rangle}),$$

for all $(x, y) \in T \times T$, and in this way we can consider the linear form ξ_{d_g}. Observe that if $[x, y]$ is a monotonicity interval of g, then we have

$$\xi_{d_g}(y - x) = d_g(x, y) = \text{Var}_d(g|_{\langle x,y\rangle}) = d(g(x), g(y)) = \xi_d(\epsilon_{\#1}(g)(y - x)),$$

and, since $\mathcal{I}_g$ generates $S_1(T, \mathbb{R})$ (see (6)), we obtain

$$\xi_{d_g} = \xi_d \circ \epsilon_{\#1}(g).$$

So, for a given piecewise monotone map $f : T \to T$, we conclude from (10) that

$$\xi_d \circ \epsilon_{\#1}(f)^n(\overrightarrow{X}) = \xi_d \circ \epsilon_{\#1}(f^n)(\overrightarrow{X}) = \xi_{d_{f^n}}(\overrightarrow{X}) = \text{Var}_d(f^n|_X),$$

for all $n \geq 0$, where X denotes an arbitrary compact and connected subset of T and $\overrightarrow{X}$ denotes the corresponding vector of $S_1(T, \mathbb{R})$ (see (3)). These equalities allow us to establish an important relationship between the determinant of the pair $(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})$ and the formal power series $V(d; X; t)$. Indeed, since $\epsilon_{\#1}(f)$ and $\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X}$ are defined both in $S_1(T, \mathbb{R})$, and $\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X} - \epsilon_{\#1}(f) = \xi_d \otimes \overrightarrow{X}$, it follows immediately that the pair $(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})$ has finite rank, and from Proposition 40

$$\begin{aligned} D_{(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})}(t) &= 1 - \sum_{n \geq 0} \xi_d \circ \epsilon_{\#1}(f)^n(\overrightarrow{X}) t^{n+1} \\ &= 1 - \sum_{n \geq 0} \text{Var}_d(f^n|_X) t^{n+1} \\ &= 1 - V(d; X; t). \end{aligned} \tag{26}$$

Observe also that we can use the determinant of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ to decompose $D_{(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})}(t)$. In fact, if $\overline{\xi}_d$ denotes an extension of ξ_d to $S_0(T, \mathbb{R})$, then we

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have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_1(T,\mathbb{R}) & \hookrightarrow & S_0(T,\mathbb{R}) & \twoheadrightarrow & \dfrac{S_0(T,\mathbb{R})}{S_1(T,\mathbb{R})} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & S_1(T,\mathbb{R}) & \hookrightarrow & S_0(T,\mathbb{R}) & \twoheadrightarrow & \dfrac{S_0(T,\mathbb{R})}{S_1(T,\mathbb{R})} & \longrightarrow & 0
\end{array}$$

where $\varphi = (\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})^n - \epsilon_{\#1}(f)^n$, and $\psi = (\theta_1 + \overline{\xi}_d \otimes \overrightarrow{X})^n - \theta_1^n$. Consequently

$$\begin{aligned}
\operatorname{Tr}((\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})^n - \epsilon_{\#1}(f)^n) &= \operatorname{Tr}((\theta_1 + \overline{\xi}_d \otimes \overrightarrow{X})^n - \theta_1^n) \\
&= \operatorname{Tr}((\theta_1 + \overline{\xi}_d \otimes \overrightarrow{X})^n - \theta_0^n) \\
&\quad - \operatorname{Tr}(\theta_1^n - \theta_0^n),
\end{aligned}$$

that is

$$\begin{aligned}
\operatorname{Tr}(\epsilon_{\#1}(f)^n, (\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})^n) &= \operatorname{Tr}(\epsilon_{\#0}(f)^n, (\epsilon_{\#1}(f) + \xi_d \otimes \overrightarrow{X})^n) \\
&\quad - \operatorname{Tr}(\epsilon_{\#0}(f)^n, \epsilon_{\#1}(f)^n),
\end{aligned}$$

and thus

$$D_{(\epsilon_{\#1}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t) = D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t) D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)^{-1}.$$

So, by (26), we obtain (compare with [**24**]) the following.

THEOREM 28. *Let (T, K) be an oriented tree, and let d be a linear semimetric on T. If f is a K-piecewise monotone map and X is compact and connected subset of T, then*

$$V(d; X; t) = 1 - \frac{D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t)}{D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)}.$$

Note that we can compute $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t)$ if we know an extension $\overline{\xi}_d$ of ξ_d to $S_0(T,\mathbb{R})$. Indeed, Proposition 40 shows that

$$\begin{aligned}
D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t) &= D_{(\theta_0,\theta_1+\overline{\xi}_d\otimes\overrightarrow{X})}(t) \\
&= \operatorname{Det}(\mathrm{id} - tN(t)),
\end{aligned}$$

where $N(t)$ is the matrix with entries in $\mathbb{R}[[t]]$ defined by

$$\begin{bmatrix}
\sum_{n\geq 0} \omega_{x_1}(\theta_0^n(y_1))t^n & \dots & \sum_{n\geq 0} \omega_{x_1}(\theta_0^n(y_l))t^n & \sum_{n\geq 0} \omega_{x_1}(\theta_0^n(\overrightarrow{X}))t^n \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{n\geq 0} \omega_{x_l}(\theta_0^n(y_1))t^n & \dots & \sum_{n\geq 0} \omega_{x_l}(\theta_0^n(y_l))t^n & \sum_{n\geq 0} \omega_{x_l}(\theta_0^n(\overrightarrow{X}))t^n \\
\sum_{n\geq 0} \overline{\xi}_d(\theta_0^n(y_1))t^n & \dots & \sum_{n\geq 0} \overline{\xi}_d(\theta_0^n(y_l))t^n & \sum_{n\geq 0} \overline{\xi}_d(\theta_0^n(\overrightarrow{X}))t^n
\end{bmatrix},$$

with $\{x_1, \dots, x_l\} = C_f$, $y_1 = f_{\#0}(x_1), \dots, y_l = f_{\#0}(x_l)$.

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Finally we are in a position to prove Theorem 24. Recall that Proposition 12 shows that the function $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ is holomorphic on $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$. The same argument can be used to prove that the function $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f)+\xi_d\otimes\overrightarrow{X})}(t)$ is also holomorphic on $\mathbb{D}$. Indeed, it is easy to see that all entries of $N(t)$ are holomorphic on $\mathbb{D}$. Thus, as an immediate consequence of Theorem 28, we conclude that $V(d; X; t)$ has a meromorphic extension to $\mathbb{D}$ whose poles are contained in the set of zeros of $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$.

Assume now that $s(d) > 1$. In this case, since the radius of convergence of $V(d; T; t)$ is precisely $s(d)^{-1}$, the meromorphic function $V(d; T; t)$ has a pole in $\{t \in \mathbb{C} : |t| = s(d)^{-1}\}$. Since the coefficients of the formal power series $V(d; T; t)$ are positives, a standard argument (see [**13**]) shows that $s(d)^{-1}$ is a pole of $V(d; T; t)$.

6. *Topological entropy*

The aim of this section is to extend to tree maps an important relationship between topological entropy and the growth rate of the sequence $N^-_{f^n}$, introduced in §2. Denoting the topological entropy of a piecewise monotone tree map f by $h_t(f)$, the main theorem of this section can be stated as follows.

THEOREM 29. *Let (T, K) be a pointed tree and let $f : T \to T$ be a K-preserving piecewise strictly monotone map. Then*

$$h_t(f) = \log\left(\max\{1, \overline{\lim_n} \sqrt[n]{N^-_{f^n}}\}\right).$$

It is known that for Markov maps the topological entropy of f coincides with the logarithm of the spectral radius of its transition matrix P. Note that, from Theorem 11, a complex number t^{-1} is an eigenvalue of P if and only if $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = 0$, therefore the spectral radius of P coincides with

$$\max\{|t|^{-1} : D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = 0\}$$

and

$$h_t(f) = \log(\max\{|t|^{-1} : D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t) = 0\}). \tag{27}$$

Theorem 29 is an immediate consequence of the next result and Corollary 23. This theorem also shows that equality (27) holds for any piecewise monotone tree map.

THEOREM 30. *Let (T, K) be an oriented tree and let f be a K-preserving piecewise strictly monotone map. Then we have $h_t(f) = \log(\rho)$, where ρ is the number defined in (25).*

Remark 31. Similar results to Theorems 29 and 30 can also be provided for K-piecewise strictly monotone maps (see [**1**]).

In order to prove Theorem 30 we have to recall a well-known relationship between topological entropy and the growth rate of lap numbers due to Misiurewicz, Szlenk and Ziemian (see [**20**] and [**21**]). Let (T, K) be an oriented tree with m edges and let f be

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a K-piecewise strictly monotone map. Define the lap number $\ell(f)$ to be the number of maximal intervals of monotonicity of f. In other words

$$\ell(f) = m + \#\{x \in T\backslash K : \epsilon_{f,0}(x) = 0\}.$$

So we can define the sequence $\ell(f^n)$ and the corresponding formal power series

$$L(t) \stackrel{\text{def}}{=} \sum_{n\geq 1} \ell(f^n)t^n.$$

Note that, if f and g are piecewise strictly monotone tree maps, then

$$\ell(f \circ g) \leq \ell(f)\ell(g),$$

and therefore

$$\ell(f^{n+m}) \leq \ell(f^n)\ell(f^m)$$

for all $n, m \geq 0$. Using these inequalities, a standard argument proves that $\sqrt[n]{\ell(f^n)}$ converges to a limit s. For interval maps, Misiurewicz and Szlenk proved in [**20**] that the topological entropy of f coincides with the logarithm of s. For tree maps, the importance of the number

$$s \stackrel{\text{def}}{=} \lim_n \sqrt[n]{\ell(f^n)} \tag{28}$$

is the same. Indeed, the same arguments can be easily adapted to show that

$$h_t(f) = \log(s) \tag{29}$$

holds for piecewise strictly monotone tree maps.

6.1. *Laps and total variations (Proof of Theorem 25).* In order to prove Theorem 30 we establish a relationship between $L(t)$ and the formal power series $V(d; T; t)$ of the previous section.

Definition 32. For each $c \in T$, define the linear semimetric $d_c : T \times T \to \mathbb{R}^+$ by

$$d_c(x, y) = \#(\langle x, y\rangle \cap \{c\}) - (\#(\{x\} \cap \{c\}) + \#(\{y\} \cap \{c\}))/2.$$

Note that if $g : T \to T$ is a piecewise strictly monotone map and $c \in T$ then

$$\mathrm{Var}_{d_c}(g) = \sum_{x\in g^{-1}(c)} \mathrm{val}(x)/2. \tag{30}$$

Consider now an oriented tree (T, K) and let $f : T \to T$ be a K-preserving piecewise strictly monotone map. For each $c \in \tilde{C}_f \stackrel{\text{def}}{=} C_f\backslash K$, define the sequence $\gamma_n(c)$, by

$$\gamma_0(c) = 1; \gamma_{n+1}(c) = \begin{cases} 0 & \text{if } f^{n+1}(c) \in \tilde{C}_f \\ \gamma_n(c) & \text{if } f^{n+1}(c) \notin \tilde{C}_f, \end{cases}$$

for all $n \geq 0$, and the corresponding formal power series

$$\gamma_c(t) \stackrel{\text{def}}{=} \sum_{n\geq 0} \gamma_n(c)t^n. \tag{31}$$

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Observe that, since $f(K) \subseteq K$ and $c \notin K$, we see from (30) that

$$V(d_c; T; t) \stackrel{\text{def}}{=} \sum_{n\geq 0} \mathrm{Var}_{d_c}(f^n)t^{n+1} = \sum_{n\geq 0} \#\{x \in T : f^n(x) = c\}t^{n+1}, \tag{32}$$

for all $c \in \tilde{C}_f$.

PROPOSITION 33. *Let (T, K) be an oriented tree with m edges and let f be a K-preserving piecewise strictly monotone map. Then we have*

$$L(t) = \frac{mt}{1-t} + \sum_{c\in\tilde{C}_f} \gamma_c(t)V(d_c; t).$$

Proof. From the definition of $\ell(f^n)$ we have

$$\ell(f^n) = m + \#\{x \in T\backslash K : \epsilon_{f^n,0}(x) = 0\}.$$

Since $f(K) \subseteq K$, we have

$$\epsilon_{f^n,0}(x) = \epsilon_{f,0}(x)\epsilon_{f,0}(f(x)) \dots \epsilon_{f,0}(f^{n-1}(x)),$$

for all $x \in T\backslash K$. Thus, for each $x \in T\backslash K$, we have $\epsilon_{f^n,0}(x) = 0$ if and only if $\{x, f(x), \dots, f^{n-1}(x)\} \cap \tilde{C}_f \neq \emptyset$, and so

$$\begin{aligned}\ell(f^n) &= m + \#\Bigg(\bigcup_{c\in\tilde{C}} \bigcup_{i=0}^{n-1} \{x \in T\backslash K : f^i(x) = c\}\Bigg)\\ &= m + \sum_{c\in\tilde{C}} \sum_{i=0}^{n-1} \gamma_{n-1-i}(c)\#\{x \in T : f^i(x) = c\}\end{aligned}$$

for all $n \geq 1$, and from (32) we obtain

$$L(t) = \sum_{n\geq 1} \ell(f^n)t^n = \frac{mt}{1-t} + \sum_{c\in\tilde{C}} \gamma_c(t)V(d_c; t),$$

as desired. □

Observe that, since the functions $\gamma_c(t)$ in (31) are holomorphic in the unit disk $\mathbb{D}$, the following result is an immediate consequence of Theorem 24 and Proposition 33.

COROLLARY 34. *Let (T, K) be an oriented tree and let f be a K-preserving piecewise strictly monotone map. Then the function $L(t)$ has a meromorphic extension to the unit disk whose poles are contained in the set of zeros of $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$.*

Finally we are in a position to prove Theorem 30. From (29) it suffices to show that $\rho = s$. Since in each lap of f^n there is at most one fixed point of f^n of negative type, we conclude that $\ell(f^n) \geq N^-_{f^n}$, for all $n \geq 1$, and from Corollary 23 we obtain $s \geq \rho$.

Assume now that $s > 1$. In this case, since the radius of convergence of $L(t)$ is precisely s^{-1}, we conclude from Corollary 34 that $D_{(\epsilon_{\#0}(f),\epsilon_{\#1}(f))}(t)$ has a zero in $\{t \in \mathbb{C} : |t| = s^{-1}\}$, and from (25) we obtain $\rho \geq s$ as desired.

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A. *Appendix. Traces and determinants of pairs of linear endomorphisms*

Let U be a vector space over $\mathbb{R}$ and let $\psi : U \to U$ be a linear map with finite rank. As usual, we define the trace of ψ by

$$\mathrm{Tr}(\psi) \stackrel{\mathrm{def}}{=} \mathrm{Tr}(\psi_{|V}),$$

where V denotes an arbitrary finite-dimensional subspace of U such that $\mathrm{Im}(\psi) \subseteq V$ (it is easy to see that the definition does not depend on V). If ψ has finite rank, then there are vectors $u_1, \ldots, u_k \in U$ and linear forms $\omega_1, \ldots, \omega_k \in U^*$ such that

$$\psi = \sum_{i=1}^{k} \omega_i \otimes u_i.$$

Considering the matrix

$$M = \begin{bmatrix} \omega_1(u_1) & \ldots & \omega_1(u_k) \\ \vdots & & \vdots \\ \omega_k(u_1) & \ldots & \omega_k(u_k) \end{bmatrix}, \tag{33}$$

we have

$$\mathrm{Tr}(\psi) = \mathrm{Tr}(M).$$

More generally, if ψ has finite rank, then, for each $n \in N_1$, ψ^n has finite rank and

$$\mathrm{Tr}(\psi^n) = \mathrm{Tr}(M^n).$$

The following proposition is well known and gives an explicit method for computing the numbers $\mathrm{Tr}(\psi^n)$, with $n \in N_1$.

PROPOSITION 35. *Let ψ be an endomorphism with finite rank, and define the determinant of ψ to be the following element of $\mathbb{R}[[t]]$:*

$$D_\psi(t) \stackrel{\mathrm{def}}{=} \exp \sum_{n \geq 1} -\frac{t^n}{n} \mathrm{Tr}(\psi^n).$$

Then we have

$$D_\psi(t) = \mathrm{Det}(\mathrm{id} - tM_\psi).$$

Now we consider a more general situation. By a pair of endomorphisms (φ, ψ), we mean two subspaces V and W of the same $\mathbb{R}$-vector space U, and two linear maps $\varphi : V \to V$ and $\psi : W \to W$.

Definition 36. We say that the pair of endomorphisms (φ, ψ) has finite rank if there exist linear maps $\bar{\varphi}$, $\tilde{\varphi}$, $\bar{\psi}$ and $\tilde{\psi}$ such that the diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \hookrightarrow & V+W & \twoheadrightarrow & \dfrac{V+W}{V} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \tilde{\varphi}} & & \\ 0 & \longrightarrow & V & \hookrightarrow & V+W & \twoheadrightarrow & \dfrac{V+W}{V} & \longrightarrow & 0 \end{array}$$

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and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \hookrightarrow & V+W & \twoheadrightarrow & \dfrac{V+W}{W} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \tilde{\psi}} & & \\ 0 & \longrightarrow & W & \hookrightarrow & V+W & \twoheadrightarrow & \dfrac{V+W}{W} & \longrightarrow & 0 \end{array}$$

are commutative, and the linear maps $\bar{\psi} - \bar{\varphi}$, $\tilde{\varphi}$ and $\tilde{\psi}$ have finite rank.

For any pair of endomorphisms (φ, ψ) with finite rank, it is possible to define its trace as follows: considering linear maps $\bar{\varphi}$, $\tilde{\varphi}$, $\bar{\psi}$ and $\tilde{\psi}$ as in the previous definition, we define

$$\mathrm{Tr}(\varphi, \psi) = \mathrm{Tr}(\bar{\psi} - \bar{\varphi}) - \mathrm{Tr}(\tilde{\psi}) + \mathrm{Tr}(\tilde{\varphi})$$

(it is easy to see that the definition does not depend on $\bar{\varphi}$, $\tilde{\varphi}$, $\bar{\psi}$ and $\tilde{\psi}$).

Observe that an endomorphism $\psi : U \to U$ with finite rank can be regarded as a pair of endomorphisms with finite rank. Considering the pair $(0, \psi)$, where $0 : U \to U$ denotes the zero map, we see that ψ has finite rank if and only if the pair $(0, \psi)$ has finite rank, and $\mathrm{Tr}(0, \psi) = \mathrm{Tr}(\psi)$. More generally, if the linear maps φ and ψ both have finite ranks, then the pair (φ, ψ) has also finite rank and

$$\mathrm{Tr}(\varphi, \psi) = \mathrm{Tr}(\psi) - \mathrm{Tr}(\varphi).$$

Of course, in general, the single traces in the previous formula are not defined. The following proposition follows from the definition and will be useful for computing $\mathrm{Tr}(\varphi, \psi)$ in the general case.

PROPOSITION 37. *A pair of linear endomorphisms (φ, ψ) has finite rank if and only if there exist extensions $\bar{\varphi} : V + W \to V + W$ and $\bar{\psi} : V + W \to V + W$ of φ and ψ, respectively, such that $\mathrm{Im}(\bar{\varphi}) \subseteq V$, $\mathrm{Im}(\bar{\psi}) \subseteq W$ and $\bar{\psi} - \bar{\varphi}$ has finite rank. Furthermore, if (φ, ψ) has finite rank, then $\mathrm{Tr}(\varphi, \psi) = \mathrm{Tr}(\bar{\psi} - \bar{\varphi})$.*

Let $\psi : U \to U$ be an endomorphism with finite rank and let $U_1, \ldots, U_m$ be subspaces of U such that

$$U = \bigoplus_{i=1}^{m} U_i.$$

Considering the projections $p_{U_i} : U \to U_i$, and the injections $i_{U_i} : U_i \to U$, it is clear that, for each $i = 1, \ldots, m$, the endomorphism

$$p_{U_i} \circ \psi \circ i_{U_i} : U_i \to U_i$$

has finite rank and

$$\mathrm{Tr}(\psi) = \sum_{i=1}^{m} \mathrm{Tr}(p_{U_i} \circ \psi \circ i_{U_i}). \tag{34}$$

The next proposition can be regarded as a generalization of (34). Let (φ, ψ) be a pair of endomorphisms with finite rank, and let $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ be subspaces of V and W respectively. Assume that

$$V = \bigoplus_{i=1}^{m} V_i \quad \text{and} \quad W = \bigoplus_{i=1}^{m} W_i. \tag{35}$$

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Considering the projections

$$p_{V_i} : V \to V_i, \quad \text{and} \quad p_{W_i} : W \to W_i,$$

and the injections

$$i_{V_i} : V_i \to V \quad \text{and} \quad i_{W_i} : W_i \to W,$$

we obtain, for each $i = 1, \ldots, m$, the pair of endomorphisms

$$(p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i}).$$

The rank of this pair may be not finite; however, if

$$(V_i + W_i) \cap \sum_{j \neq i} (V_j + W_j) \tag{36}$$

is finite dimensional for each $i = 1, \ldots, m$, then we have the following.

PROPOSITION 38. *Let (φ, ψ) be a pair of endomorphisms with finite rank, and let $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ be subspaces of V and W, respectively, satisfying (35) and (36). Then, for each $i = 1, \ldots, m$, the pair $(p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i})$ has finite rank and*

$$\operatorname{Tr}(\varphi, \psi) = \sum_{i=1}^{m} \operatorname{Tr}(p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i}).$$

Let (φ, ψ) be a pair of endomorphisms having finite rank, and consider endomorphisms $\bar{\varphi}$ and $\bar{\psi}$ as in Proposition 37. Since $\bar{\psi} - \bar{\varphi}$ has finite rank, we can consider vectors $u_1, \ldots, u_k \in V + W$ and linear forms $\omega_1, \ldots, \omega_k \in (V + W)^*$ such that

$$\bar{\psi} - \bar{\varphi} = \sum_{i=1}^{k} \omega_i \otimes u_i. \tag{37}$$

More generally, we have

$$\bar{\psi}^n - \bar{\varphi}^n = \sum_{i=1}^{k} \sum_{j=1}^{n} (\omega_i \circ \bar{\psi}^{n-j}) \otimes \bar{\varphi}^{j-1}(u_i),$$

for each $n \in N_1$. This shows that $\bar{\psi}^n - \bar{\varphi}^n$ has finite rank, for each $n \in N_1$. Thus, once more from Proposition 37, we conclude that the pair (φ^n, ψ^n) has finite rank and

$$\operatorname{Tr}(\varphi^n, \psi^n) = \operatorname{Tr}(\bar{\psi}^n - \bar{\varphi}^n), \quad \text{for each } n \in \mathbb{N}_1.$$

Definition 39. Let (φ, ψ) be a pair of endomorphisms having finite rank. We define the determinant of (φ, ψ) to be the following element of $\mathbb{R}[[t]]$:

$$D_{(\varphi,\psi)}(t) \stackrel{\text{def}}{=} \exp \sum_{n \geq 1} -\frac{t^n}{n} \operatorname{Tr}(\varphi^n, \psi^n).$$

Observe that, if ψ has finite rank, then

$$D_{(0,\psi)}(t) = D_{\psi}(t).$$

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If φ and ψ both have finite rank, then

$$D_{(\varphi,\psi)}(t) = D_{\psi}(t)D_{\varphi}(t)^{-1}.$$

So, in these cases, we can use Proposition 35 to compute $D_{(\varphi,\psi)}(t)$. Obviously, in the general case, Proposition 35 does not allow us to compute $D_{(\varphi,\psi)}(t)$ (in general, $D_{\varphi}(t)$ and $D_{\psi}(t)$ are not defined).

In order to compute $D_{(\varphi,\psi)}(t)$ in the general case, we generalize Proposition 35. Let $\bar{\varphi}$ and $\bar{\psi}$ be endomorphisms as in Proposition 37. Considering vectors $u_1, \ldots, u_k \in V + W$ and linear forms $\omega_1, \ldots, \omega_k \in (V + W)^*$ as in (37), we define the matrix

$$M(t) = \begin{bmatrix} \sum_{n\geq 0} \omega_1(\bar{\varphi}^n(u_1))t^n & \cdots & \sum_{n\geq 0} \omega_1(\bar{\varphi}^n(u_k))t^n \\ \vdots & \ddots & \vdots \\ \sum_{n\geq 0} \omega_k(\bar{\varphi}^n(u_1))t^n & \cdots & \sum_{n\geq 0} \omega_k(\bar{\varphi}^n(u_k))t^n \end{bmatrix}, \tag{38}$$

with coefficients in $\mathbb{R}[[t]]$. Observe that, if we identify an endomorphism with finite rank $\psi : U \to U$ with the corresponding pair of finite rank $(0, \psi)$, then the matrix $M(t)$ of (38) coincides with the matrix M defined in (33). Thus, the next proposition, which gives an explicit method for computing $D_{(\varphi,\psi)}(t)$, can be regarded as a generalization of Proposition 35.

PROPOSITION 40. *Let (φ, ψ) be a pair of endomorphisms having finite rank. Then we have*

$$D_{(\varphi,\psi)}(t) = \mathrm{Det}(\mathrm{id} - tM(t)).$$

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