So it appears that I did not deserve full marks in my school geometry exam, but they cannot take them back from me now. Aristotle asserted that a good education is the best insurance for old age, but Euclidean geometry is quite enough for me!

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105.30 A Cinderella theorem in circle geometry

The 'circle theorems' have an established place in the English school geometry curriculum, although, as their converses are not in the syllabus, it turns out that there is not much one can do with them apart from chasing angles around a diagram. However, there is a neglected circle theorem which is rarely mentioned and of which, I suspect, many teachers are not even aware. It turns out, that this 'poor relation' of the circle theorems is really quite useful. I was reminded of it when I processed Des MacHale's paper [1]. I will begin by explaining what the theorem says and then prove it.

In the circle AB and CD are two chords and X and Y are two points on the circumference. Then $AB = CD$ if, and only if, $\angle BXA = \angle DYC$.

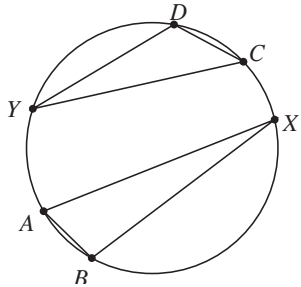


FIGURE 1

At first sight this looks very much like 'angles in the same segment' but the important difference is that the chords are distinct; they can even overlap. In plain language, what is being said is that two chords are equal in length if, and only if, they subtend the same angles at the circumference.

Of course, we really ought to be more careful in the exposition of this result. Any chord which is not a diameter actually subtends two angles at the circumference – one in the major segment which is acute and one in the minor segment which is obtuse. So obviously we must be careful to select the correct pair of angles when working with this theorem. For convenience, the diagrams will denote the acute case.

Usually this theorem is stated in the form 'equal chords subtend equal angles', which makes the point that 'angle size' and 'chord length' are intimately related. One can even bring the concept of arc length into play,

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since it is obvious that this is related to chord length.

What this depends on is the fact that a circle is congruent to itself. This means that we can rotate a circle about the centre, taking with it chords and angles which do not change.

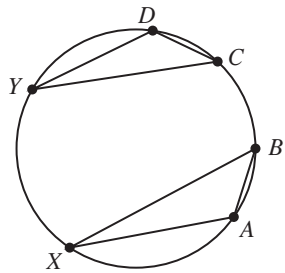


FIGURE 2

Suppose that $AB = CD$. Rotate ΔABX about the centre of the circle so that B coincides with D.

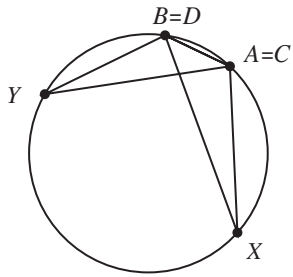


FIGURE 3

Then, since $AB = CD$, A coincides with C.

By angles in the same segment, $\angle DYC = \angle BXA$, as required.

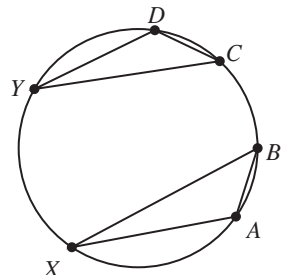


FIGURE 4

Now suppose that $\angle DYC = \angle BXA$.

Rotate ΔABX about the centre of the circle so that B coincides with D.

Suppose that A moves to a point A'.

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Then $\angle DYC = \angle BXA'$.

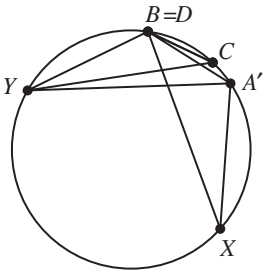


FIGURE 5

By angles in the same segment, $\angle BXA' = \angle BYA'$.
Hence $\angle BYC = \angle DYC = \angle BXA' = \angle BYA'$.
This is clearly impossible unless A' coincides with C.
Hence $AB = CD$, as required.

The argument from angles to sides is much more delicate than the one from sides to angles.

A simpler argument uses the generalised sine rule $a = 2R \sin A$, where R is the circumradius. This means that for any triangle in the circle, the value of a determines the opposite angle, which can be either acute or obtuse as mentioned earlier. The sine rule itself is derived from angles in the same segment, which links the two approaches.

Next I show how this approach is useful in proving the theorems in [1].

Theorem 1: Given $AC = BD$ then

$$\angle BDA = \angle BCA \text{ (angles in same segment),}$$

$$\angle CDA = \angle BCD \text{ (angles on equal chords).}$$

It follows that $\angle CDB = \angle ACD$. Hence $CP = DP$ and so $AP = BP$.

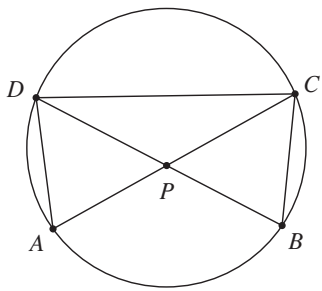


FIGURE 6

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Theorem 2: Given $AB = CD$ then

$$\angle ADB = \angle DBC$$

$$\angle CDA = \angle CBA \text{ (same segment).}$$

It follows that $\angle CDB = \angle DBA$. Hence $PD = PB$ and so $PC = PA$.

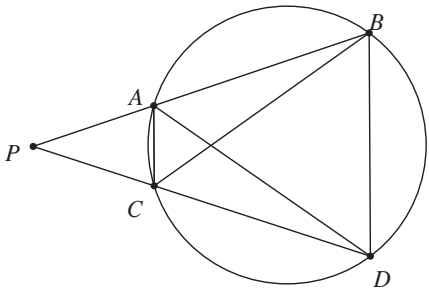


FIGURE 7

Theorem 3: Given $AB = DC$ then

$$\angle BCA = \angle BDA = \angle DBC = \angle DAC \text{ (angles subtended by equal chords).}$$

Hence $BC \,//\, AD$ so $ABCD$ is a trapezium (with equal base angles). Also ΔPDA and ΔPBC are isosceles so $AC = DB$.

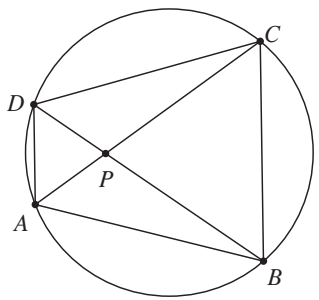


FIGURE 8

Here are a couple of problems for the reader to practise using this Cinderella amongst theorems.

1. A hexagon $ABCDEF$ is inscribed in a circle. If AB is parallel to DE and BC is parallel to EF, prove that CD is parallel to FA.

2. Two circles intersect at P and Q. Point X is a point on one circle, and Y, Z are points on the other circle such that XPY and XQZ are straight lines. Prove that the length of YZ is constant as X varies.

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105.31 On the Brocard disc

The Brocard disc and the incentre

For a triangle ABC with circumcentre O and symmedian point K, the circle with diameter OK is called the *Brocard circle*, named after the French geometer Henri Brocard (1845-1922). Other notable triangle centres that lie on the Brocard circle are the two Brocard points Ω_1 and Ω_2. These are the points in ABC which have the equal angle property:

$$\angle\Omega_1AB = \angle\Omega_1BC = \angle\Omega_1CA \text{ and } \angle\Omega_2BA = \angle\Omega_2CB = \angle\Omega_2AC.$$

The two angles formed by Ω_1 and Ω_2 are equal and this common angle is the *Brocard angle* ω of ABC. For more properties of the Brocard circle and ω, the interested reader is referred to [1]. In this Note we show that the incentre I lies within the Brocard disc.

The triangle OKI and the proof

To prove that I is in the Brocard disc, it is sufficient to show the inequality

$$OK^2 \geqslant OI^2 + KI^2. \tag{1}$$

This result was found by the second author and the original proof was given by lengthy trigonometric calculations, using cotangent metric and areal calculus. The simple proof which follows, found by the first author, is based on calculating the sides of the triangle OKI in terms of the circumradius R, inradius r and semi-perimeter s of the triangle ABC with sides a, b and c.

For a point $P = (x : y : z)$ in areal coordinates, that is, with normalised barycentric coordinates, and any point Q, their squared distance is [2]

$$PQ^2 = xQA^2 + yQB^2 + zQC^2 - yza^2 - zxb^2 - xyc^2. \tag{2}$$

In particular, if Q is the circumcentre, since $x + y + z = 1$, we have

$$OP^2 = R^2 - yza^2 - zxb^2 - xyc^2. \tag{3}$$

The symmedian point has barycentric coordinates [1] $K\left(a^2 : b^2 : c^2\right)$. Hence by (3)

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}. \tag{4}$$

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