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Magnetohydrodynamic wave mixing in shear flows: Hamiltonian equations and wave action

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Abstract. Magnetohydrodynamic wave interactions in a linear shear flow are investigated using the Lagrangian fluid displacement ξ and entropy perturbation ΔS, in which a spatial Fourier solution is obtained in the frame of the background shear flow (Kelvin's method). The equations reduce to three coupled oscillator equations, with time-dependent coefficients and with source terms proportional to the entropy perturbation. In the absence of entropy perturbations, the system admits a wave action conservation integral consisting of positive and negative energy waves. Variational and Hamiltonian forms of the equations are obtained. Examples of wave amplification phenomena and sharp resonant-type wave interactions are obtained. Implications for the interaction of magnetohydrodynamic waves in the shear flow between fast, polar coronal-hole solar wind and slow, streamer belt solar wind are discussed.

1. Introduction

The shear between fast, polar coronal-hole solar wind with speeds of ∼750–800 km s^{-1} and slow, streamer belt solar wind with speeds of ∼350–400 km s^{-1} was observed by the NASA/ESA Ulysses mission (McComas et al. 1998). The Ulysses spacecraft passed from slow, dense solar wind characteristic of the streamer belt, into fast, less dense flow from the northern polar coronal hole over a short time period (of the order of a few days) in late March (1995) (McComas et al. 1998). At the time, the spacecraft was at 1.35 AU and ∼19°N heliographic latitude. The crossing was characterized by several scales. There was a sharp discontinuity at the center of the interface of $\sim 10^5$ km, followed by a larger ramp in the slow solar wind of $\sim 10^7$ km. The transition was characterized by several spatial scales. On the shortest scale ($\sim 10^5$ km), the stream interface is a tangential discontinuity, where the proton and core electron densities and ion and electron pressures all drop, while the magnetic pressure increases in order to maintain pressure balance as one crosses into the faster wind on the northern side of the discontinuity. Following the $\sim 10^7$ km ramp in the slow solar wind, there is a more gradual increase of the wind speed on a scale of $\sim 10^8$ km up to the high-speed flow values for coronal-hole solar wind.

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The interaction of magnetohydrodynamic (MHD) waves in shear flows, such as that observed between fast and slow solar wind, has been developed by a number of authors (e.g., Chagelishvili et al. 1997; Poedts et al. 1998; Kaghashvili 1999; Bodo et al. 2001; Gogoberidze et al. 2004; Rogava et al. 2003 and others). The approach used by these authors is a non-modal approach originally devised by Kelvin (1887), in which the interaction of the spatial Fourier harmonics (SFHs) of the waves in the fluid frame are investigated. The background fluid velocity $\mathbf{u}_0$ and magnetic field $\mathbf{B}_0$ have the form

$$\mathbf{u}_0 = (u_0 + Ay, 0, 0), \quad \mathbf{B}_0 = (B_0, 0, 0), \tag{1.1}$$

and the background density and pressure ρ_0 and p_0 are constant. The SFHs of the perturbed fluid variables $(\delta\rho, \delta\mathbf{u}, \delta\mathbf{B})$ in the model satisfy coupled, linear, first-order differential equations with time-dependent coupling coefficients dependent on the shear parameter A. Gogoberidze et al. (2004) showed that the equations admit a wave action integral. They used a scattering matrix formalism and discussed the role of positive and negative (canonical) energy waves in the wave action integral and discussed the phenomenon of wave amplification in flows with a sufficiently large shear parameter.

The main aim of this paper is to show how these and related results are obtained using the Lagrangian fluid displacement $\xi(\mathbf{x}, t)$ and Lagrangian entropy perturbation ΔS to describe the waves. We use the Lagrangian variational approach of Webb et al. (2005b, c). A preliminary account of MHD wave interactions in a linear shear flow using these methods is given in Webb et al. (2005a). We reduce the equations to normal form. A variational formulation of the equations reveals that the wave action conservation equation is a consequence of Noether's theorem. Two distinct variational formulations are possible depending on whether one considers the evolution of the wave amplitudes in which the $\mathbf{k}$ vector of the waves evolves according to the ray equations of geometrical MHD optics or whether one uses an eigenvector expansion for the waves about a uniform background state with no shear in which $\mathbf{k}$ is constant. One of the main differences between the present approach and the work of Gogoberidze et al. (2004) is that we consider the wave interactions using systems of six first-order differential equations for the normal variables $\{c_j\}$, corresponding to the backward and forward magnetoacoustic and Alfvén modes with entropy wave source terms, whereas Gogoberidze et al. (2004) used a system of three coupled second-order differential equations describing the normal modes $\{\psi_j\}$ $(j = 1, 2, 3)$ in which the $\mathbf{k}$ vector of the waves evolves along the rays.

Section 2 presents non-Wentzel–Kramers–Brouillon (non-WKB) equations for linear waves in a non-uniform background flow, in terms of the Lagrangian fluid displacement ξ and Lagrangian entropy perturbation ΔS (e.g., Webb et al. 2005a). These equations are slight generalizations of the Frieman and Rotenberg (1960) equations for linear MHD waves in a non-uniform background flow including the effects of entropy perturbations, which have formed the basis of many studies of linear MHD stability (e.g., Holm et al. 1985; Van der Holst et al. 1999). The form of the wave equations for the sheared background flow (1.1) are obtained, both in the stationary frame and in the frame of the background flow.

In Sec. 3, the wave equations in the fluid frame in the model shear flow (1.1) are reduced to three coupled second-order oscillator equations for the spatial Fourier transform $\tilde{\xi}(\mathbf{k}, t)$ of $\xi(\mathbf{x}', t)$ where $\mathbf{x}' = \mathbf{x} - \mathbf{u}t$ is the position coordinate in the fluid frame, with entropy wave source terms proportional to $\Delta\tilde{S}(\mathbf{k})$. The equations are

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reduced to normal form based on the eigenvalues and eigenvectors of the matrix $\bar{\mathbf{C}}$ characteristic of the background state (1.1) with no fluid velocity shear (Sec. 3) and also based on the matrix $\mathbf{C}$ including the effects of the shear (Sec. 4). The normal mode equations consist of six coupled first-order ordinary differential equations, with time-dependent coefficients describing the interaction of the different eigenmodes. Variational and Hamiltonian formulations of the normal mode equations are obtained. In the absence of entropy perturbations ΔS, the normal mode equations admit a wave action integral, which is shown to be a consequence of Noether's theorem. The normal mode equations of Secs 3 and 4 describe the same physics and the normal variables $\{c_j\}$ and $\{c'_j\}$ in the two eigensystems are related by a unitary matrix transformation (Sec. 5). The time evolution of the normal variables, in the absence of entropy waves, can be described in terms of scattering matrix theory (Sec. 5.1). Section 6 shows the connection between the normal mode formulation of Gogoberidze et al. (2004), involving three coupled second-order oscillator equations describing the slow and fast magnetoacoustic modes and the Alfvén mode, and the six coupled first-order normal mode equations used in the present analysis. Section 7 presents a variety of numerical examples of wave interactions, emphasizing the important role of the wave action integral. Estimates of the parameters and wave transformation time scales in the shear between fast, coronal-hole solar wind and slower, streamer belt solar wind are discussed. Section 8 concludes with a summary and discussion.

2. Model and perturbation equations

In this section we describe the MHD wave equations for linear, non-WKB waves in a non-uniform background flow (e.g., Webb et al. 2005b) that form the basis of our analysis (Sec. 2.1). The equations for the waves for the background flow (1.1) are described in Sec. 2.2.

2.1. MHD wave equations

By using a Lagrangian variational approach, Webb et al. (2005b) showed that linear MHD waves propagating in a given non-uniform background flow may be described by the equations

$$\mathbf{L}(\xi) + \mathbf{R}(\xi, \Delta S) = 0, \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\Delta S = 0, \tag{2.2}$$

where

$$\mathbf{L}(\xi) = \xi_{tt} + 2\mathbf{u}\cdot\nabla(\xi_t) + \mathbf{u}\mathbf{u} : \nabla\nabla\xi - (a^2+b^2)\nabla(\nabla\cdot\xi) + \mathbf{b}\cdot\nabla(\nabla\xi)\cdot\mathbf{b} - \mathbf{b}\mathbf{b} : \nabla\nabla\xi + [\mathbf{b}\cdot\nabla(\nabla\cdot\xi)]\mathbf{b} \tag{2.3}$$

corresponds to the second derivatives of the Lagrangian displacement ξ of the fluid element describing the waves. Here

$$a = \left(\frac{\partial p}{\partial \rho}\right)_S^{1/2}, \quad \mathbf{b} = \frac{\mathbf{B}}{\sqrt{\mu\rho}} \tag{2.4}$$

denote the adiabatic sound speed a and the Alfvén velocity $\mathbf{b}$, respectively. The gas density, fluid velocity and pressure are denoted by ρ, $\mathbf{u}$ and p; $\mathbf{B}$ is the magnetic

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induction and μ is the magnetic permeability. The term $\mathbf{R}$ in (2.1) given by

$$\mathbf{R}(\xi,\Delta S)=\frac{1}{\rho}\Big\{\nabla\cdot\xi\Big[\nabla(p-a^2\rho)+\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu}-\nabla\Big(\frac{B^2}{2\mu}\Big)\Big]$$
$$-[\nabla\xi+(\nabla\xi)^t]\cdot\nabla\Big(p+\frac{B^2}{2\mu}\Big)+\nabla(p_S\Delta S)$$
$$+\nabla\Big(\frac{\mathbf{BB}}{\mu}\Big):\nabla\xi+\rho\xi\cdot\nabla\nabla\phi\Big\}, \tag{2.5}$$

corresponds to lower order derivatives of ξ, terms linear in ξ (the gravitational term) and terms independent of ξ (the entropy wave contribution). Here $\phi(\mathbf{x})$ denotes an external gravitational potential, and

$$p=\rho\frac{\partial\varepsilon}{\partial\rho}-\varepsilon,\quad T=\frac{1}{\rho}\frac{\partial\varepsilon}{\partial S},\quad h=\frac{\varepsilon+p}{\rho}=\frac{\partial\varepsilon}{\partial\rho} \tag{2.6}$$

are the standard expressions for the gas pressure p, temperature T and enthalpy h in terms of the internal energy density $\varepsilon(\rho,S)$. Equation (2.2) expresses the fact that the Lagrangian entropy perturbation ΔS is adverted with the background flow.

Equations (2.1)–(2.5) are generalizations of the Frieman and Rotenberg (1960) equations used to study the stability of steady MHD background flows in an external gravitational potential $\phi(\mathbf{x})$. Frieman and Rotenberg effectively generalized the energy principle of Bernstein et al. (1958) who studied the stability of magnetostatic equilibria. Equations (2.1)–(2.5) also apply for non-steady background flows, and include the effects of the Lagrangian entropy perturbation ΔS.

A discussion of the characteristic manifolds of (2.1) and (2.2) for both steady and non-steady flows is given by Webb et al. (2005b). The characteristic manifolds for (2.1) and (2.2) are manifolds on which the initial value problem for the waves (the Cauchy problem) does not have a unique solution. The characteristic manifolds of (2.1) and (2.2) are the same as the characteristic manifolds of the fully nonlinear MHD equations, and correspond to the Alfvén waves, fast and slow magnetoacoustic waves and the entropy wave.

2.2. The model

We use the model shear flow considered by Chagelishvili et al. (1997), Poedts et al. (1998), Kaghasvhili (1999), Gogoberidze et al. (2004) and others. For this model shear flow,

$$\mathbf{u}_0=(u^x(y),0,0),\quad \mathbf{B}_0=(B_0,0,0),\quad u^x(y)=u_0+Ay\equiv u. \tag{2.7}$$

The background gas pressure p_0, density ρ_0 and entropy S_0 are assumed to be constant. The effects of gravity are neglected (i.e. $\phi(\mathbf{x})=0$ in the model).

For the above model background flow, the linear wave interaction equations (2.1) and (2.2) reduce to the coupled equations

$$\xi^i_{tt}+2u\xi^i_{xt}+(u^2-b^2)\xi^i_{xx}-(a^2+b^2)\nabla_i(\nabla\cdot\xi)$$
$$+b^2[\nabla_i(\xi^x_x)+\delta^i_1\partial_x(\nabla\cdot\xi)]+\frac{p}{\rho}\nabla_i\Delta\bar{S}=0, \tag{2.8}$$

$$\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\Big)\Delta\bar{S}=0, \tag{2.9}$$

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where $u = u^x$ is the background flow speed, $\bar{S} = S/C_v$ and C_v is the specific heat of the gas at constant volume.

In Kelvin's method (Kelvin 1887), the next step is to transform to the co-moving fluid frame, using the change of independent variables

$$x' = x - ut, \quad y' = y, \quad z' = z, \quad t' = t. \tag{2.10}$$

Using the independent variables (2.10), the wave equation system (2.8) and (2.9) becomes

$$\xi^x_{t't'} - a^2\xi^x_{x'x'} - a^2(\xi^x_{y'x'} + \xi^z_{z'x'} - At\xi^y_{x'x'}) + \frac{p}{\rho}\frac{\partial \Delta\bar{S}}{\partial x'} = 0, \tag{2.11}$$

$$\xi^y_{t't'} - b^2\xi^y_{x'x'} - (b^2+a^2)(\xi^y_{y'y'} + A^2t^2\xi^y_{x'x'} - 2At\xi^y_{x'y'}) - a^2(\xi^x_{x'y'} - At\xi^x_{x'x'}) - (b^2+a^2)(\xi^z_{y'z'} - At\xi^z_{x'x'}) + \frac{p}{\rho}\left(\frac{\partial}{\partial y'} - At\frac{\partial}{\partial x'}\right)\Delta\bar{S} = 0, \tag{2.12}$$

$$\xi^z_{t't'} - b^2\xi^z_{x'x'} - a^2\xi^x_{x'z'} - (a^2+b^2)(\xi^y_{z'y'} - At\xi^y_{x'z'} + \xi^z_{z'z'}) + \frac{p}{\rho}\frac{\partial \Delta\bar{S}}{\partial z'} = 0, \tag{2.13}$$

in which $\Delta\bar{S}(x', y', z')$ is advected with the background flow.

In Appendix A, we list the transformations between the Lagrangian perturbation variables ξ and ΔS, and the Eulerian perturbations $\delta\rho, \delta\mathbf{u}, \delta\mathbf{B}$ and δp.

3. Reduction to a coupled oscillator system

Since the coefficients of the derivatives in (2.11)–(2.13) are either constants or depend only on time t, the system can be reduced to an ordinary differential equation system with independent variable t by using the Fourier transform representation

$$\xi^j(\mathbf{x}', t) = \int_{-\infty}^{\infty} d^3k \exp(i\mathbf{k}\cdot\mathbf{x}')\tilde{\xi}^j(\mathbf{k}, t). \tag{3.1}$$

Using the Fourier representation (3.1) in (2.11)–(2.13) we obtain the ordinary differential equation system

$$\frac{d^2\tilde{\boldsymbol{\xi}}}{dt^2} + \mathbf{A}\tilde{\boldsymbol{\xi}} = -i\frac{p}{\rho}\tilde{\mathbf{k}}\Delta\tilde{S}, \tag{3.2}$$

where

$$\tilde{\mathbf{k}} = (k_x, k_y - Atk_x, k_z)^{\mathrm{T}} \equiv (k_x, \bar{k}_y, k_z)^{\mathrm{T}}, \tag{3.3}$$

and the matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{pmatrix} a^2k_x^2 & a^2k_x\bar{k}_y & a^2k_xk_z \\ a^2k_x\bar{k}_y & b^2k_x^2 + (b^2+a^2)\bar{k}_y^2 & (a^2+b^2)k_z\bar{k}_y \\ a^2k_xk_z & (a^2+b^2)k_z\bar{k}_y & b^2k_x^2 + (a^2+b^2)k_z^2 \end{pmatrix}. \tag{3.4}$$

It is interesting to note that the ray equations of geometrical MHD optics in the linear shear flow (2.7) have integrals

$$k_x = k_{x0}, \quad k_y = k_{y0} - Atk_x, \quad k_z = k_{z0}, \quad \omega = \omega_0, \tag{3.5}$$

where the subscript 0 denotes the initial point. These integrals apply for each of the MHD eigenmodes (the Alfvén, magnetoacoustic and entropy waves). Note

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that $\bar{k}_y$ and $\tilde{\mathbf{k}}$ in (3.3) are constant on the ray trajectories. Further details of the characteristics and rays are described in Appendix B. In (3.2), the entropy perturbation

$$\Delta\tilde{S} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} d^3k \Delta\bar{S}\exp(-i\mathbf{k}\cdot\mathbf{x}') \tag{3.6}$$

depends only on $\mathbf{k} = (k_x, k_y, k_z)^{\mathrm{T}}$.

Equations (3.2) describe the interaction of three coupled oscillators. Similar, related equations using an Eulerian perturbation approach have been obtained by Chagelishvili et al. (1997), Poedts et al. (1998), Kaghashvili (1999), Rogava et al. (2003) and Gogoberidze et al. (2004). The system (3.2) differs from previous analyses by the inclusion of an entropy perturbation source term proportional to $\Delta\tilde{S}$ on the right-hand side of (3.2). The relations between ξ, ΔS and the Eulerian perturbations are described in Appendix A.

Below we reduce the equation system (3.2) to normal form. To this end, we set

$$\mathbf{w} = (\tilde{\xi}^x, \tilde{\xi}^y, \tilde{\xi}^z, \tilde{\xi}^x_t, \tilde{\xi}^y_t, \tilde{\xi}^z_t)^{\mathrm{T}}. \tag{3.7}$$

Using $\mathbf{w}$ as the dependent variables the second-order differential equation system (3.2) reduces to the first-order system

$$\frac{d\mathbf{w}}{dt} + \mathbf{C}\mathbf{w} = \mathbf{Q}, \tag{3.8}$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{O}_3 & -\mathbf{I}_3 \\ \mathbf{A} & \mathbf{O}_3 \end{pmatrix}. \tag{3.9}$$

In (3.9) $\mathbf{O}_3$ and $\mathbf{I}_3$ are the zero and unit 3×3 matrices respectively, and the matrix $\mathbf{A}$ is given by (3.4). The source term $\mathbf{Q}$ in (3.8) due to the entropy wave has the form

$$\mathbf{Q} = -i\frac{p}{\rho}\Delta\tilde{S}(0,0,0,k_x,\bar{k}_y,k_z)^{\mathrm{T}}, \tag{3.10}$$

where $\bar{k}_y = k_y - Atk_x$.

One can carry out the normal mode analysis of the first-order differential equation system (3.8) in two different ways. In this section we carry out the normal mode analysis by splitting the matrix $\mathbf{C}$ up into a time independent matrix $\bar{\mathbf{C}}$ corresponding to the background flow (1.1) without shear, plus a time-dependent matrix $\delta\mathbf{C}$ dependent on the shear parameter A. In Sec. 4, we carry out a similar normal mode analysis based on the complete matrix $\mathbf{C}$. Both of these approaches are equivalent, but view the evolution of the perturbations from a different perspective. The detailed proof of the equivalence of the two systems and their relationship to each other are described in Sec. 5.

3.1. Reduction to normal form

The matrix $\mathbf{C}$ in (3.9) contains both a background component independent of the shear parameter A, and other terms dependent on A. To separate these we write

$$\mathbf{C} = \bar{\mathbf{C}} + \delta\mathbf{C}, \quad \bar{\mathbf{C}} = \begin{pmatrix} \mathbf{O}_3 & -\mathbf{I}_3 \\ \bar{\mathbf{A}} & \mathbf{O}_3 \end{pmatrix}, \quad \delta\mathbf{C} = \begin{pmatrix} \mathbf{O}_3 & \mathbf{O}_3 \\ \delta\mathbf{A} & \mathbf{O}_3 \end{pmatrix}, \tag{3.11}$$

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where $\bar{\mathbf{C}}$ contains the background effects, and $\delta\mathbf{C}$ the effects of the shear. Here, $\bar{\mathbf{A}}$ is given by (3.4) but with the replacement $\bar{k}_y \to k_y$ and $\delta\mathbf{A}$ is given by

$$\delta\mathbf{A} = -Atk_x \begin{pmatrix} 0 & a^2k_x & 0 \\ a^2k_x & (a^2+b^2)(2k_y - Atk_x) & (a^2+b^2)k_z \\ 0 & (a^2+b^2)k_z & 0 \end{pmatrix}. \tag{3.12}$$

Note that the matrix $\delta\mathbf{A}$ contains both linear and quadratic terms in t and is dependent on the shear parameter A, whereas the matrix $\bar{\mathbf{A}}$ is independent of the shear parameter A. Using the decompositions (3.11) and (3.12) for $\mathbf{C}$, the first-order differential equation system (3.8) reduces to the system

$$\frac{d\mathbf{w}}{dt} + \bar{\mathbf{C}}\mathbf{w} + \delta\mathbf{C}\mathbf{w} = \mathbf{Q}, \tag{3.13}$$

where $\delta\mathbf{C}$ contains the effects of the shear.

PROPOSITION 3.1. *The first-order differential equation system (3.13) describing MHD waves in the shear flow (1.1) has solutions of the form*

$$\mathbf{w} = {\sum_s}' a_s(\boldsymbol{k}, t)\mathbf{R}_s, \tag{3.14}$$

where the primed superscript denotes summation over a complete set of right eigenvectors $\{\mathbf{R}_j\}$ *of the matrix* $\bar{\mathbf{C}}$ *(i.e. for* $s = 1, 2, 3, 5, 6, 7$ *since we reserve the index* $s = 4$ *for the entropy wave). The left and right eigenvectors of the matrix* $\bar{\mathbf{C}}$, $\mathbf{L}_j$ *and* $\mathbf{R}_j$ *satisfy the eigen equations*

$$(\bar{\mathbf{C}} - i\omega_j\mathbf{I}_6)\cdot\mathbf{R}_j = 0, \quad \mathbf{L}_j\cdot(\bar{\mathbf{C}} - i\omega_j\mathbf{I}_6) = 0, \quad \det(\mathbf{C} - i\omega_j\mathbf{I}_6) = 0, \tag{3.15}$$

where the eigenvectors are normalized so that $\mathbf{L}_j\cdot\mathbf{R}_s = \delta_{js}$. *The wave amplitudes* $\{a_j(\mathbf{k}, t)\}$ *satisfy the coupled wave interaction equations*

$$\frac{da_j}{dt} + i\omega_j a_j + {\sum_s}' \bar{\Lambda}_{js}a_s = \mathbf{L}_j\cdot\mathbf{Q} \equiv q_j, \quad j = 1, 2, 3, 5, 6, 7, \tag{3.16}$$

where the sum over s *is for* $s = 1, 2, 3, 5, 6, 7$ *and the wave interaction coefficients* $\bar{\Lambda}_{js}$ *are given by*

$$\bar{\Lambda}_{js} = \mathbf{L}_j\cdot\delta\mathbf{C}\cdot\mathbf{R}_s, \tag{3.17}$$

and $\mathbf{Q}$ *denotes the entropy wave source term (3.10). The detailed form of the wave interaction coefficients* $\{\bar{\Lambda}_{js}\}$ *in (3.16) depends on the normalization constants for the eigenvectors, which are chosen to satisfy* $\mathbf{L}_j\cdot\mathbf{R}_s = \delta_{js}$.

The eigenvalues $\{\omega_j\}$ *satisfy the eigenvalue equation*

$$\det(\bar{\mathbf{C}} - i\omega\mathbf{I}_6) \equiv \det(\bar{\mathbf{A}} - \omega^2\mathbf{I}_3) = 0. \tag{3.18}$$

The eigenvalues $\omega_j = k\lambda_j$, *where* $\lambda = \omega/k$ *denotes the phase velocity and* $k = |\mathbf{k}|$. *In particular,*

$$\det(\bar{\mathbf{A}} - \omega^2\mathbf{I}_3) = -k^6(\lambda^2 - b_n^2)(\lambda^4 - (a^2+b^2)\lambda^2 + a^2b_n^2) = 0. \tag{3.19}$$

The eigenvalue solutions of (3.19) for the $\{\lambda_j\}$ *(note* $\omega_j = k\lambda_j$*) are given by*

$$\lambda_1 = -c_f, \quad \lambda_2 = -b_n, \quad \lambda_3 = -c_s, \quad \lambda_5 = c_s, \quad \lambda_6 = b_n, \quad \lambda_7 = c_f. \tag{3.20}$$

The eigenvelocities (3.20) correspond to the backward and forward fast and slow magnetoacoustic waves $(\lambda_1, \lambda_3, \lambda_5, \lambda_7)$ *and the backward and forward Alfvén waves*

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(λ_2 and λ_6). The speed $\lambda_4 = 0$ corresponds to the entropy wave. The magnetoacoustic dispersion equation

$$c^4 - (a^2 + b^2)c^2 + a^2 b_n^2 = 0 \tag{3.21}$$

has solutions $c = \pm c_f$ and $c = \pm c_s$ corresponding to the fast and slow magnetoacoustic waves and $b_n = \mathbf{b}\cdot\mathbf{n}$ is the phase speed of the Alfvén wave with wave normal $\mathbf{n} = \mathbf{k}/k$.

The right and left eigenvector solutions of (3.15) for the Alfvén waves are given by

$$\mathbf{R}_j = N_r^{(j)}[\mathbf{U}_j - i\omega_j \mathbf{V}_j], \quad \mathbf{L}_j = N_l^{(j)}[-i\omega_j \mathbf{U}_j^{\mathrm{T}} + \mathbf{V}_j^{\mathrm{T}}], \quad j = 2, 6,$$

$$\mathbf{U}_j = (0, n^z, -n^y, 0, 0, 0)^{\mathrm{T}}, \quad \mathbf{V}_j = (0, 0, 0, 0, n^z, -n^y)^{\mathrm{T}}, \tag{3.22}$$

where $\omega_2 = -kb_n$ and $\omega_6 = kb_n$. The normalization condition $\mathbf{L}_j \cdot \mathbf{R}_j = 1$ for the eigenvectors implies

$$N_l^{(j)} N_r^{(j)} = \frac{i}{2\omega_j \Delta_j}, \quad \Delta_j = 1 - (n^x)^2, \ j = 2, 6. \tag{3.23}$$

Similarly, the left and right eigenvectors for the magnetoacoustic waves have the form

$$\mathbf{R}_j = N_r^{(j)}[\mathbf{U}_j - i\omega_j \mathbf{V}_j], \quad \mathbf{L}_j = N_l^{(j)}[-i\omega_j \mathbf{U}_j^{\mathrm{T}} + \mathbf{V}_j^{\mathrm{T}}],$$

$$\mathbf{U}_j = [(\lambda_j^2 - b^2)n^x, \lambda_j^2 n^y, \lambda_j^2 n^z, 0, 0, 0]^{\mathrm{T}},$$

$$\mathbf{V}_j = [0, 0, 0, (\lambda_j^2 - b^2)n^x, \lambda_j^2 n^y, \lambda_j^2 n^z]^{\mathrm{T}}, \quad j = 1, 3, 5, 7. \tag{3.24}$$

The normalization conditions $\mathbf{L}_j \cdot \mathbf{R}_s = \delta_{js}$ are satisfied if we choose

$$N_r^{(j)} N_l^{(j)} = \frac{i}{2\omega_j \Delta_j}, \quad \Delta_j = (\lambda_j^2 - b_n^2)^2 + b_n^2 b_\perp^2, \quad j = 1, 3, 5, 7, \tag{3.25}$$

where $b_\perp^2 = b^2 - b_n^2$. It is of interest to note that

$$\bar{\Lambda}_{js} = \mathbf{l}_j \cdot \delta\mathbf{A} \cdot \mathbf{r}_s, \tag{3.26}$$

where

$$\mathbf{r}_s = (R_s^1, R_s^2, R_s^3)^{\mathrm{T}}, \quad \mathbf{l}_j = (L_j^4, L_j^5, L_j^6), \tag{3.27}$$

and R_s^p and L_s^p refer to the components of $\mathbf{R}_s$ and $\mathbf{L}_s$, respectively. The wave interaction coefficients (3.17) or (3.26) can be seen to be pure imaginary, i.e.

$$\bar{\Lambda}_{js} = i\bar{\Omega}_{js}, \tag{3.28}$$

where the $\bar{\Omega}_{js}$ are real.

Proof. Substitution of the solution ansatz (3.14) for $\mathbf{w}$ in (3.13) and taking the scalar product of the resultant equation with $\mathbf{L}_j$ yields the wave interaction equations (3.16). Equation (3.16) is, in effect, a compatibility condition on the solution ansatz (3.14) in order that (3.14) is a solution of (3.13), and is known as the Fredholm alternative condition. The detailed form of the wave interaction coefficients $\bar{\Lambda}_{js}$ depends on the choice of the normalization constants $N_l^{(j)}$ and $N_r^{(j)}$ in (3.23) and (3.25). □

Proposition 3.2. *For the choice $N_r^{(j)} = 1$ of the right eigenvectors for both the Alfvén and magnetoacoustic eigenvectors in (3.22) and (3.24) the wave interaction*

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coefficients (3.17) have the form $\bar{\Lambda}_{js} = i\bar{\Omega}_{js}$ *listed below. The Alfvén wave interaction coefficients are given by*

$$\bar{\Omega}_{22} = \bar{\Omega}_{26} = \frac{A^2t^2k^2(n^x)^2(n^z)^2(a^2+b^2)}{2\omega_2\Delta_2},$$

$$\bar{\Omega}_{2j} = -\bar{\Omega}_{6j} = \frac{Atk^2n^xn^z\lambda_j^2[b_n^2 - \lambda_j^2 + At(a^2+b^2)n^xn^y]}{2\omega_2\Delta_2}, \quad j = 1,3,5,7. \tag{3.29}$$

The magnetoacoustic wave interaction coefficients $\bar{\Lambda}_{js} = i\bar{\Omega}_{js}$ *in (3.16) are given by*

$$\bar{\Omega}_{j2} = \bar{\Omega}_{j6} = -\frac{Atk^2n^xn^z\lambda_j^2[(\lambda_j^2 - b_n^2) - (a^2+b^2)Atn^xn^y]}{2\omega_j\Delta_j},$$

$$\bar{\Omega}_{js} = -\frac{Atk^2n^xn^y\lambda_j^2\lambda_s^2[\lambda_j^2 + \lambda_s^2 - 2b_n^2 - At(a^2+b^2)n^xn^y]}{2\omega_j\Delta_j}, \tag{3.30}$$

where $j, s = 1,3,5,7$. *The wave interaction coefficients* $\bar{\Omega}_{js}$ *have the symmetries*

$$\bar{\Omega}_{js} = -\bar{\Omega}_{j's} \quad \textit{where } j + j' = 8 \tag{3.31}$$

corresponding to a change of sign of the characteristic velocities for backward and forward waves. The source terms q_j *in (3.16) due to the entropy wave perturbations have the form*

$$q_2 = -q_6 = -\frac{Atkn^xn^z}{2\omega_2\Delta_2}\frac{p\Delta\bar{S}}{\rho},$$

$$q_j = \frac{k[\lambda_j^2 - b_n^2 - At\lambda_j^2n^xn^y]}{2\omega_j\Delta_j}\frac{p\Delta\bar{S}}{\rho}, \quad j = 1,3,5,7. \tag{3.32}$$

Note that $q_{j'} = -q_j$ *for* $j + j' = 8$ *due to the wave-reversal symmetry for backward and forward propagating waves.*

Comment. For the choice $N_r^{(j)} = 1$, (3.23) requires $N_l^{(j)} = i/2\omega_j\Delta_j$. The derivation of (3.29)–(3.31) follows from either (3.17) or (3.26), and the entropy wave source terms q_j can be determined from (3.16).

PROPOSITION 3.3 (Physical constraints). *The wave amplitudes* $a_j(\mathbf{k},t)$ *in the eigenvector expansion (3.14) satisfy the constraints*

$$a_j^*(\mathbf{k},t) = -a_{j'}(-\mathbf{k},t) \quad \textit{where } j + j' = 8. \tag{3.33}$$

Proof. The physical perturbation vector

$$\hat{\mathbf{w}} = (\xi^x, \xi^y, \xi^z, \dot{\xi}^x, \dot{\xi}^y, \dot{\xi}^z) \tag{3.34}$$

is necessarily a real function of (x,y,z,t). The spatial Fourier transform of $\hat{\mathbf{w}}$ consequently has the property

$$\mathbf{w}(\mathbf{k},t)^* = \mathbf{w}(-\mathbf{k},t). \tag{3.35}$$

Using the eigenvector expansion (3.14) and the orthonormality of the left and right eigenvectors $\{\mathbf{L}_j\}$ and $\{\mathbf{R}_j\}$ in (3.35) implies

$$a_j(\mathbf{k},t)^* = \sum_p{}' \alpha_{jp}a_p(-\mathbf{k},t) \quad \text{where } \alpha_{jp} = \mathbf{L}_j^*(\mathbf{k},t)\cdot\mathbf{R}_p(-\mathbf{k},t). \tag{3.36}$$

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Using the eigenvector results (3.22)–(3.25) we find that $\alpha_{jp} = -\delta_{pj'}$ where $j + j' = 8$ (i.e. the index j' corresponds to the oppositely directed wave mode of the same type as j). This in turn, implies that the wave amplitudes $a_j(\mathbf{k}, t)$ satisfy (3.33). □

Comment. The result (3.33) is independent of the detailed choice of the normalization constants $N_l^{(j)}$ and $N_r^{(j)}$ for the left and right eigenvectors. The result (3.33) is also a key property in the variational formulation of the wave interaction equations (Proposition 3.6).

3.2. Wave action and Hamiltonian formulation

The wave amplitudes $\{a_j\}$ in the normal mode equations (3.16) depend on the choice of the normalization constants $N_r^{(j)}$ and $N_l^{(j)}$ for the right and left eigenvectors, subject to the constraint $\mathbf{L}_j \cdot \mathbf{R}_s = \delta_{js}$ used in the analysis. In (3.16) and (3.17) we have arbitrarily chosen $N_r^{(j)} = 1$. An alternative normalization is possible based on the wave action and wave energy densities of the different eigenmodes. This latter normalization has the advantage of reducing the equation system to Hamiltonian form, and has the added bonus of revealing extra symmetries in the wave interaction coefficients (see, e.g., Sagdeev and Galeev (1969) for a similar discussion for three wave resonant interactions in MHD). For the normalization with $N_r^{(j)} = 1$, the wave energy densities of the Alfvén and magnetoacoustic eigenmodes can be written in the form

$$E_w^{(j)} = \rho |a_j|^2 \omega_j^2 \Delta_j, \quad j = 1, 2, 3, 5, 6, 7. \tag{3.37}$$

The corresponding wave action densities A_j are given by

$$A_j = \frac{E_w^{(j)}}{\omega_j} = \rho \omega_j |a_j|^2 \Delta_j, \quad j = 1, 2, 3, 5, 6, 7. \tag{3.38}$$

The detailed derivation of the results (3.37) and (3.38) depends on the relationships between the Eulerian perturbations $\delta\rho$, $\delta\mathbf{u}$, δp and $\delta\mathbf{B}$ for the different wave modes and the Lagrangian displacement ξ (see Appendix C).

To reduce the system (3.16) to Hamiltonian form, we introduce new wave amplitudes $\{c_j\}$ such that

$$c_j = \nu_j a_j, \quad \nu_j = \sqrt{\rho |\omega_j| \Delta_j}. \tag{3.39}$$

In terms of the new wave amplitudes, the wave action densities A_j in (3.38) have the form

$$A_j = \rho \omega_j |a_j|^2 \Delta_j = \sigma_j |c_j|^2 \quad \text{where } \sigma_j = \mathrm{sgn}(\omega_j) \tag{3.40}$$

is the sign of the frequency of the jth MHD eigenmode. The wave amplitudes $\{c_j\}$ are the natural Hamiltonian variables for the system (e.g., Sagdeev and Galeev 1969).

PROPOSITION 3.4. *The wave interaction equations (3.16) may be written in terms of the Hamiltonian variables $\{c_j(t)\}$ in the form*

$$\frac{dc_j}{dt} + \omega_j c_j + \Lambda_{js} c_s = -\Lambda_{j4} \Delta \tilde{S}. \tag{3.41}$$

The wave interaction coefficients Λ_{js} ($s \neq 4$) are pure imaginary, i.e.

$$\Lambda_{js} = i\Omega_{js} \quad (s \neq 4). \tag{3.42}$$

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The Ω_{js} are real, and possess the symmetries

$$\sigma_j\Omega_{js} = \sigma_s\Omega_{sj}, \quad \Omega_{j's} = -\Omega_{js}, \quad j' + j = 8, \quad \Omega_{js'} = \Omega_{js}, \quad s + s' = 8. \tag{3.43}$$

Thus, the wave interaction equations (3.16) have a more symmetric form when written in terms of the canonical variables $\{c_j(t)\}$. These symmetries play an important role in the variational formulation of the equations. The symmetries (3.43) imply that $|\Omega_{js}| = |\Omega_{sj}|$.

Proof. The Ω_{js} ($s \neq 4$) are related to the $\bar{\Omega}_{js}$ coefficients (3.29) and (3.30) by the equations

$$\Omega_{js} = \frac{\nu_j\bar{\Omega}_{js}}{\nu_s} \equiv \frac{\sigma_j f_{js}}{2(|\omega_j|\Delta_j|\omega_s|\Delta_s)^{1/2}}, \tag{3.44}$$

where the f_{js} are symmetric in the indices j and s. This implies that $|\Omega_{js}|$ is symmetric in the indices j and s. The coefficients

$$f_{js} = -Atk^2n^xn^y\lambda_j^2\lambda_s^2[\lambda_j^2 + \lambda_s^2 - 2b_n^2 - At(a^2 + b^2)n^xn^y] \tag{3.45}$$

($j, s = 1, 3, 5, 7$) describe the magnetoacoustic–magnetoacoustic wave interactions; the coefficients

$$f_{66} = f_{22} = f_{26} = (At)^2k^2(n^xn^z)^2(a^2 + b^2) \tag{3.46}$$

describe the Alfvén–Alfvén wave interactions, and

$$f_{j2} = f_{j6} = -Atk^2n^xn^z\lambda_j^2[\lambda_j^2 - b_n^2 - (a^2 + b^2)Atn^xn^y] \tag{3.47}$$

describes the Alfvén–magnetoacoustic wave interactions.

The entropy wave interaction coefficients Λ_{j4} in (3.41) are

$$\Lambda_{j4} = -\frac{a^2k[\lambda_j^2 - b_n^2 - At\lambda_j^2n^xn^y]\rho^{1/2}}{2(|\omega_j|\Delta_j)^{1/2}\gamma}\sigma_j, \quad j = 1, 3, 5, 7,$$

$$\Lambda_{j4} = \frac{a^2Atkn^xn^z\rho^{1/2}}{2(|\omega_j|\Delta_j)^{1/2}\gamma}\sigma_j, \quad j = 2, 6, \tag{3.48}$$

where γ is the adiabatic index of the gas. Note that $\Lambda_{j4} = -\Lambda_{j'4}$ where $j + j' = 8$ in (3.48), corresponding to the wave reversal symmetry associated with backward and forward moving waves of the same type. □

Proposition 3.5 (Wave action integral). *If there are no entropy perturbations (i.e. $\Delta S = 0$) the wave interaction equations (3.41) possess the wave action integral*

$$\sum_j{}' \sigma_j|c_j|^2 \equiv \sum_j{}' A_j = N = \text{constant}, \tag{3.49}$$

where A_j is the wave action of the jth wave mode.

Proof. To derive the wave action integral, we first note that the wave interaction equations (3.41) can be written in the form

$$\frac{dc_j}{dt} + i\omega_jc_j + i\Omega_{js}c_s = -\Lambda_{j4}\Delta\tilde{S}, \quad 1 \leqslant j, s \leqslant 7, \ j, s \neq 4, \tag{3.50}$$

where the entropy wave interaction coefficients Λ_{j4} are given by (3.48).

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From (3.50) and its complex conjugate we obtain the equation

$$\sum_j{}' \sigma_j \frac{d|c_j|^2}{dt} - 2\sum_j{}' \sum_s{}' \sigma_j \Omega_{js} \,\mathrm{Im}(c_s c_j^*) + 2\Delta\tilde{S} \sum_j{}' \sigma_j \Lambda_{j4} \,\mathrm{Re}(c_j) = 0, \qquad (3.51)$$

where $\sum_j'$ means sum over all indices $1 \leqslant j \leqslant 7$ but $j \neq 4$. The second term in (3.51) can be shown to be zero by using the symmetry property $\sigma_j \Omega_{js} = \sigma_s \Omega_{sj}$ (see (3.43)). Hence, (3.51) reduces to

$$\sum_j{}' \sigma_j \frac{d|c_j|^2}{dt} + 2\Delta\tilde{S} \sum_j{}' \sigma_j \Lambda_{j4} \,\mathrm{Re}(c_j) = 0. \qquad (3.52)$$

Thus, if there are no entropy perturbations ($\Delta\tilde{S} = 0$), (3.52) integrates to yield the wave action integral (3.49).

Comment. The wave action integral (3.49) was also derived by Gogoberidze et al. (2004) using an Eulerian perturbation approach, but has a different form to (3.49) due to the different variables used in their analysis. They used variables ψ_j related to the ray-system analysis of Sec. 4 (see also Sec. 6 for more discussion). The result (3.52) shows that the wave action integral no longer applies in the presence of entropy perturbations. The wave action A_j is sometimes referred to as the quasi-particle number density (e.g., Sagdeev and Galeev 1969) or as the canonical wave energy density (e.g., Dewar 1970, 1977). Note that $A_j = \sigma_j |c_j|^2$ is negative if $\sigma_j = \mathrm{sgn}(\omega_j) < 0$, and in this case the wave is said to be a negative energy wave. Since the total number of quasi-particles N is conserved, then the presence of both negative and positive energy waves in the same system in many cases implies wave instability and growth, since the growth of the positive energy waves is balanced by the growth of the negative energy waves in such a way as to conserve the total wave action. Gogoberidze et al. (2004) discuss the phenomenon of wave over-reflection in the high-shear regime, which is associated with the growth of both positive and negative energy waves. The source of the wave energy is of course the background flow, which in the linear approximation used here is assumed not to be affected by the loss of energy to the waves. In a more consistent treatment, total energy of the waves and the background must be conserved, implying that there will be limitations on the wave growth.

Proposition 3.6 (Hamilton's Principle and shear flows). *The wave interaction equations (3.41) or (3.50) written in terms of the Hamiltonian variables $\{c_j(t)\}$ result from the action principle $\delta J = 0$, where the action has the form*

$$J = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3k \; \mathscr{L}, \qquad (3.53)$$

and

$$\mathscr{L} = i \sum_p{}' \left[\sigma_p c_p^* \left(\frac{dc_p}{dt} + i\omega_p c_p + \sum_s{}' \Lambda_{ps} c_s \right) + \sigma_p \Lambda_{p4} (c_p^* - c_p) \Delta\tilde{S} \right] \qquad (3.54)$$

is the Lagrangian density in $\mathbf{k}$*-space.*

Proof. Using the physical reality constraints $c_j(\mathbf{k}, t)^* = -c_{j'}(-\mathbf{k}, t)$ where $j + j' = 8$ (see (3.33)), the symmetry properties $\sigma_j \Lambda_{jp} = \sigma_p \Lambda_{pj}$ from (3.43), and noting

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$\Lambda_{ps}(\mathbf{k},t) = \Lambda_{ps}(-\mathbf{k},t)$, in (3.53) and (3.54) we obtain

$$\frac{\delta J}{\delta c_p^*} = 2i\left(\frac{dc_p}{dt} + i\omega_p c_p + \sum_s{}' \Lambda_{ps} c_s + \Lambda_{p4}\Delta\tilde{S}\right) = 0, \tag{3.55}$$

for extremal variations of the action (3.53). Thus, extremal variations of the action (3.53) give the wave interaction equations (3.41) or (3.50). □

PROPOSITION 3.7. *The wave interaction equations (3.41) may be written in the Hamiltonian form*

$$\frac{dc_j}{dt} = \frac{\delta H}{\delta \pi_j}, \quad \frac{d\pi_j}{dt} = -\frac{\delta H}{\delta c_j}, \tag{3.56}$$

where

$$\pi_j = \frac{\partial \mathscr{L}}{\partial c_{j,t}} = i\sigma_j c_j^* \tag{3.57}$$

is the canonical momentum conjugate to the canonical coordinate c_j. The Hamiltonian functional H is given by

$$H = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3k\, \mathscr{H}, \tag{3.58}$$

where the Hamiltonian density $\mathscr{H}$ is given by the Legendre transformation

$$\mathscr{H} = \sum_j{}' \pi_j \frac{dc_j}{dt} - \mathscr{L}. \tag{3.59}$$

Proof. The proof is a straightforward consequence of the action principle (3.53). The Hamiltonian density $\mathscr{H}$ in (3.59) can be written in the alternative forms

$$\begin{aligned}\mathscr{H} &= \left(\sum_j{}' |\omega_j||c_j|^2 - i\sigma_j \sum_s{}' \Lambda_{js} c_j^* c_s - i\sigma_j \Lambda_{j4}(c_j^* - c_j)\Delta\tilde{S}\right) \\ &= -i\left(\sum_j{}' \omega_j c_j \pi_j - i\sum_s{}' \Lambda_{js}\pi_j c_s - \Lambda_{j4}(i\pi_j + \sigma_j c_j)\Delta\tilde{S}\right).\end{aligned} \tag{3.60}$$

The second form of $\mathscr{H}$ in (3.60) is written in terms of the canonical conjugate coordinates c_j and π_j. In (3.60) the first term in the Hamiltonian $|\omega_j||c_j|^2 \equiv \sigma_j\omega_j c_j c_j^*$ is the Hamiltonian one obtains in the absence of wave interactions. The second and third terms in $\mathscr{H}$ represent the effects of wave interactions due to the shear flow. Note that the first form for the Hamiltonian density in (3.60) is real. □

Comment. The wave action invariant (3.49) is in fact a special case of the invariant

$$\sum_j{}' \sigma_j d_j c_j^* = \text{constant}, \tag{3.61}$$

where d_j and c_j are any two solutions of the system (3.50) with $\Delta S = 0$. This result is a consequence of Noether's theorem for the variational principle (3.53) corresponding to the fact that the equation system with $\Delta S = 0$ admits the canonical, infinitesimal Lie symmetry $t' = t$ and $c_j' = c_j + \epsilon d_j$ (see, e.g., Bluman and Kumei (1989) and Olver (1993) for accounts of Noether's first theorem). The wave action invariant (3.49) results if we choose $d_j = c_j$ in (3.61).

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3.3. Special cases

In this section we consider special cases of the wave interaction coefficients Ω_{js} and Λ_{js} in (3.43)–(3.48) that are intrinsically related to irregular behavior of the MHD eigensystem. First note that the normalization constant $\Delta_2 = 1 - (n^x)^2 = 0$ if $|n^x| = 1$ (i.e. $|k_x| = k$), which suggests that we need to look more carefully at the limit as $k_\perp/k \to 0$ where $\mathbf{k}_\perp = (0, k_y, k_z)$. The Eulerian eigenvectors of MHD are well defined, in general, in this limit, and this corresponds to wave propagation along the magnetic field, for which at least two of the MHD eigenspeeds coincide. In a low beta plasma in which $a < b$, the slow mode speed $c_s = a$ but the Alfvén and fast speeds coincide, i.e. $c_f = b_n = b$. Similarly, in a high beta plasma, with $a > b$, the slow and Alfvén eigenspeeds coincide: $c_s = b_n = b$ and $c_f = a$. Despite the fact that two of the MHD eigenspeeds coincide, the corresponding eigenvectors are distinct (e.g., Roe and Balsara 1996), but special care is needed in the normalization of the Eulerian eigenvectors for $(\delta\rho, \delta\mathbf{u}, \delta\mathbf{B}, \delta p)$ at the triple point for the case $a = b$, where the slow, fast and Alfvén speeds coincide (e.g., Brio and Wu 1988; Roe and Balsara 1996).

For wave propagation perpendicular to the magnetic field, $k_x = 0$. In this case $c_s = b_n = 0$ (i.e. both the slow speed and Alfvén speeds are zero) and

$$c_f^2 = a^2 + V_{\rm A}^2, \tag{3.62}$$

where $V_{\rm A} = B_0/\sqrt{\mu\rho_0}$ is the magnitude of the total Alfvén velocity. Thus $c_s = b_n = 0$, implying that the slow mode, the Alfvén mode and the entropy wave have zero phase speed relative to the fluid in this limit. Despite the occurrence of multiple eigenvalues, the eigenvectors are complete and form an orthogonal set. However, the present analysis shows that the wave interaction coefficients $\{\Omega_{js}\}$ $(s \neq 4)$ are all zero in this case because $n^x = k_x/k = 0$. In other words, there are no wave interactions between the MHD waves in the shear flow (1.1), except for the interaction between the magnetoacoustic and entropy waves for wave propagation perpendicular to $\mathbf{B}$.

A further interesting limit is the cold gas limit as $a \to 0$ (e.g., Roe and Balsara 1996; Falle and Hartquist 2002). Roe and Balsara (1996) point out that the slow mode eigenvectors in this limit collapse onto a single eigenvector, implying that the MHD equations are ill-posed. However, Falle and Hartquist (2002), who restrict themselves to the case where there are no entropy waves, show that a complete set of right eigenvectors for the slow mode can be constructed by using the eigenvectors of the matrices $\mathbf{A}$ and $\mathbf{A}^2$, corresponding to the slow mode eigenspeed $\lambda = 0$, where $\mathbf{A}$ is the basic matrix describing Eulerian one-dimensional MHD. Falle and Hartquist construct a complete set of eigenvectors for the slow mode by first determining the eigenvector $\mathbf{r}_1$ where $\mathbf{A} \cdot \mathbf{r}_1 = 0$, and subsequently determining the solution of $\mathbf{A} \cdot \mathbf{r}_2 = \mathbf{r}_1$ (note $\mathbf{r}_2$ is an eigenvector of $\mathbf{A}^2$, with zero eigenvalue, i.e. $\mathbf{A}^2 \cdot \mathbf{r}_2 = \mathbf{A} \cdot \mathbf{r}_1 = 0$). The canonical form of the matrix $\mathbf{A}$ using the right eigenvectors as base vectors has the Jordan canonical form. The net upshot of the analysis of Falle and Hartquist (2002) is that the initial value problem for linear MHD waves in one space dimension admits a secularly growing Jordan slow mode, which grows linearly in time. They also discuss the generation of the slow magnetoacoustic waves from nonlinear, steepend fast magnetoacoustic waves. Their numerical simulations, with $a \neq 0$ but small, show that there is a slow mode instability that grows linearly in time at early times for fairly general initial data (there are some restrictions),

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and they discuss its importance for star formation and cloud clumping in giant molecular clouds. However, the density enhancements they observe are due in part to nonlinear wave interactions.

If one restricts the analysis to the linearized initial value problem, then the problem appears to be ill-posed, since the solution does not depend continuously on the initial data. Apart from these mathematical considerations it is also a moot point whether the MHD equations can be physically valid for a cold gas, since the MHD equations are only strictly valid if there are enough collisions in the plasma to ensure that the pressure tensor is isotropic, yet not too many otherwise dissipation due to finite conductivity must be included. Clearly, zero frequency modes (i.e. the slow mode, with zero phase speed) violate these constraints.

In this subsection, we first consider the limiting case where $k_\perp/k \to 0$ corresponding to wave propagation along or nearly along the magnetic field. For this purpose, it is convenient to use spherical polar coordinates in $\mathbf{k}$-space, with the polar axis along the magnetic field. The wave normal $\mathbf{n} = \mathbf{k}/k$ is given by

$$\mathbf{n} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \tag{3.63}$$

where θ is the polar angle, and ϕ is the azimuthal angle measured in a right-handed sense from the y-axis (Sec. 3.3.1). We then consider the form of the wave interaction coefficients for the cold gas case ($a = 0$) following the analysis of Falle and Hartquist (2002) (Sec. 3.3.2). The net upshot is that there is a secular Jordan mode describing the slow mode interactions in the cold gas limit, but that these modes are separated off from a second system of equations describing the interaction of the Alfvén and fast magnetoacoustic modes.

3.3.1. Parallel propagation limit ($\theta \to 0$). We consider two cases separately: (a) the low-beta plasma case $a < b$; and (b) the high-beta plasma case $a > b$. The case for $a = b$ is more problematical, because of the coincidence of three eigenspeeds at the triple point, where the fast, slow and Alfvén speeds coincide for parallel propagation (note, however, that an analysis of this case can also be carried out by setting $a = b$ and $\mathbf{k} = (k_x, 0, 0)$ in the matrix $\mathbf{C}$ in the derivation of the wave interaction equations). A key equation in the analysis is the magnetosonic dispersion equation (3.21), which can be written in the form

$$(c^2 - a^2)(c^2 - b_n^2) = c^2 b^2 \sin^2\theta. \tag{3.64}$$

This form of the dispersion equation is useful in determining the lowest-order behavior of $(c_f^2 - b_n^2)$ and $(c_s^2 - b_n^2)$ in the limit as $\theta \to 0$. These results are important in taking the limit as $\theta \to 0$ in the wave interaction coefficients Ω_{js} in (3.43) in the following analysis.

Case $a < b$. In this case $c_f \to b$ and $c_s \to a$ as $\theta \to 0$. Note that the limiting forms of the wave interaction coefficients Ω_{js} still depend on the azimuthal angle ϕ in $\mathbf{k}$-space. Using (3.64) and (3.43)–(3.48) the Alfvén–Alfvén wave interaction coefficients in the limit as $\theta \to 0$ become

$$\Omega_{22} = -\frac{(At)^2 k(a^2 + b^2)\sin^2\phi}{2b}, \quad \Omega_{26} = \Omega_{22}, \quad \Omega_{62} = \Omega_{66} = -\Omega_{22}. \tag{3.65}$$

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Note that there is effectively only one basic wave interaction coefficient Ω_{22} in (3.65), and the other Alfvén–Alfvén interaction coefficients are determined by the symmetry relations (3.47).

Similarly, the basic magnetoacoustic–magnetoacoustic wave interaction coefficients are

$$\Omega_{11} = -\frac{(At)^2 k(a^2+b^2)\cos^2\phi}{2b}, \quad \Omega_{13} = -\frac{(At)ka}{2}\left(\frac{a}{b}\right)^{1/2}\cos\phi, \quad \Omega_{33} = 0. \tag{3.66}$$

The remaining interaction coefficients of this type can be determined by using the symmetry relations (3.43) for the Ω_{js}. The basic Alfvén–magnetoacoustic wave interaction coefficients are

$$\Omega_{12} = -\frac{(At)^2 k(a^2+b^2)\cos\phi\sin\phi}{2b}, \quad \Omega_{23} = -\frac{Atka}{2}\left(\frac{a}{b}\right)^{1/2}\sin\phi. \tag{3.67}$$

The remaining interaction coefficients of this type can be determined from the symmetry relations (3.43).

Case $a > b$. In the limit as $\theta \to 0$, $c_f \to a$, $b_n \to b$ and $c_s \to b$. Thus, in the high-beta plasma case, the fast wave is a sound wave and the slow mode is sometimes referred to as the pseudo-Alfvén mode. Taking the limit as $\theta \to 0$ in (3.43)–(3.48), we obtain

$$\Omega_{22} = -\frac{(At)^2(a^2+b^2)k\sin^2\phi}{2b},$$

$$\Omega_{11} = 0, \quad \Omega_{13} = \frac{Atka}{2}\left(\frac{a}{b}\right)^{1/2}\cos\phi, \quad \Omega_{33} = -\frac{(At)^2(a^2+b^2)k\cos^2\phi}{2b},$$

$$\Omega_{12} = \frac{Atka}{2}\left(\frac{a}{b}\right)^{1/2}\sin\phi, \quad \Omega_{23} = -\frac{(At)^2(a^2+b^2)k\cos\phi\sin\phi}{2b} \tag{3.68}$$

for the basic wave interaction coefficients. The remaining interaction coefficients Ω_{js} follow from (3.43).

A comparison of the wave interaction coefficients (3.65)–(3.67) for the low-beta case ($a < b$) with the interaction coefficients (3.68) for $a > b$ reveals that the non-zero self-wave interaction coefficients for the magnetoacoustic modes correspond to the eigenspeed $\lambda = b$. The results indicate that there is an interchange in the role of the fast and slow mode waves as one passes through the point $a = b$ in parameter space. In general, one expects a strong interaction between the different wave modes in the vicinity of the triple point in the case where $a \simeq b$.

3.3.2. Cold gas limit ($a = 0$).

Proposition 3.8. *In the cold gas limit* ($a = 0$), *the wave interaction equations (3.8) for* $\mathbf{w} = (\xi, \dot{\xi})$ *possess solutions of the form*

$$\mathbf{w} = \sum_{j=1,2,6,7} c_j(t)\mathbf{R}_j + c_3(t)\mathbf{R}_3 + c_5(t)\mathbf{R}_5. \tag{3.69}$$

In (3.69), the vectors $\{\mathbf{R}_j : j = 1,2,6,7\}$ *are the right eigenvectors of the matrix* $\bar{\mathbf{C}}$ *(i.e.* $\bar{\mathbf{C}}\mathbf{R}_j = i\omega_j\mathbf{R}_j$*) for the background state (1.1) with no shear and* $a = 0$. *These eigenvectors correspond to the backward and forward fast magnetoacoustic modes* ($\omega_j = k\lambda_j$, $j = 1,7$, $\lambda_1 = -b$ *and* $\lambda_7 = b$), *and the backward and forward Alfvén*

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waves ($\omega_2 = -kb_n$, $\omega_6 = kb_n$, $b_n = bn^x$). *The vectors* $\mathbf{R}_3$ *and* $\mathbf{R}_5$ *are right eigenvectors of the matrix* $\bar{\mathbf{C}}^2$ *with eigenvalue* $\omega_3 = \omega_5 = 0$, *such that*

$$\bar{\mathbf{C}} \cdot \mathbf{R}_3 = 0 \quad \text{and} \quad \bar{\mathbf{C}} \cdot \mathbf{R}_5 = \mathbf{R}_3, \tag{3.70}$$

and correspond to the slow magnetoacoustic modes. The wave amplitudes $\{c_j\}$ *satisfy two separate first-order differential equation systems. The first system describes the interaction of the fast magnetoacoustic and Alfvén waves for* $\{c_j(t) : j = 1, 2, 6, 7\}$ *and the second system describes the slow magnetoacoustic wave amplitudes* $\{c_j(t) : j = 3, 5\}$. *The fast magnetoacoustic and Alfvén wave system is*

$$\begin{aligned}
&\frac{dc_1}{dt} + i\omega_1 c_1 + i[\Omega_{11}(c_1 + c_7) + \Omega_{12}(c_2 + c_6)] = 0,\\
&\frac{dc_7}{dt} - i\omega_1 c_7 - i[\Omega_{11}(c_1 + c_7) + \Omega_{12}(c_2 + c_6)] = 0,\\
&\frac{dc_2}{dt} + i\omega_2 c_2 + i[\Omega_{12}(c_1 + c_7) + \Omega_{22}(c_2 + c_6)] = 0,\\
&\frac{dc_6}{dt} - i\omega_2 c_6 - i[\Omega_{12}(c_1 + c_7) + \Omega_{22}(c_2 + c_6)] = 0.
\end{aligned} \tag{3.71}$$

The wave interaction coefficients Ω_{11}, Ω_{12} *and* Ω_{22} *are given by*

$$\begin{aligned}
\Omega_{11} &= \frac{Atkbn^x n^y[2 - 2(n^x)^2 - Atn^x n^y]}{2[1 - (n^x)^2]},\\
\Omega_{12} &= \frac{Atkb(n^x)^{1/2} n^z[1 - (n^x)^2 - Atn^x n^y]}{2[1 - (n^x)^2]},\\
\Omega_{22} &= -\frac{(At)^2 kb(n^z)^2 n^x}{2[1 - (n^x)^2]}.
\end{aligned} \tag{3.72}$$

The slow mode amplitudes $\{c_3(t), c_5(t)\}$ *satisfy the independent differential equation system*

$$\frac{dc_3}{dt} + c_5 = 0 \quad \textit{and} \quad \frac{dc_5}{dt} = 0, \tag{3.73}$$

with solutions

$$c_5(t) = c_5(0) \quad \textit{and} \quad c_3(t) = c_3(0) - c_5(0)t. \tag{3.74}$$

The solution for $c_3(t)$ *describes the so-called Jordan mode (Falle and Hartquist 2002).*

Comment 1. The matrix $\bar{\mathbf{C}}$ is similar to the matrix

$$\tilde{\mathbf{C}} = \mathbf{L} \cdot \bar{\mathbf{C}} \cdot \mathbf{R}, \tag{3.75}$$

where

$$\begin{aligned}
\mathbf{L} &= (\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_5, \mathbf{L}_6, \mathbf{L}_7)^{\mathrm{T}},\\
\mathbf{R} &= (\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_5, \mathbf{R}_6, \mathbf{R}_7).
\end{aligned} \tag{3.76}$$

The rows of the matrix $\mathbf{L}$ consist of the left eigenvectors $\{\mathbf{L}_j\}$ and the columns of $\mathbf{R}$ consist of the right eigenvectors $\{\mathbf{R}_j\}$. The matrices $\mathbf{L}$ and $\mathbf{R}$ are inverses of each other: $\mathbf{L} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{L} = \mathbf{I}$. $\mathbf{L}_3$ and $\mathbf{L}_5$ are the left eigenvectors of $\bar{\mathbf{C}}^2$: $\mathbf{L}_3 \cdot \bar{\mathbf{C}}^2 = 0$ and $\mathbf{L}_5 \cdot \bar{\mathbf{C}}^2 = 0$ with eigenvalue $\omega_3 = \omega_5 = 0$. They satisfy the equations $\mathbf{L}_3\bar{\mathbf{C}} = \mathbf{L}_5$ and $\mathbf{L}_5\bar{\mathbf{C}} = 0$. The eigenvectors $\{\mathbf{L}_j\}$ and $\{\mathbf{R}_j\}$ for $j = 1, 2, 6, 7$ are normalized such that

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the wave action densities $A_j = \sigma_j |c_j|^2$ where $\sigma_j = \mathrm{sgn}(\lambda_j)$ satisfy the wave action integral (3.49). The matrix $\tilde{\mathbf{C}}$ has the Jordan canonical form

$$\tilde{\mathbf{C}} = \begin{pmatrix} i\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\omega_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega_7 \end{pmatrix}, \tag{3.77}$$

where $\omega_7 = -\omega_1 = kb$ and $\omega_6 = -\omega_2 = kb_n \equiv kbn^x$.

Comment 2. The eigenvectors $\mathbf{R}_3$ and $\mathbf{R}_5$ are

$$\mathbf{R}_3 = (-1, 0, 0, 0, 0, 0)^{\mathrm{T}} \quad \text{and} \quad \mathbf{R}_5 = (0, 0, 0, 1, 0, 0)^{\mathrm{T}}, \tag{3.78}$$

and $\mathbf{L}_3 = \mathbf{R}_3^{\mathrm{T}}$ and $\mathbf{L}_5 = \mathbf{R}_5^{\mathrm{T}}$.

Outline of proof. The proof of the proposition consists of first obtaining solutions of (3.70) for $\mathbf{R}_3$ and $\mathbf{R}_5$. The matrix $\bar{\mathbf{C}}$ is given by setting $a^2 = p_0 = 0$ and setting the shear parameter $A = 0$ in the matrix $\mathbf{A}$ in (3.4) to obtain $\bar{\mathbf{A}}$, and then determining $\bar{\mathbf{C}}$ from (3.11). The required solutions of (3.70) for $\mathbf{R}_3$ and $\mathbf{R}_5$ are given by (3.78). From (3.70) $\bar{\mathbf{C}}^2 \cdot \mathbf{R}_j = 0$ for $j = 3, 5$. The motivation for the equations (3.70) is due to the fact that the matrix $\bar{\mathbf{C}}$, by the Cayley–Hamilton theorem, must satisfy the characteristic polynomial matrix equation

$$\bar{\mathbf{C}}^2(\bar{\mathbf{C}} - i\omega_2\mathbf{I}_6)(\bar{\mathbf{C}} - i\omega_6\mathbf{I}_6)(\bar{\mathbf{C}} - i\omega_1\mathbf{I}_6)(\bar{\mathbf{C}} - i\omega_7\mathbf{I}_6) = 0, \tag{3.79}$$

where $\omega_j = k\lambda_j$. From (3.79) it is clear that the $\omega = 0$ subspace corresponds to the eigenvector space associated with $\bar{\mathbf{C}}^2 \cdot \mathbf{R} = 0$. This subspace is spanned by $\mathbf{R}_3$ and $\mathbf{R}_5$. The eigenvectors of $\bar{\mathbf{C}}$ corresponding to the Alfvén modes ($\mathbf{R}_2$ and $\mathbf{R}_6$) and the fast modes $\mathbf{R}_1$ and $\mathbf{R}_7$ can be obtained by setting $a = 0$ in the eigenvector solutions (3.22) and (3.24).

The Jordan canonical form of the matrix $\tilde{\mathbf{C}}$ is a consequence of changing from the original set of base vectors in which $\mathbf{w} = (\xi^x, \xi^y, \xi^z, \dot{\xi}^x, \dot{\xi}^y, \dot{\xi}^z)$ to the base consisting of the eigenvectors $\{\mathbf{R}_j : j = 1, 2, 3, 5, 6, 7\}$.

The derivation of the wave interaction equations (3.71) and (3.72) for the fast magnetoacoustic waves and the Alfvén waves, and the independent equation system (3.73) for the slow mode waves, follows by substituting the solution ansatz (3.69) into the first-order differential equation system (3.13) for $\mathbf{w}$ to obtain the vector differential equation system

$$\sum_{s=1,2,6,7} \left(\frac{dc_s}{dt} + i\omega_s c_s \right) \mathbf{R}_s + \frac{dc_3}{dt}\mathbf{R}_3 + \frac{dc_5}{dt}\mathbf{R}_5 + c_5\mathbf{R}_3 + \sum_{s \neq 4} c_s \delta\mathbf{C} \cdot \mathbf{R}_s = 0, \tag{3.80}$$

where the entropy wave source term $\mathbf{Q} = 0$ for the case $a^2 = 0$. Taking the scalar product of (3.80) with the left eigenvectors $\mathbf{L}_j$ ($j = 1, 2, 6, 7$) gives the differential equation system (3.71) for the fast magnetoacoustic and Alfvén waves. Similarly, taking the scalar product of (3.80) with the eigenvectors $\mathbf{L}_3$ and $\mathbf{L}_5$ gives the evolution equations for the slow mode amplitudes $c_3(t)$ and $c_5(t)$. The wave interaction coefficients Ω_{js} are determined by noting that $\Lambda_{js} = i\Omega_{js}$ and $\Lambda_{js} = \mathbf{L}_j \cdot \delta\mathbf{C} \cdot \mathbf{R}_s$.

Comment. The solutions for the case of a small ($a \approx 0$) and $a = 0$, are different. This implies that the linearized initial value problem in the cold gas limit is ill-posed.

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4. Wave interactions in the ray system

In this section, we reconsider the coupled oscillator system (3.2) in which the solution is expanded using the eigenvectors and eigenvalues of the matrix $\mathbf{C}$ in (3.9). This is different from Sec. 3, where the matrices $\mathbf{C}$ and $\mathbf{A}$ were written in the forms $\mathbf{C} = \bar{\mathbf{C}} + \delta\mathbf{C}$ and $\mathbf{A} = \bar{\mathbf{A}} + \delta\mathbf{A}$, where $\delta\mathbf{C}$ and $\delta\mathbf{A}$ contained the effects of the shear and $\bar{\mathbf{C}}$ and $\bar{\mathbf{A}}$ are constant matrices describing the background state. By using the eigensystems of the matrices $\mathbf{C}$ and $\mathbf{A}$ we are effectively considering the wave interactions in a system moving along the ray trajectories in $\mathbf{k}$-space, in which $\bar{k}_y = k_y - Atk_x$, $k_x = k_{x0}$, $k_z = k_{z0}$ gives the variation of the $\mathbf{k}$ vector along the rays (note that $\mathbf{k}$ varies this way for all the eigenmodes, see Appendix B).

4.1. Mode separation equations

From (3.7)–(3.10) the equation system for $\mathbf{w}$ has the form

$$\frac{d\mathbf{w}}{dt} + \mathbf{C}\mathbf{w} = \mathbf{Q}, \tag{4.1}$$

where $\mathbf{w}$ is the six-dimensional vector (3.7) describing $\tilde{\xi}$ and $\tilde{\xi}_t$. To reduce (4.1) to normal form, we search for solutions of (4.1) of the form

$$\mathbf{w} = \sum_{\substack{s=1 \\ s\neq 4}}^{7} a'_s(t)\mathbf{R}'_s(t), \tag{4.2}$$

where the $\mathbf{R}'_j$ and $\mathbf{L}'_j$ are the right and left eigenvectors of the matrix $\mathbf{C}$ satisfying the equations

$$(\mathbf{C} - i\omega'_j\mathbf{I}_6)\cdot\mathbf{R}'_j = 0, \quad \mathbf{L}'_j\cdot(\mathbf{C} - i\omega'_j\mathbf{I}_6) = 0, \quad \det(\mathbf{C} - i\omega'\mathbf{I}_6) = 0, \tag{4.3}$$

normalized such that $\mathbf{L}'_j\cdot\mathbf{R}'_s = \delta_{js}$. The eigenvalues ω'_j in (4.3) are the solutions of the determinantal equation $\det(\mathbf{C} - i\omega'\mathbf{I}_6) = 0$ in (4.3). Substituting the solution ansatz (4.2) for $\mathbf{w}$ in (4.1) and taking the scalar product of the resultant equation with $\mathbf{L}'_j$ yields the wave interaction equations

$$\frac{da'_j}{dt} + i\omega'_j a'_j + \sum_{\substack{s=1 \\ s\neq 4}}^{7} \bar{\Lambda}'_{js}a'_s = \mathbf{L}'_j\cdot\mathbf{Q} \equiv q'_j, \quad j = 1,2,3,5,6,7, \tag{4.4}$$

where the wave interaction coefficients Λ_{js} are given by

$$\bar{\Lambda}'_{js} = \mathbf{L}'_j\cdot\frac{d\mathbf{R}'_s}{dt}. \tag{4.5}$$

Equations (4.4) and (4.5) are analogous to (3.16) and (3.17), except that the interaction coefficients Λ'_{js} have a different form than in (3.17). Also, the eigenvalues $\{\lambda_j\}$ and the eigenvectors used in the analysis of Sec. 3 were time independent, whereas the eigenvectors and eigenvalues are time dependent in the present analysis.

By differentiating the right eigenvector equation in (4.3) with respect to t and taking the scalar product of the resultant equation with respect to $\mathbf{L}'_j$ we obtain

$$\bar{\Lambda}'_{js} = \frac{i}{\omega'_j - \omega'_s}\mathbf{L}'_j\cdot\frac{d\delta\mathbf{C}}{dt}\cdot\mathbf{R}'_s, \ \omega'_j \neq \omega'_s, \tag{4.6}$$

$$\frac{d\omega'_s}{dt} = -i\mathbf{L}'_s\cdot\frac{d\delta\mathbf{C}}{dt}\cdot\mathbf{R}'_s, \tag{4.7}$$

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where $\delta\mathbf{C}$ is given by (3.11) and (3.12). Equations (4.6) give an alternative method to calculate the wave interaction coefficients for cases where $\omega'_j \neq \omega'_s$.

From (4.3), the eigenvalues $\{\omega'_j\}$ satisfy the eigenvalue equation

$$\det(\mathbf{C} - i\omega'\mathbf{I}_6) \equiv \det(\mathbf{A} - \omega'^2\mathbf{I}_3) = 0, \tag{4.8}$$

or, evaluating the last determinant in (4.8) we obtain

$$\det(\mathbf{A} - \omega'^2\mathbf{I}_3) = -k'^6(\lambda^2 - b_n^2)(\lambda^4 - (a^2+b^2)\lambda^2 + a^2b_n^2) = 0, \tag{4.9}$$

where $\lambda = \omega'/k'$ is the characteristic wave phase velocity $k' = |\mathbf{k}'| = (k_x^2 + \bar{k}_y^2 + k_z^2)^{1/2}$, and $b_n = bn'^x = \mathbf{b}\cdot\mathbf{n}'$ is the Alfvén phase speed with $\mathbf{n}' = \mathbf{k}'/k'$ is the wave normal. Equations (4.8) and (4.9) are exactly analogous to (3.18) and (3.19) based on the background state with no fluid shear. Since the matrices $\mathbf{C}$ and $\mathbf{A}$ are also exactly analogous to $\bar{\mathbf{C}}$ and $\bar{\mathbf{A}}$ except that k_y is replaced by $\bar{k}_y = k_y - Atk_x$, it follows that the eigenvectors and eigenvalues of the two systems have the same algebraic form, except that k_y is replaced by $\bar{k}_y$ to obtain the eigensolutions of the matrices $\mathbf{C}$ and $\mathbf{A}$.

From (4.5) the Alfvén wave mixing coefficients $\bar{\Lambda}'_{2j}$ and $\bar{\Lambda}'_{6j}$ are given by

$$\bar{\Lambda}'_{22} = \bar{\Lambda}'_{66} = -\frac{A(n^x)^3 n^y}{\Delta_2}, \quad \bar{\Lambda}'_{26} = \Lambda'_{62} = 0,$$

$$\bar{\Lambda}'_{2j} = \frac{An^z\lambda_j^2(\lambda_j - b_n)}{2b\Delta_2}, \quad \bar{\Lambda}'_{6j} = -\frac{An^z\lambda_j^2(\lambda_j + b_n)}{2b\Delta_6}, \quad j = 1,3,5,7, \tag{4.10}$$

where in an abuse of mathematical notation we have dropped the primed superscripts on $\mathbf{n}$, λ_j and Δ_j (i.e. these quantities are given by the results in Sec. 3, except that k_y is replaced by $\bar{k}_y$). In (4.10), we have used the normalization $N_r^{(j)} = 1$ for the right eigenvectors (cf. Sec. 3.1).

Similarly, the magnetoacoustic wave mixing coefficients are

$$\bar{\Lambda}'_{j2} = \frac{An^x n^z\lambda_j(\lambda_j - b_n)}{2\Delta_j}, \quad \bar{\Lambda}'_{j6} = \frac{An^x n^z\lambda_j(\lambda_j + b_n)}{2\Delta_j},$$

$$\bar{\Lambda}'_{js} = -\frac{An^x n^y}{2\lambda_j\Delta_j}\left[b_n^2(\lambda_j+\lambda_s)(\lambda_s^2+\lambda_j^2-b^2)\right.$$

$$\left. + \frac{2a^2b_n^2(\lambda_j^2-b_n^2)(\lambda_j+\lambda_s) + \lambda_s(\lambda_s^2-b_n^2)(\lambda_s^2\lambda_j^2 - a^2b_n^2)}{2\lambda_s^2 - (a^2+b^2)}\right], \tag{4.11}$$

where $j, s = 1,3,5,7$. One can show that

$$\bar{\Lambda}'_{jj} = \frac{1}{2}\frac{d}{dt}\ln(|\omega'_j|\Delta_j), \tag{4.12}$$

where $j = 1(1)7, j \neq 4$. The derivation of (4.6), (4.7) and (4.12) are given in Appendix D.

The entropy wave source terms in the wave interaction equations due to the entropy waves are

$$q'_2 = q'_6 = 0, \quad q'_j = \frac{p_0(\lambda_j^2 - b_n^2)}{2\lambda_j\Delta_j}\Delta\tilde{S}, \quad j = 1,3,5,7. \tag{4.13}$$

Thus, the entropy wave source terms are zero for the Alfvén mode equations (4.4) for $j = 2$ and $j = 6$.

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The formulae (4.10)–(4.12) for the wave mixing coefficients $\bar{\Lambda}'_{js}$, and the entropy wave source terms q'_j and ω'_j are the basic coefficients in the wave mixing equations (4.4). Note that the Alfvén wave interaction coefficients $\bar{\Lambda}'_{26} = \bar{\Lambda}'_{62} = 0$, implying that there is no direct coupling between the backward and forward Alfvén eigenmodes in this formulation, although there are indirect couplings via the magnetoacoustic wave interactions. Note that the wave interaction coefficients $\bar{\Lambda}'_{js}$ in the ray system in (4.10)–(4.12) are real, whereas the wave interaction coefficients $\Lambda_{js} = i\Omega_{js}$ based on the constant background state with no shear used in Sec. 3 are pure imaginary.

In the next section we re-scale the wave amplitudes $\{a'_j\}$ to obtain wave interaction equations, in a more symmetric form, which are shown to possess a wave action integral and Hamiltonian principle.

4.2. Wave action and Hamiltonian formulation

Following the approach of Sec. 3.2, we introduce new, re-scaled wave amplitudes $\{c'_j\}$ of the form

$$c'_j = \nu'_j a'_j, \quad \nu'_j = \sqrt{\rho|\omega'_j|\Delta'_j}, \tag{4.14}$$

where the primed superscripts emphasize that ω'_j and Δ'_j are evaluated using $\bar{k}_y = k_y - Atk_x$ to replace k_y in the corresponding formulae of Sec. 3. The wave interaction equations (4.4) in the new variables $\{c'_j\}$ are of the form

$$\frac{dc'_j}{dt} + i\omega'_j c'_j + \sum_{s\neq j}{}' \Lambda'_{js} c'_s = \hat{q}_j = -\Lambda'_{j4}\Delta\tilde{S}, \tag{4.15}$$

where

$$\Lambda'_{js} = \bar{\Lambda}'_{js}\nu'_j/\nu'_s, \quad j \neq s. \tag{4.16}$$

The 'self-wave' interaction coefficient Λ'_{jj} is given by

$$\Lambda'_{jj} = \bar{\Lambda}'_{jj} - \frac{1}{2}\frac{d}{dt}\ln(|\omega'_j|\Delta'_j) = 0, \tag{4.17}$$

by virtue of the result (4.12) for $\bar{\Lambda}'_{jj}$. Thus, $\Lambda'_{jj} = 0$.

Using the wave interaction coefficients $\bar{\Lambda}'_{js}$ from (4.10)–(4.12) in (4.16), we obtain the Alfvén wave mixing coefficients

$$\Lambda'_{22} = \Lambda'_{66} = \Lambda'_{26} = \Lambda'_{62} = 0,$$

$$\Lambda'_{2j} = \frac{kAn^x n^z \lambda_j^2(\lambda_j - b_n)}{2\sqrt{|\omega_2|\Delta_2|\omega_j|\Delta_j}}, \quad \Lambda'_{6j} = -\frac{kAn^x n^z \lambda_j^2(\lambda_j + b_n)}{2\sqrt{|\omega_6|\Delta_6|\omega_j|\Delta_j}}, \tag{4.18}$$

where $j = 1, 3, 5, 7$. Equations (4.18) imply that the Alfvén modes c'_2 and c'_6 interact with the magnetoacoustic waves, but not with each other.

Similarly, the magnetoacoustic wave mixing coefficients $\{\Lambda'_{js}\}$ in (4.15) are given by

$$\Lambda'_{j2} = \frac{\sigma_j Akn^x n^z \lambda_j^2(\lambda_j - b_n)}{2\sqrt{|\omega_j|\Delta_j|\omega_2|\Delta_2}}, \quad \Lambda'_{j6} = \frac{\sigma_j Akn^x n^z \lambda_j^2(\lambda_j + b_n)}{2\sqrt{|\omega_j|\Delta_j|\omega_6|\Delta_6}},$$

$$\Lambda'_{js} = \frac{\sigma_j Akn^x n^y[2(a^2 + b_\perp^2)\lambda_s^2\lambda_j^2 - a^2 b_n^2(\lambda_s^2 + \lambda_j^2)]}{2(\lambda_j - \lambda_s)\sqrt{|\omega_j|\Delta_j|\omega_s|\Delta_s}}, \tag{4.19}$$

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where $b_\perp^2 = b^2 - b_n^2$, $j, s = 1, 3, 5, 7$, $j \neq s$, and $\sigma_j = \mathrm{sgn}(\lambda_j)$ corresponds to the sign of the canonical wave energy density. In (4.19) we have used the form of the wave mixing coefficients $\bar{\Lambda}'_{js}$ ($j, s = 1, 3, 5, 7$) derived from (4.6) rather than the equivalent formulae from (4.11) (see Appendix D for further details).

The wave interaction coefficients $\{\Lambda'_{js}\}$ in (4.18) and (4.19) have the symmetry properties

$$\sigma_j \Lambda'_{js} = -\sigma_s \Lambda'_{sj} \quad \text{and} \quad \Lambda'_{j's'} = \Lambda'_{js}, \quad j + j' = 8, \ s + s' = 8. \tag{4.20}$$

These symmetries are analogous to the symmetries (3.43) satisfied by the Ω_{js} mixing coefficients in Sec. 3.

The entropy wave source terms in the symmetrized wave mixing equations (4.15) are given by

$$\hat{q}_2 = \hat{q}_6 = 0,$$

$$\hat{q}_j = \frac{\sigma_j k(\lambda_j^2 - b_n^2) p \sqrt{\rho}}{2\sqrt{|\omega_j| \Delta_j}} \Delta\tilde{S} \equiv -\Lambda'_{j4} \Delta\tilde{S}, \quad j = 1, 3, 5, 7. \tag{4.21}$$

Again in (4.21) ω_j, Δ_j, λ_j and k refer to the values of these variables when k_y is replaced by $\bar{k}_y = k_y - Atk_x$ in the formulae of Sec. 3. Equation (4.21) also defines the magnetoacoustic–entropy wave interaction coefficient Λ'_{j4}.

4.2.1. Wave action integral. By combining the wave interaction equations (4.15) with their complex conjugates, and using the symmetry property $\sigma_j \Lambda'_{js} = -\sigma_s \Lambda'_{sj}$ of the wave interaction coefficients listed in (4.20) we obtain the equation

$$\frac{d}{dt}\left(\sum_j{}' \sigma_j |c'_j|^2\right) = -\sum_j{}' \sigma_j \Lambda'_{j4} \Delta\tilde{S}(c'_j + c'^*_j), \tag{4.22}$$

where the source on the right-hand side is due to the entropy wave. Thus, if $\Delta\tilde{S} = 0$, we obtain the wave action integral

$$\sum_j{}' \sigma_j |c'_j|^2 = N = \text{constant}. \tag{4.23}$$

The integral (4.23) is clearly equivalent to the wave action integral (3.49) obtained in Sec. 3, in which the analysis was based on the constant background state eigenmodes with no shear. Since $c'_j(0) = c_j(0)$, the total quasi-particle number density N in (4.23) has the same value as in (3.49). However, the physical interpretation is different, since the $\mathbf{k}$ vector evolves according to the ray equations in this section (i.e. $\bar{k}_y = k_y - Atk_x$), whereas $\mathbf{k}$ is constant in (3.49).

4.2.2. Hamilton's principle. The wave interaction equations (4.15) can be derived from a variational principle of the form (3.53) in which the Lagrangian density $\mathscr{L}'$ has the form

$$\mathscr{L}' = i\left[\sum_j{}' \sigma_j c'^*_j \left(\frac{dc'_j}{dt} + i\omega'_j c'_j + \sum_s{}' \Lambda'_{js} c'_s\right) + \sigma_j \Lambda'_{j4} \Delta\tilde{S}(c'^*_j - c'_j)\right]. \tag{4.24}$$

In (4.15) and (4.24) the entropy wave source term and entropy wave interaction coefficients Λ'_{j4} are defined in (4.21).

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Using the Legendre transformation (3.59) and the canonical coordinates c'_j and $\pi'_j = i\sigma_j c'^*_j$ (see (3.57)), we obtain the Hamiltonian $\mathscr{H}'$ for the system in the form

$$\mathscr{H}' = \sum_j{}' |\omega'_j||c'_j|^2 - i\sum_s{}' \sigma_j \Lambda'_{js} c'^*_j c'_s - i\sigma_j \Lambda'_{j4}(c'^*_j - c'_j)\Delta\tilde{S}$$

$$= -i\left(\sum_j{}' \omega'_j c'_j \pi'_j - i\sum_s{}' \Lambda'_{js}\pi'_j c'_s - \Lambda'_{j4}\Delta\tilde{S}(i\pi'_j + \sigma_j c'_j)\right). \tag{4.25}$$

It is straightforward to verify that the (c'_j, π'_j) satisfy Hamilton's equations (3.56).

The fact that there are two versions of Hamilton's equations describing the wave interactions, based on the constant-state canonical formulation of (3.53) et seq. and the ray-based canonical Hamiltonian approach in (4.24) and (4.25), implies that there exists a canonical transformation relating the two equivalent systems. The ray-based Hamiltonian system has the attractive feature that strong wave interactions occur when two dispersion branches ω'_j and ω'_k approach each other, thus allowing for wave transformations between the two branches (e.g., Chagelishvili et al. 1997; Bodo et al. 2001; Kaghashvili 2002; Gogoberidze et al. 2004) (see also (4.6)).

5. Transformations

So far, we have developed two different canonical formulations for MHD wave interactions in shear flows, namely: (a) a wave decomposition based on the MHD eigenvalues and eigenvectors characteristic of the background state, without shear (Sec. 3), which we call the K-system, since $\mathbf{k}$ remains constant; and (b) the wave eigenmode decomposition based on the ray equations in which the $\mathbf{k}$ vector of the waves evolves with time along the rays of the system (i.e. $k'_x = k_x$, $k'_y(t) = k_y - Atk_x$, $k'_z = k_z$ along the rays of the system (Appendix B)), which we refer to as the R-system. In this latter approach (Sec. 4) both the wave amplitudes and eigenvectors evolve in time.

We use the Hamiltonian variables $\{c_j\}$ and $\{c'_j\}$ introduced in Secs 3 and 4 to describe the waves. The wave state vector $\mathbf{w} = (\xi, \dot{\xi})$ is expanded in the form

$$\mathbf{w} = \sum c_j \mathbf{R}_j = \sum c'_j \mathbf{R}'_j, \tag{5.1}$$

where the $\mathbf{R}_j$ and $\mathbf{R}'_j$ are the right eigenvectors of the matrices $\bar{\mathbf{C}}$ and $\mathbf{C}$, respectively, and $\mathbf{L}_j$ and $\mathbf{L}'_j$ are the corresponding right eigenvectors normalized so that

$$\mathbf{L}_j \cdot \mathbf{R}_s = \delta_{js} \quad \text{and} \quad \mathbf{L}'_j \cdot \mathbf{R}'_s = \delta_{js}. \tag{5.2}$$

Since we use the canonical wave amplitudes c_j and c'_j, the left and right eigenvectors $\mathbf{L}_j$ and $\mathbf{R}_j$ are given by the formulae

$$\mathbf{R}_j = \frac{1}{\sqrt{\rho|\omega_j|\Delta_j}}(\mathbf{U}_j - i\omega_j \mathbf{V}_j),$$

$$\mathbf{L}_j = \frac{\rho\sigma_j}{2\sqrt{\rho|\omega_j|\Delta_j}}(\omega_j \mathbf{U}_j + i\mathbf{V}_j). \tag{5.3}$$

Here, the vectors $\mathbf{U}_j$ and $\mathbf{V}_j$, ω_j and Δ_j have been defined in (3.22)–(3.25). Thus, the right eigenvectors have been normalized so that $N_r^{(j)} = 1/\sqrt{\rho|\omega_j|\Delta_j}$ and $N_l^{(j)}$

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is determined by the normalization condition (3.23). This is different than the normalization used in Sec. 3.1, where $N_r^{(j)} = 1$ was used. Similar formulae to (5.3) apply for the left and right eigenvectors $\mathbf{L}'_j$ and $\mathbf{R}'_j$ in the R-system (i.e. primed superscripts are added to all variables in this system).

Using the orthogonality relations (5.2) we obtain the relations

$$c'_s = P_{sj} c_j, \quad c_s = Q_{sj} c'_j \tag{5.4}$$

between the wave amplitudes in the two systems, where the transformation matrices P_{sj} and Q_{sj} are given by

$$P_{sj} = \mathbf{L}'_s \cdot \mathbf{R}_j \quad \text{and} \quad Q_{sj} = \mathbf{L}_s \cdot \mathbf{R}'_j. \tag{5.5}$$

Note that the matrices $\mathbf{P}$ and $\mathbf{Q}$ are inverses, i.e. $\mathbf{PQ} = \mathbf{QP} = \mathbf{I}_6$. The right and left eigenvectors transform as

$$\mathbf{R}'_j = Q_{sj}\mathbf{R}_s, \quad \mathbf{R}_j = P_{sj}\mathbf{R}'_s,$$

$$\mathbf{L}'_j = P_{js}\mathbf{L}_s, \quad \mathbf{L}_j = Q_{js}\mathbf{L}'_s. \tag{5.6}$$

Thus, $\mathbf{P}$ and $\mathbf{Q}$ describe the transformation of vectors between frames.

Using the results (5.3) for the right and left eigenvectors in the K-system and the corresponding results for the eigenvectors in the R-system, we obtain the formulae

$$P_{sj} = \frac{\sigma_s(\omega'_s + \omega_j)\mathbf{U}'_s \cdot \mathbf{U}_j}{2\sqrt{|\omega'_s|\Delta'_s|\omega_j|\Delta_j}}, \quad Q_{sj} = \frac{\sigma_s(\omega_s + \omega'_j)\mathbf{U}_s \cdot \mathbf{U}'_j}{2\sqrt{|\omega_s|\Delta_s|\omega'_j|\Delta'_j}}, \tag{5.7}$$

for the transformation matrices $\mathbf{P}$ and $\mathbf{Q}$, where $\sigma_s = \mathrm{sgn}(\lambda_s)$ is the sign of the wave phase velocity for mode s. From (5.7) it is straightforward to show that

$$\sigma_p P_{pj} = \sigma_j Q_{jp} \quad \text{or} \quad Q_{jp} = \sigma_j \sigma_p P_{pj}. \tag{5.8}$$

Equations (5.8), together with the result that $\mathbf{P}$ and $\mathbf{Q}$ are inverse matrices, implies that these transformations are unitary transformations (note that to properly define a unitary transformation in the present case requires a definition of the scalar product (inner product), and the difference between covariant and contravariant vector fields (see e.g., Sec. 5.1, where we define the scattering matrix)).

PROPOSITION 5.1. *The wave action integral (3.49) in the K-system and the corresponding integral (4.23) in the R-system are equivalent. This is a consequence of the unitary property (5.8) of the transformations between the two frames.*

Proof. To prove the above assertion, we note

$$\sigma_j c_j c_j^* = \sigma_j Q_{jp} c'_p Q_{js} c'^*_s = \sigma_p (P_{pj} Q_{js}) c'_p c'^*_s = \sigma_p c'_p c'^*_p = N = \text{constant}, \tag{5.9}$$

where we use the Einstein summation convention for repeated indices. In the proof of (5.9) we have consecutively used the transformations (5.4) for the wave amplitudes c_j, the unitary property (5.8), and the fact that $\mathbf{P}$ and $\mathbf{Q}$ are inverse matrices.

A further property of the matrices $\mathbf{P}$ and $\mathbf{Q}$ is the wave-velocity reversal symmetry

$$P_{s'j'} = P_{sj}, \quad Q_{s'j'} = Q_{sj}, \quad j + j' = 8, \; s + s' = 8, \tag{5.10}$$

where we use the convention (3.20) for the wave eigenmode velocities.

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PROPOSITION 5.2. *The wave evolution equations (3.41) and (4.15) in the K- and R-systems are related by the identities*

$$\frac{dc'_j}{dt} + i\omega'_j c'_j + \Lambda'_{js} c'_s + \Lambda'_{j4}\Delta\tilde{S} = P_{js}\left(\frac{dc_s}{dt} + i\omega_s c_s + \Lambda_{sp} c_p + \Lambda_{s4}\Delta\tilde{S}\right), \tag{5.11}$$

$$\Lambda'_{j4} = P_{js}\Lambda_{s4}. \tag{5.12}$$

Furthermore,

$$\Lambda_{sa} = i(Q_{sp}P_{pa}\omega'_p - \omega_s\delta_{sa}), \tag{5.13}$$

$$P_{js}(\Lambda_{sa}c_a + i\omega_s c_s) = i\omega'_j c'_j. \tag{5.14}$$

In (5.14) j is fixed.

Proof. Perhaps the simplest proof of (5.11) is first to note that

$$\mathbf{L}'_j \cdot \left(\frac{d\mathbf{w}}{dt} + \mathbf{C}\cdot\mathbf{w}\right) = \frac{dc'_j}{dt} + i\omega'_j c'_j + \Lambda'_{js}c'_s + \Lambda'_{j4}\Delta\tilde{S}. \tag{5.15}$$

However, using the transformation formula $\mathbf{L}'_j = P_{js}\mathbf{L}_s$ (see (5.6)) in (5.15) gives the result (5.11). The result (5.12) is a straightforward consequence of the transformation of the entropy wave source terms between frames. The identity (5.11) connects the wave evolution equations between the two frames.

The proof of (5.13) follows from the definition (3.17) of the wave interaction coefficients Λ_{sa} in the K-system, namely,

$$\Lambda_{sa} = \mathbf{L}_s \cdot (\mathbf{C} - \bar{\mathbf{C}}) \cdot \mathbf{R}_a, \tag{5.16}$$

where $\delta\mathbf{C} = \mathbf{C} - \bar{\mathbf{C}}$ is the difference of the matrices $\mathbf{C}$ and $\bar{\mathbf{C}}$ (see (3.11)). Taking into account the fact that $\mathbf{L}_s$ and $\mathbf{R}_a$ are eigenvectors of $\bar{\mathbf{C}}$ we obtain

$$\Lambda_{sa} = \mathbf{L}_s \cdot \mathbf{C} \cdot \mathbf{R}_a - i\omega_s\delta_{sa}. \tag{5.17}$$

The result (5.13) now follows by expanding the eigenvectors $\mathbf{L}_s$ and $\mathbf{R}_a$ in (5.17) in terms of the eigenvectors $\mathbf{L}'_j$ and $\mathbf{R}'_j$ using the transformations (5.6), and by noting that $\mathbf{L}'_j$ and $\mathbf{R}'_j$ are eigenvectors of $\mathbf{C}$. The related identity (5.14) is obtained by premultiplying (5.13) by $P_{js}c_a$, summing over s and a, and noting that $\mathbf{P}\cdot\mathbf{Q} = \mathbf{I}$.

PROPOSITION 5.3. *The Lagrangians $\mathscr{L}$ and $\mathscr{L}'$ in the action principles (3.53) and (3.54) and in (4.24) are equal, but the corresponding Hamiltonian densities $\mathscr{H}$ and $\mathscr{H}'$ constructed by Legendre transformation are not equal. In particular,*

$$\mathscr{L}' = \mathscr{L}, \tag{5.18}$$

$$\mathscr{H}' = \mathscr{H} - i\sigma_j\Lambda'_{js}c'^{*}_j c'_s \tag{5.19}$$

relate the Lagrangian and Hamiltonian densities.

Proof. To prove that $\mathscr{L}' = \mathscr{L}$ we first note from (4.24) that $\mathscr{L}'$ can be written in the form

$$\mathscr{L}' = i\sigma_j c'^{*}_j\left(\frac{dc'_j}{dt} + i\omega'_j c'_j + \Lambda'_{jp}c'_p + \Lambda'_{j4}\Delta\tilde{S}\right) - i\sigma_j\Lambda'_{j4}\Delta\tilde{S}c'_j. \tag{5.20}$$

Using the results (5.11) and (5.12) for the transformation of the wave evolution operators, and the transformations (5.4) for the wave amplitudes c_j and c'_j in (5.20) we obtain

$$\mathscr{L}' = i\left[\sigma_j P_{ja}c^{*}_a P_{js}\left(\frac{dc_s}{dt} + i\omega_s c_s + \Lambda_{sp}c_p + \Lambda_{s4}\Delta\tilde{S}\right) - \sigma_j P_{ja}\Lambda_{a4}P_{jb}c_b\Delta\tilde{S}\right]. \tag{5.21}$$

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Finally, using the unitary property (5.8) and noting that $\mathbf{P}$ and $\mathbf{Q}$ are inverse matrices, the expression (5.21) for $\mathscr{L}'$ reduces to the result (3.54) for $\mathscr{L}$.

We now indicate the proof of (5.19) relating $\mathscr{H}'$ and $\mathscr{H}$. Using the identity (5.14) in the expression (3.59) for $\mathscr{H}$ we find

$$\mathscr{H} = \sigma_j\omega_j c_j c_j^* - i\sigma_j\Lambda_{js}c_j^* c_s - i\sigma_j\Lambda_{j4}(c_j^* - c_j)\Delta\tilde{S}$$
$$\equiv \sigma_j\omega_j' c_j' c_j'^* - i\sigma_s\Lambda_{s4}'(c_s'^* - c_s')\Delta\tilde{S}. \tag{5.22}$$

Comparing the result (5.22) for $\mathscr{H}$ with the expression (4.25) for $\mathscr{H}'$ gives the result (5.19) relating $\mathscr{H}'$ and $\mathscr{H}$.

5.1. The scattering matrix

The concept of the scattering matrix is well developed in quantum field theory (e.g., Landau and Lifshitz 1965; Barut 1967; Weinberg 1995). The definition of the scattering matrix $S_\alpha{}^\beta(t)$ (or, more precisely, the scattering tensor) is the transformation that relates the initial data $c_\beta(0)$ of the wave interaction equations at time $t = 0$ to their corresponding values $c_\beta(t)$ at time t. Thus,

$$c_\alpha(t) = S_\alpha{}^\beta(t)c_\beta(0). \tag{5.23}$$

In (5.23) the labels α and β correspond to covariant and contravariant indices, where the metric tensor $g_{\alpha\beta}$ has been chosen as

$$g_{\alpha\beta} = \sigma_\alpha\delta_{\alpha\beta} \equiv \mathrm{diag}(-1,-1,-1,1,1,1). \tag{5.24}$$

Thus, the metric tensor takes into account whether the canonical wave energy is negative ($\sigma_\alpha = -1$) or positive ($\sigma_\alpha = 1$). We also introduce the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle = u_\alpha v^{*\alpha} \equiv g^{\alpha\beta}u_\alpha v_\beta^*, \tag{5.25}$$

where the superscript $*$ denotes the complex conjugate.

One important constraint on the scattering matrix $S_\alpha{}^\beta(t)$ is that it should conserve the wave action, or canonical wave energy integral (3.49) in the case where there are no entropy perturbations $\Delta\tilde{S}$. Using the definitions (5.23)–(5.25) we obtain

$$\langle \mathbf{c}, \mathbf{c}\rangle = c_\alpha c^{*\alpha} = g^{\alpha\beta}c_\alpha c_\beta^* = \sigma_\alpha c_\alpha c_\alpha^* = N. \tag{5.26}$$

Thus, we identify the wave action integral with the inner product $\langle \mathbf{c}, \mathbf{c}\rangle$. Using the transformation formulae (5.23) relating $c_\alpha(t)$ to $c_\alpha(0)$ we find that $\langle \mathbf{c}(t), \mathbf{c}(t)\rangle = \langle \mathbf{c}(0), \mathbf{c}(0)\rangle$ is satisfied if the scattering matrix $S_\alpha{}^\beta(t)$ is a unitary matrix, i.e.

$$S^{*j}{}_p S_j{}^q = S_p{}^j S^{*q}{}_j = \delta^q{}_p. \tag{5.27}$$

In matrix notation, (5.27) gives

$$\mathbf{S}^{*\mathrm{T}}\mathbf{S} = \mathbf{S}\mathbf{S}^{*\mathrm{T}} = \mathbf{I}, \tag{5.28}$$

which implies that $\mathbf{S}$ is a unitary matrix, for which $\mathbf{S}^{-1} = \mathbf{S}^{*\mathrm{T}}$. In essence, the scattering matrix $S_\alpha{}^\beta(t)$ gives the solution of the wave interaction equations for the initial data $c_k(0) = \delta_{k\beta}$. Since the wave interaction equations are linear the scattering matrix (5.23) gives the solution for $c_\alpha(t)$ for general initial data $c_\beta(0)$.

6. Coupled oscillator formulation

So far, we have developed the form of the wave interaction equations as six coupled first-order differential equations using either the Hamiltonian variables $\{c_j\}$ (Sec. 3)

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or the ray-based Hamiltonian variables $\{c'_j\}$ (Sec. 4). Gogoberidze et al. (2004) also studied MHD wave interactions in a linear shear flow, using Kelvin's method. Their analysis was carried out by reducing the wave interaction equations to three coupled, second-order oscillator equations for the normal modes. One of the aims in this section is to show the connection between their analysis and the present analysis. We show that the wave interaction equations can be written as three coupled, linear oscillator equations of second order, by using the normal modes of the matrix $\mathbf{A}$, which leads directly to the Gogoberidze et al. (2004) equations. We also show that a similar analysis can be carried out using the normal modes of the matrix $\bar{\mathbf{A}}$, for which there is no shear. The variational form of the equations are also discussed.

6.1. Normal mode equations for $\mathbf{A}$

The starting point for our analysis is the system of second-order equations (3.2) for the Fourier transform of the Lagrangian displacement $\tilde{\xi}$. We look for solutions of (3.2) for $\tilde{\xi}$ of the form

$$\tilde{\xi} = \sum_{j=1}^{3} \psi_j \mathbf{r}_j, \tag{6.1}$$

where $\{\mathbf{r}_j\}$ and $\{\mathbf{l}_j\}$ are the right and left eigenvectors of the matrix $\mathbf{A}$ in (3.4), corresponding to the eigenvalue ω_j^2 ($j = 1, 2, 3$), i.e.

$$(\mathbf{A} - \omega_j^2 \mathbf{l}_3) \cdot \mathbf{r}_j = 0, \quad \mathbf{l}_j \cdot (\mathbf{A} - \omega_j^2 \mathbf{l}_3) = 0, \quad \det(\mathbf{A} - \omega^2 \mathbf{l}) = 0, \quad j = 1, 2, 3, \tag{6.2}$$

where $\mathbf{l}_j \cdot \mathbf{r}_s = \delta_{js}$. The eigenvalues $\omega^2 = \omega_j^2 = k'^2 \lambda_j^2$ where the λ_j correspond to the fast magnetoacoustic, Alfvén wave and slow magnetoacoustic waves, respectively, for $j = 1, 2, 3$ as in (3.20), except that we use $\mathbf{k}'$ to replace $\mathbf{k}$ in the formulas (see also (4.9) et seq.). In other words, in a convenient abuse of mathematical notation, we drop the superscript $'$ on all quantities in this section, unless explicitly stated otherwise. The right and left eigenvectors $\{\mathbf{r}_j\}$ and $\{\mathbf{l}_j\}$ of the matrix $\mathbf{A}$ have the form

$$\mathbf{r}_j = \mathbf{l}_j^{\mathrm{T}} = \frac{1}{\sqrt{\Delta_j}} (u_{j1}, u_{j2}, u_{j3})^{\mathrm{T}}, \tag{6.3}$$

where

$$\mathbf{U}_j = (u_{j1}, u_{j2}, u_{j3}, 0, 0, 0)^{\mathrm{T}}, \quad \mathbf{V}_j = (0, 0, 0, u_{j1}, u_{j2}, u_{j3})^{\mathrm{T}}, \tag{6.4}$$

and Δ_j are defined in (3.22)–(3.25), but with $\mathbf{k}$ replaced by $\mathbf{k}'$.

Using the eigenvector expansion (6.1) for $\tilde{\xi}$ in (3.2), and taking the scalar product of (3.2) with the left eigenvector $\mathbf{l}_j$, we obtain normal mode equations for ψ_j of the form

$$\ddot{\psi}_j + \omega_j^2 \psi_j + \mathbf{l}_j \cdot \ddot{\mathbf{r}}_p \psi_p + 2(\mathbf{l}_j \cdot \dot{\mathbf{r}}_p) \dot{\psi}_p = \mathbf{l}_j \cdot \mathbf{Q}_\xi, \tag{6.5}$$

($j = 1, 2, 3$), where we use the Einstein summation convention for the repeated indices p and $\mathbf{Q}_\xi$ is the entropy source term in (3.2). In (6.5), $\dot{\psi} \equiv d\psi/dt$ and $\ddot{\psi} = d^2\psi/dt^2$ denote the first and second time derivatives of ψ. As $\mathbf{l}_j = \mathbf{r}_j^{\mathrm{T}}$ (note the matrix $\mathbf{A}$ is symmetric), we find

$$\mathbf{l}_j \cdot \dot{\mathbf{r}}_j = 0, \quad \mathbf{l}_j \cdot \ddot{\mathbf{r}}_j = -\dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j, \tag{6.6}$$

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and the system (6.5) reduces to

$$\ddot{\psi}_j + (\omega_j^2 - \theta_j)\psi_j = \mathbf{l}_j \cdot \mathbf{Q}_\xi - \sum_{p\neq j} \Upsilon_{jp}\psi_p - 2\sum_{p\neq j}\nu_{jp}\dot{\psi}_p, \tag{6.7}$$

where

$$\Upsilon_{jp} = \mathbf{l}_j \cdot \ddot{\mathbf{r}}_p, \quad \nu_{jp} = \mathbf{l}_j \cdot \dot{\mathbf{r}}_p, \quad \theta_j = \Upsilon_{jj} = -\dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j. \tag{6.8}$$

Equations (6.8) are equivalent to the normal mode equations of Gogoberidze et al. (2004), supplemented by entropy wave source terms. Note, however, that the normal mode variables ψ_j in their paper are expressed in terms of the Eulerian perturbations $(\delta\rho, \delta\mathbf{u}, \delta\mathbf{B})$. The detailed demonstration that the variables ψ_j above correspond to the normal variables of Gogoberidze et al. (2004) (their equations (29)–(31)) can be found in Appendix E. Gogoberidze et al. use scattering matrix ideas, coupled with the wave action conservation equation, to determine the transformation coefficients between the different wave modes. Similar ideas have been used, for example, in scattering problems in quantum mechanics (e.g., Landau and Lifshitz 1965), and for wave coupling and transformations in inhomogeneous plasmas (e.g., Swanson 1989, 1998).

An inspection of the form of the eigenvectors $\mathbf{R}_j$ and $\mathbf{L}_j$ for the matrix $\mathbf{C}$, and the eigenvectors $\mathbf{r}_j$ and $\mathbf{l}_j$ of the matrix $\mathbf{A}$ reveals that

$$\psi'_j = \frac{1}{\sqrt{\rho|\omega'_j|}}(c'_j + c'_{j'}); \tag{6.9}$$

the superscript $'$ on ω'_j and c'_j emphasizes the fact that the quantities depend on $\mathbf{k}'$, and the superscript $'$ on j' corresponds to the backward wave corresponding to j (i.e. $j + j' = 8$).

6.2. Normal modes for $\bar{\mathbf{A}}$

In this section we consider the coupled oscillator equations for the normal modes of the matrix $\bar{\mathbf{A}}$ in which there is no shear. Since ρ and ω_j are constants for the normal modes based on $\bar{\mathbf{A}}$, then we can set $u_j = c_j + c_{j'}$ as the normal variables (see (6.9)). The second-order differential equations governing the normal variables u_j in this case can be obtained by combining (3.49) for c_j and $c_{j'}$.

The starting point for the analysis is the wave interaction equations (3.41) for the $\{c_j\}$, namely,

$$\frac{dc_j}{dt} + i\omega_j c_j + i\Omega_{js}c_s = -\Lambda_{j4}\Delta\tilde{S}, \tag{6.10}$$

where $1 \le j, s \le 7$, $j, s \neq 4$. Taking into account the symmetries of the wave interaction coefficients $\{\Omega_{js}\}$ and noting that $\Lambda_{j'4} = -\Lambda_{j4}$ $(j + j' = 8)$, the wave interaction equations (6.10) can be written in the form

$$\frac{dc_j}{dt} + i\omega_j c_j + i\Omega_{js}(c_s + c_{s'}) = -\Lambda_{j4}\Delta\tilde{S}, \tag{6.11}$$

$$\frac{dc_{j'}}{dt} - i\omega_j c_{j'} - i\Omega_{js}(c_s + c_{s'}) = \Lambda_{j4}\Delta\tilde{S}, \tag{6.12}$$

where $j, s = 1, 2, 3$ and $j' = 8 - j$ and $s' = 8 - s$. Note that

$$\Omega_{js} = \Omega_{sj} \quad \text{for } j, s = 1, 2, 3. \tag{6.13}$$

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To further analyze (6.11) and (6.12), we introduce new variables

$$u_j = c_j + c_{j'}, \quad v_j = c_{j'} - c_j, \quad j = 1, 2, 3, \tag{6.14}$$

where $j' = 8 - j$ (e.g., $u_1 = c_7 + c_1$ and $v_1 = c_7 - c_1$). By adding and subtracting (6.11) from (6.12) we obtain the system of equations

$$\dot{u}_j - i\omega_j v_j = 0, \tag{6.15}$$

$$\dot{v}_j - i\omega_j u_j - 2i\Omega_{js} u_s = 2\Lambda_{j4}\Delta\tilde{S}, \tag{6.16}$$

where $j, s = 1, 2, 3$, and $\dot{w} \equiv dw/dt$.

Differentiating (6.15) with respect to t and eliminating $\dot{v}_j$ using (6.16) we obtain the coupled oscillator system

$$E_j(\mathbf{u}) = \ddot{u}_j + \omega_j(\omega_j u_j + 2\Omega_{js} u_s - 2i\Lambda_{j4}\Delta\tilde{S}) = 0, \tag{6.17}$$

where $j, s = 1, 2, 3$. The above analysis shows that the six first-order differential equations (6.10) for the $\{c_j\}$ can be reduced to a system of three coupled, second-order oscillator equations (6.17) for the $\{u_j : j = 1, 2, 3\}$. Below, we discuss the variational formulation of the coupled oscillator system (6.17).

PROPOSITION 6.1. *The coupled oscillator system (6.17) can be obtained by extremizing the action*

$$J = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3k \, \mathscr{L}, \tag{6.18}$$

where

$$\mathscr{L} = \frac{1}{2}\left(-\frac{\dot{u}_j \dot{u}_j^*}{\omega_j} + \omega_j u_j u_j^* + 2\Omega_{js} u_s u_j^* - 2i\Lambda_{j4}(u_j^* - u_j)\Delta\tilde{S}\right) \tag{6.19}$$

is the Lagrangian density for the system. In particular,

$$\frac{\delta J}{\delta u_k^*} = 2\left[\frac{\partial \mathscr{L}}{\partial u_k^*} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_k^*}\right)\right] = \frac{1}{\omega_k} E_k(\mathbf{u}) = 0, \quad k = 1, 2, 3, \tag{6.20}$$

for extremal variations of the action. Thus, the variational principle (6.18)–(6.20) gives the coupled oscillator equations (6.17).

Comment. Further development of the variational principle of Proposition 6.1 is possible, but will not be pursued further here.

7. Illustrative examples

In this section, we consider some simple examples of MHD wave interactions in shear flows. We restrict our attention mainly to waves with $k_z = 0$. For $k_z = 0$ the wave interaction equations split up into two separate, non-interacting subsystems. One system describes the Alfvén waves and the other system describes the fast and slow magnetoacoustic waves. As discussed in Secs 3 and 4, there are two Hamiltonian forms of the equations. The first Hamiltonian form (Sec. 3) is based on the eigenvector decomposition using the background state with no shear. In this formulation, the $\mathbf{k}$ vector of the wave, $\mathbf{k} = (k_x, k_y, k_z)^{\mathrm{T}}$, does not change. We call this system the K-system. The second Hamiltonian form (Sec. 4) is based on an eigenvector decomposition about a background state, including the fluid velocity shear. In this formulation, the wavevector $\mathbf{k}' = (k_x, k_y', k_z)^{\mathrm{T}}$, where $k_y' = k_y - Atk_x$,

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evolves according to the ray equations of geometrical MHD optics, which we call the R-system. Both systems are useful for describing the wave interactions. The mathematical form of the equations is simpler in the K-system but the physics is simpler in the R-system.

7.1. Wave interaction equations ($k_z = 0$)

The K-system. The wave interactions in the K-system are described by (3.41)–(3.48). For $k_z = 0$ the equations reduce to the system

$$\frac{dc_2}{dt} + i\omega_2 c_2 = 0,$$
$$\frac{dc_6}{dt} - i\omega_2 c_6 = 0 \tag{7.1}$$

describing the Alfvén waves and the system

$$\frac{dc_1}{dt} + i\omega_1 c_1 + i\Omega_{11}(c_1 + c_7) + i\Omega_{13}(c_3 + c_5) = -\Lambda_{14}\Delta\tilde{S},$$
$$\frac{dc_7}{dt} - i\omega_1 c_7 - i\Omega_{11}(c_1 + c_7) - i\Omega_{13}(c_3 + c_5) = \Lambda_{14}\Delta\tilde{S},$$
$$\frac{dc_3}{dt} + i\omega_3 c_3 + i\Omega_{13}(c_1 + c_7) + i\Omega_{33}(c_3 + c_5) = -\Lambda_{34}\Delta\tilde{S},$$
$$\frac{dc_5}{dt} - i\omega_3 c_5 - i\Omega_{13}(c_1 + c_7) - i\Omega_{33}(c_3 + c_5) = \Lambda_{34}\Delta\tilde{S} \tag{7.2}$$

describing the magnetoacoustic modes. The detailed form of the wave interaction coefficients $\{\Omega_{js}\}$ can be obtained from (3.41)–(3.47). In particular,

$$\Omega_{11} = -\frac{f_{11}}{2|\omega_1|\Delta_1}, \quad \Omega_{13} = -\frac{f_{13}}{2\sqrt{|\omega_1\Delta_1|\omega_3|\Delta_3}}, \quad \Omega_{33} = -\frac{f_{33}}{2|\omega_3|\Delta_3},$$
$$f_{11} = -Atk^2 n^x n^y c_f^4[2c_f^2 - 2b_n^2 - At(a^2 + b^2)n^x n^y],$$
$$f_{13} = -Atk^2 n^x n^y c_f^2 c_s^2[c_f^2 + c_s^2 - 2b_n^2 - At(a^2 + b^2)n^x n^y],$$
$$f_{33} = -Atk^2 n^x n^y c_s^4[2c_s^2 - 2b_n^2 - At(a^2 + b^2)n^x n^y]. \tag{7.3}$$

The detailed form of the entropy wave coefficients $\{\Lambda_{j4}\}$ is given by (3.48), but we restrict the examples to cases where there are no entropy waves. Equations (7.1) and (7.2) show that the backward and forward Alfvén waves c_2 and c_6 do not interact with the magnetoacoustic waves or with each other if $k_z = 0$.

The R-system. In the R-system, the wave interaction equations and coefficients are given by (4.15)–(4.21). For $k^z = 0$, these equations reduce to the system

$$\frac{dc_2'}{dt} + i\omega_2' c_2' = 0,$$
$$\frac{dc_6'}{dt} - i\omega_2' c_6' = 0 \tag{7.4}$$

describing the Alfvén waves where $\omega_2' = -k_x b$, and the system

$$\frac{dc_1'}{dt} + i\omega_1' c_1' + \Lambda_{13}' c_3' + \Lambda_{15}' c_5' + \Lambda_{17}' c_7' = -\Lambda_{14}'\Delta\tilde{S},$$
$$\frac{dc_7'}{dt} - i\omega_1' c_7' + \Lambda_{17}' c_1' + \Lambda_{15}' c_3' + \Lambda_{13}' c_5' = \Lambda_{14}'\Delta\tilde{S},$$

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$$\frac{dc_3'}{dt} + i\omega_3' c_3' - \Lambda_{13}' c_1' + \Lambda_{35}' c_5' + \Lambda_{15}' c_7' = -\Lambda_{34}' \Delta\tilde{S},$$

$$\frac{dc_5'}{dt} - i\omega_3' c_5' + \Lambda_{15}' c_1' + \Lambda_{35}' c_3' - \Lambda_{13}' c_7' = \Lambda_{34}' \Delta\tilde{S} \tag{7.5}$$

describing the magnetoacoustic waves. The wave interaction coefficients $\{\Lambda_{js}'\}$ $(s \neq 4)$ are given by

$$\Lambda_{13}' = \frac{An^x n^y a^2 b_n^2 (a^2 + b^2 - 2b_n^2)}{2(c_f - c_s)\sqrt{c_f c_s \Delta_f \Delta_s}},$$

$$\Lambda_{15}' = \frac{An^x n^y a^2 b_n^2 (a^2 + b^2 - 2b_n^2)}{2(c_f + c_s)\sqrt{c_f c_s \Delta_f \Delta_s}},$$

$$\Lambda_{17}' = \frac{An^x n^y c_f^2 (c_f^2 - b_n^2)}{2\Delta_f},$$

$$\Lambda_{35}' = \frac{An^x n^y c_s^2 (c_s^2 - b_n^2)}{2\Delta_s}, \tag{7.6}$$

where

$$\Delta_f = (c_f^2 - b_n^2)(c_f^2 - c_s^2) \quad \text{and} \quad \Delta_s = (b_n^2 - c_s^2)(c_f^2 - c_s^2). \tag{7.7}$$

In (7.6) and (7.7), $\mathbf{n}$, c_f and c_s are calculated according to the ray system, i.e. we use $\mathbf{n} = \mathbf{k}'/k'$ where $\mathbf{k}' = (k_x, k_y - Atk_x, k_z)$ in calculating $\mathbf{n}$, c_f, c_s, b_n, Δ_f and Δ_s.

Equations (7.4) and (7.5) show that the Alfvén modes c_2' and c_6' do not interact with each other, or with the magnetoacoustic waves that satisfy the system (7.5)–(7.7). Note that the R-system (7.5)–(7.7) for the magnetoacoustic waves looks more complicated than the corresponding K-system (7.2), but the two systems are equivalent, and describe the same initial value problem (i.e. $c_j'(0) = c_j(0)$). The transformations between the $\{c_s'\}$ and the $\{c_j\}$ from (5.4) are $c_s' = P_{sj} c_j$, where $P_{sj} = \mathbf{L}_s' \cdot \mathbf{R}_j$.

7.2. Cold gas case ($a = 0$, $k_z = 0$)

In the cold gas case ($a = 0$), the Jordan mode analysis of Sec. 3 implies that the slow mode amplitudes c_3 and c_5 (note c_3 and c_5 do not necessarily correspond to backward and forward slow mode waves, since the slow mode speed is zero in this limit) are governed by the Jordan mode equations:

$$\frac{dc_3}{dt} + c_5 = 0 \quad \text{and} \quad \frac{dc_5}{dt} = 0. \tag{7.8}$$

The slow mode equations (7.2) for the case of small a ($a \approx 0$) decouple from the fast mode equations, and have the approximate form

$$\frac{dc_3}{dt} + i\omega_3 c_3 = 0 \quad \text{and} \quad \frac{dc_5}{dt} - i\omega_3 c_5 = 0, \tag{7.9}$$

where $\omega_3 = kc_s$ and $c_s \approx an^x$. Equations (7.8) have distinctly different solutions than (7.9). The Jordan mode equations (7.8) have a secularly growing mode solution, whereas the solutions of (7.9) in the limit as $a \to 0$ have constant solutions for c_3 and c_5.

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The fast magnetoacoustic mode equations (from (7.2) as $a \to 0$ or from (3.71) with $k_z = 0$) reduce to the equation system

$$\frac{dc_1}{dt} + i\omega_1 c_1 + i\Omega_{11}(c_1 + c_7) = 0, \tag{7.10}$$

$$\frac{dc_7}{dt} - i\omega_1 c_7 - i\Omega_{11}(c_1 + c_7) = 0, \tag{7.11}$$

where

$$\omega_1 = -kb, \quad \Omega_{11} = \frac{Atn^x n^y kb[2 - 2(n^x)^2 - Atn^x n^y]}{2(n^y)^2}. \tag{7.12}$$

Equations (7.10)–(7.12) apply for the K-system.

From (7.5) the wave amplitudes $\{c'_j\}$ in the R-system for the fast mode waves satisfy the coupled differential equation system

$$\frac{dc'_1}{dt} + i\omega'_1 c'_1 + \Lambda'_{17} c'_7 = 0, \tag{7.13}$$

$$\frac{dc'_7}{dt} - i\omega'_1 c'_7 + \Lambda'_{17} c'_1 = 0, \tag{7.14}$$

where

$$\omega'_1 = -k'b \quad \text{and} \quad \Lambda'_{17} = \frac{An'^x n'^y}{2}. \tag{7.15}$$

Using the transformations (5.4) in the cold gas limit, with $k_z = 0$, we find

$$\begin{pmatrix} c'_1 \\ c'_7 \end{pmatrix} = \begin{pmatrix} \dfrac{k' + k}{2\sqrt{kk'}} & \dfrac{k' - k}{2\sqrt{kk'}} \\ \dfrac{k' - k}{2\sqrt{kk'}} & \dfrac{k' + k}{2\sqrt{kk'}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_7 \end{pmatrix} = \mathbf{P} \begin{pmatrix} c_1 \\ c_7 \end{pmatrix}, \tag{7.16}$$

and $c'_j = c_j$ for $j = 2, 3, 5, 6$. Note that $P_{js} = \mathbf{L}'_j \cdot \mathbf{R}_s$ and $\det(\mathbf{P}) = 1$. Here $k' = [k_x^2 + (k_y - Atk_x)^2]^{1/2}$ is the magnitude of $\mathbf{k}'$ in the R-system. The transformations (7.16) converts system (7.13) and (7.14) into system (7.10) and (7.11).

7.2.1. Exact solution of (7.10) and (7.11). Introducing the new variables

$$u = c_7 + c_1 \quad \text{and} \quad v = c_7 - c_1, \tag{7.17}$$

the system (7.10) and (7.11) reduces to the coupled equation system

$$\frac{du}{dt} - i\omega_1 v = 0 \quad \text{and} \quad \frac{dv}{dt} - i(\omega_1 + 2\Omega_{11})u = 0. \tag{7.18}$$

Equations (7.18) can be combined to yield the second-order differential equation

$$\frac{d^2u}{dt^2} + \Omega^2(t)u = 0, \tag{7.19}$$

where

$$\Omega^2(t) = k^2b^2[(n^x)^2 + (Atn^x - n^y)^2] \equiv k'^2b^2. \tag{7.20}$$

Equation (7.19) also follows directly from the second-order differential equation system (6.17) in the cold gas limit with $k_z = 0$.

Equation (7.18) has the form of a simple harmonic oscillator equation with a time-dependent frequency $\Omega(t)$. Equations of this type were studied by Ermakov (1880). Generalized Ermakov systems have been used to describe time-dependent

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harmonic oscillators in quantum mechanics, and also to describe the motion of a charged particle in a time-dependent magnetic field (e.g., Lutzky 1978; Leach 1991). Setting $u = r\exp(i\theta)$ in (7.19), the resulting two differential equations for r and θ have the integral

$$r^2\dot{\theta} = h \equiv kb(|c_7|^2 - |c_1|^2) = \text{constant}, \tag{7.21}$$

and $r(t)$ satisfies the equation

$$\ddot{r} - r\dot{\theta}^2 = -\Omega(t)^2 r. \tag{7.22}$$

Note that (7.21) and (7.22) are basic equations for particle motion in a central force field $-\Omega^2\mathbf{r}$. The wave action conservation integral (7.21) in this analogy appears as the angular momentum conservation law (i.e. constant areal velocity), which is related to the so-called Lewis invariant and the Pinney equation. The Pinney equation results from using $\dot{\theta} = h/r^2$ from (7.21) in (7.22), resulting in a second-order differential equation for $r = r(t)$.

Equation (7.19) has exact solutions of the form

$$u = \exp\left(-i\frac{\nu z}{2}\right)\left[a_1 M\left(\chi, \frac{1}{2}, i\nu z\right) + a_2 U\left(\chi, \frac{1}{2}, i\nu z\right)\right], \tag{7.23}$$

where a_1 and a_2 are arbitrary constants and

$$\nu = \frac{kb}{An^x}, \quad \chi = \frac{1}{4}\left(1 + i\frac{k_x b}{A}\right), \quad z = (Atn^x - n^y)^2. \tag{7.24}$$

In (7.23) $M(a,b,z)$ and $U(a,b,z)$ are independent solutions of Kummer's confluent hypergeometric equation (Abramowitz and Stegun 1965, ch. 13, p. 504). Since the parameter b in $M(a,b,z)$ is $b = 1/2$, then the solution can also be expressed in terms of parabolic cylinder functions (Abramowitz and Stegun 1965, ch. 19, p. 686). A discussion of scattering matrix theory for c_1 and c_7 based on WKB solutions of (7.19) is given in Appendix F. Some of the results of this theory are verified by the numerical simulations described below.

7.3. Solution examples

Figure 1 shows the evolution of the backward and forward fast mode solutions for c_1 and c_7 for the case of a cold background gas for the K-system (7.10)–(7.12) as functions of the normalized time variable $\tau = k_x bt$. The $\mathbf{k}$ vector $\mathbf{k} = k_x(1,2,0)$ and the normalized shear parameter $\tilde{A} = A/(k_x b) = 0.1$, and the initial data for the wave amplitudes are $c_7(0) = 1$ and $c_1(0) = 0$ (i.e. the initial data consists of a forward wave only). The figure shows that both c_1 and c_7 pass through a slowly evolving phase for $0 < t < t_c$ where $t_c = k_y/(Ak_x)$ or $\tau_c = k_y/(k_x\tilde{A}) = 20$. At this time $k'_y = k_y - Atk_x = 0$ and the frequency of the waves in the ray system $\omega' = k'b \equiv \Omega = k_x b$ is at a minimum. At late times both $|c_1(t)|$ and $|c_7(t)|$ increase with t like $\sqrt{k'(t)/k} \sim \sqrt{Atn^x}$ as $t \to \infty$, as suggested by WKB analysis of the system for the case of weak shear with $|\tilde{A}| \ll 1$ (Appendix F), but in such a way as to conserve the total wave action (7.21). Figure 2 shows the wave action densities $A_7 = |c_7(t)|^2$, $A_1 = -|c_1(t)|^2$ in the K-system and their sum $A_1 + A_7$ as functions of τ. The figure shows that, although A_1 and A_7 have a complicated oscillatory structure at $\tau > 20$ (note $k'_y = 0$ at $\tau = 20$), their sum $A_1 + A_7$ is conserved, as required by wave action conservation (7.21). The WKB analysis of Appendix F indicates that the wave action profiles in Fig. 2 have local maxima when $\sin(2\theta) = 1$ where $\theta = \int_0^t k'b\,dt$.

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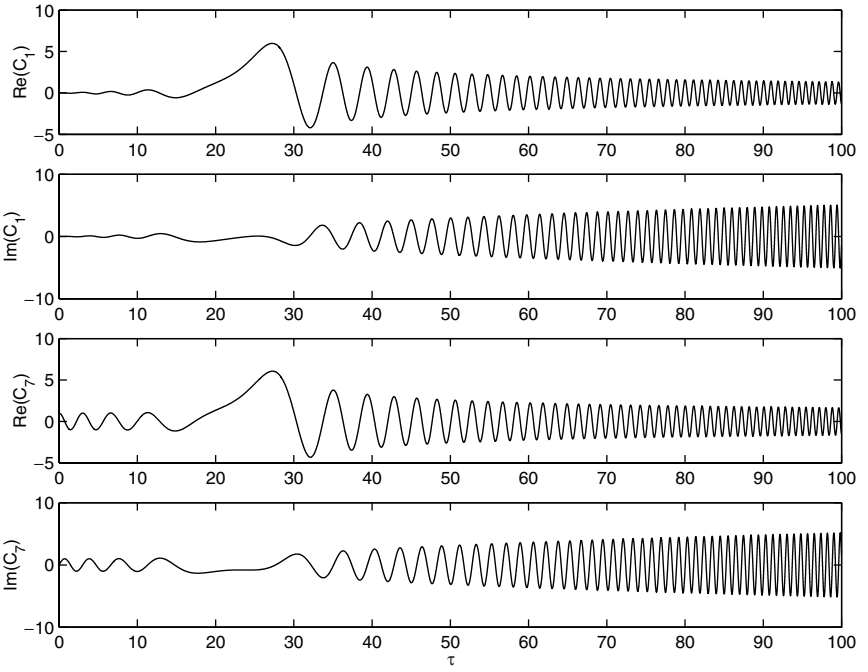


Figure 1. Real and imaginary parts of $c_j(t)$ $(j = 1, 7)$ in the K-system versus $\tau = k_x bt$ for the cold gas case: $\mathbf{k} = k_x(1, 2, 0)^{\mathrm{T}}$; $\tilde{A} = A/(k_x b) = 0.1$; $c_7(0) = 1$; and $c_1(0) = 0$.

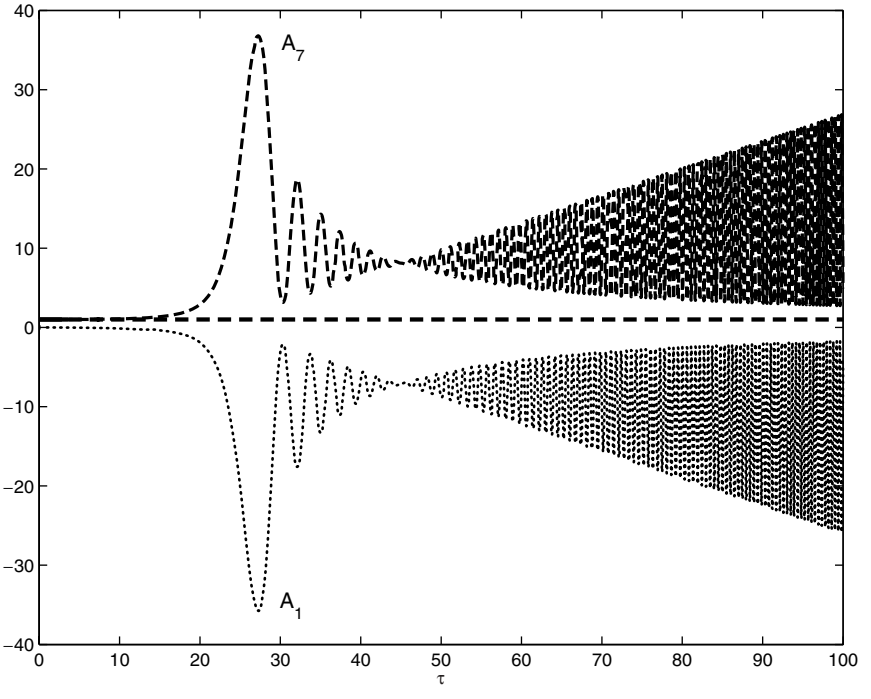


Figure 2. The wave action densities A_1 and A_7 in the K-system versus $\tau = k_x bt$, and their sum $A_{\mathrm{sum}} = A_1 + A_7$ for the same parameters as in Fig. 1: $\mathbf{k} = k_x(1, 2, 0)^{\mathrm{T}}$; $\tilde{A} = A/(k_x b) = 0.1$; $c_7(0) = 1$; and $c_1(0) = 0$.

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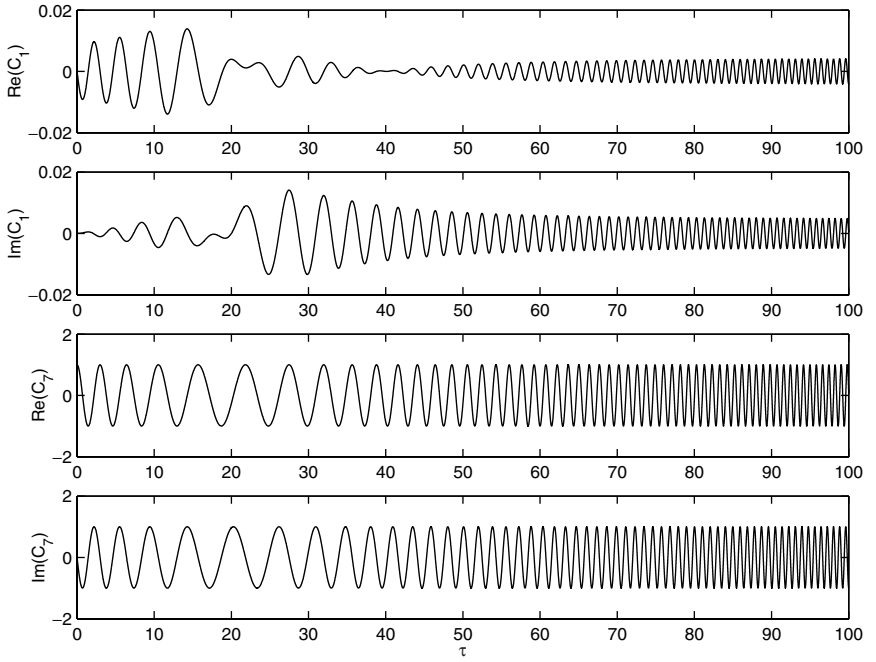


Figure 3. Real and imaginary parts of $c_j'(t)$ $(j = 1, 7)$ in the R-system versus $\tau = k_x bt$ for the cold gas case for the same parameters as in Fig. 1: $\mathbf{k} = k_x(1, 2, 0)^{\mathrm{T}}$; $\tilde{A} = A/(k_x b) = 0.1$; $c_7(0) = 1$; and $c_1(0) = 0$.

Figure 3 shows solutions of the R-system (7.13) and (7.14) for the same initial data and parameter values as in Fig. 1 (i.e. $c_7'(0) = 1$ and $c_1'(0) = 0$). The solutions $c_1'(t)$ and $c_7'(t)$ can also be obtained by using the transformations (7.16) between the K- and R-systems. The profiles of $c_1'(t)$ and $c_7'(t)$ evolve through a low-frequency region for $0 < t < t_{\mathrm{c}}$, where $t = t_{\mathrm{c}}$ when $k_y' = 0$ (i.e. $\tau = 20$). The corresponding wave action densities $A_7' = |c_7'(t)|^2$ and $A_1' = -|c_1'(t)|^2$ are shown in Fig. 4, as well as their sum, which is constant. The wave action densities A_1' and A_7' appear to be essentially constant. However, the analysis of Appendix F shows that the wave action densities A_1' and A_7' do oscillate, but the oscillation amplitude is $O(\tilde{A}^2) \approx 0.01$ about their means, which is too small to be seen in the figure. According to WKB theory (Appendix F) $|c_1'| \sim |\tilde{A}(k_x^2 k_y/4k^3)|$ and $|c_7'| \sim 1 + O(\tilde{A})$ at late times as $\tau \to \infty$, where $\tilde{A} = A/(k_x b)$ is the shear parameter.

The results in Figs 1–4 show that the ray system (Figs 3 and 4) provide a more transparent physical picture of the wave interactions than the K-system results in Figs 1 and 2. In particular, the wave action densities in the R-system evolve slowly in time and are bounded, whereas the action densities in the K-system oscillate wildly at late times, and grow linearly with t at late times.

Figure 5 shows the evolution of the backward and forward fast mode waves in the R-system for the cold gas case ($a = 0$) governed by (7.13) and (7.14) for the initial data $c_7(0) = 1$ and $c_1(0) = 0$ and the shear parameter $\tilde{A} = A/(k_x b) = 1$. In this case $k_y' = k_y - Atk_x = 0$ at time $\tau = k_x bt = 2$. There is a substantially stronger generation of the backward wave than in Fig. 3 where $\tilde{A} = 0.1$. Figure 6 shows the corresponding wave action densities A_1' and A_7' and their sum $A_{\mathrm{sum}} = A_1' + A_7'$ as functions of τ. The results in Fig. 6 show the phenomenon of wave amplification

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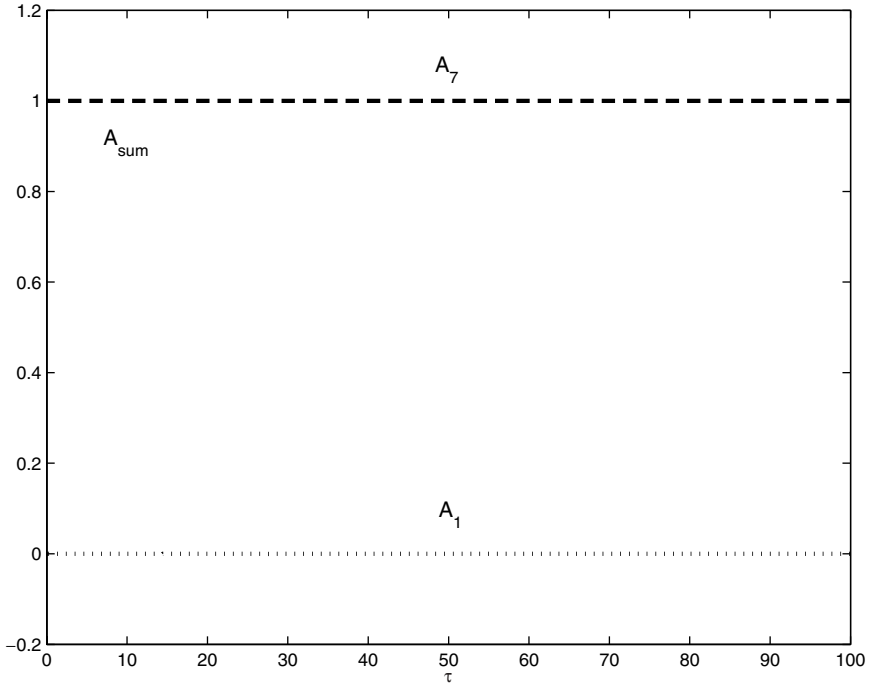


Figure 4. Wave action densities A_1' and A_7' and $A_{\rm sum} = A_1' + A_7'$ in the R-system versus $\tau = k_x bt$ for the cold gas case for the same parameters as in Fig. 1: $\mathbf{k} = k_x(1, 2, 0)^{\rm T}$; $\tilde{A} = A/(k_x b) = 0.1$; $c_7(0) = 1$; and $c_1(0) = 0$.

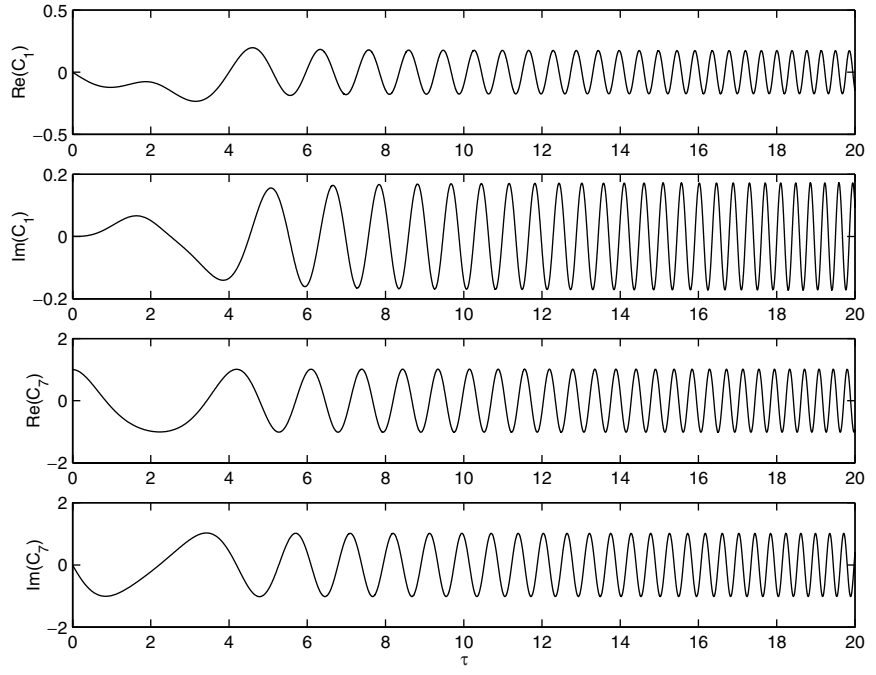


Figure 5. Real and imaginary parts of $c_j'(t)$ $(j = 1, 7)$ in the R-system versus $\tau = k_x bt$ for the same parameters as in Fig. 1, except that $\tilde{A} = A/(k_x b) = 1$.

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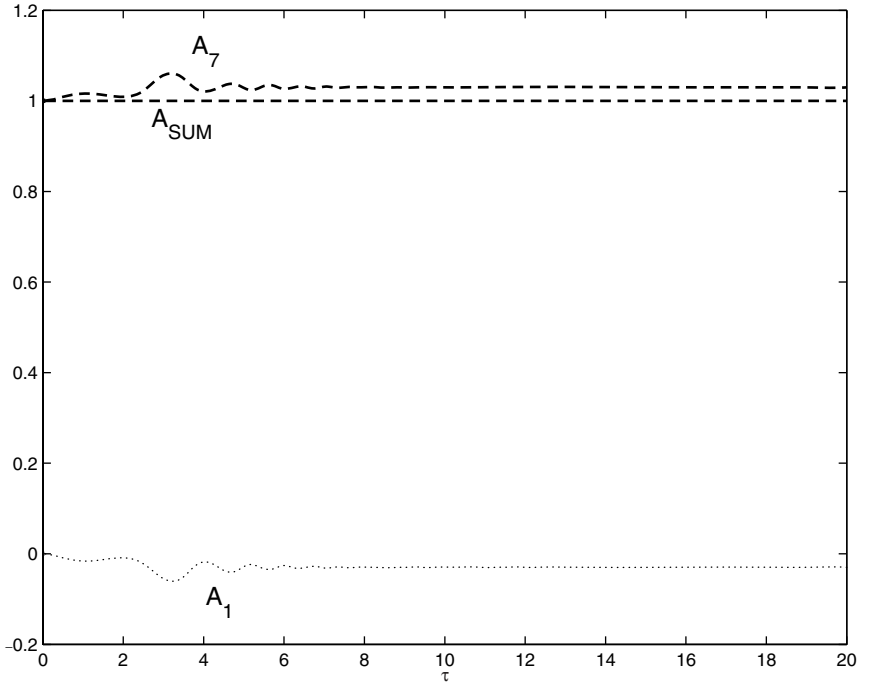


Figure 6. Wave action densities A'_j $(j = 1, 7)$ in the R-system versus $\tau = k_x bt$ and their sum $A_{\rm sum} = A'_1 + A'_7$ for the same parameter values as in Fig. 5.

(the action density of the initial forward wave $A'_7 = 1$ at $t = 0$, but $A'_7 > 1$ at late times). There is a phase of fast wave evolution and interaction at early times ($\tau < 6$) that evolves to essentially constant solutions for A'_1 and A'_7 at late times. Similar results of wave amplification are illustrated in Figs 7 and 8 for the same initial data as in Figs 5 and 6, except the shear parameter $\tilde{A} = 10$. Figure 7 shows the wave amplitudes c'_1 and c'_7 as functions of $\tau = k_x bt$. Figure 8 shows the corresponding wave action densities A'_1, A'_7 and their sum $A_{\rm sum} = A'_1 + A'_7$. The main point to note is the substantial increase in the wave amplification due to the increased shear in Fig. 8 compared to the corresponding results in Fig. 6. Note also the higher frequencies in the wave amplitudes c'_1 and c'_7 due to the increased shear in Fig. 7 (where $\tilde{A} = 10$) as compared with the smaller shear example ($\tilde{A} = 1$) in Fig. 5.

Figures 9–11 show features of the interaction between the backward and forward fast and slow magnetoacoustic modes, based on the wave interaction equation systems (7.2) and (7.5). Figure 9 shows the evolution of the $\{c_j(t) : j = 1, 3, 5, 7\}$ in the K-system versus $\tau = k_x bt$. The initial data are $c_1(0) = 0.1$, $c_j(0) = 0$ $(j = 3, 5, 7)$. In this example $(a/b)^2 = 0.99$ and $\mathbf{k} = k_x(1, 2, 0)$ and $\tilde{A} = A/(k_x b) = 0.1$. The wave action densities in the K-system are displayed in Fig. 10. Significant evolution of the wave action densities occurs when $\tau \approx 20$ $(k'_y = 0)$. The sum of the wave action densities $A_{\rm sum} = -0.01$ throughout the evolution. Notice that the action densities A_j oscillate at $\tau \geq 20$. In addition, A_1 and A_3 show a linear drift or trend in their means $\langle A_1 \rangle$ and $\langle A_3 \rangle$ with increasing time, suggesting strong mode coupling between c_1 and c_3. Figure 11 shows the evolution of the wave action densities $\{A'_j : j = 1, 3, 5, 7\}$ in the R-system for the same values of the parameters and initial data as in Fig. 9. For $\tau < 20$, the dominant interaction is principally

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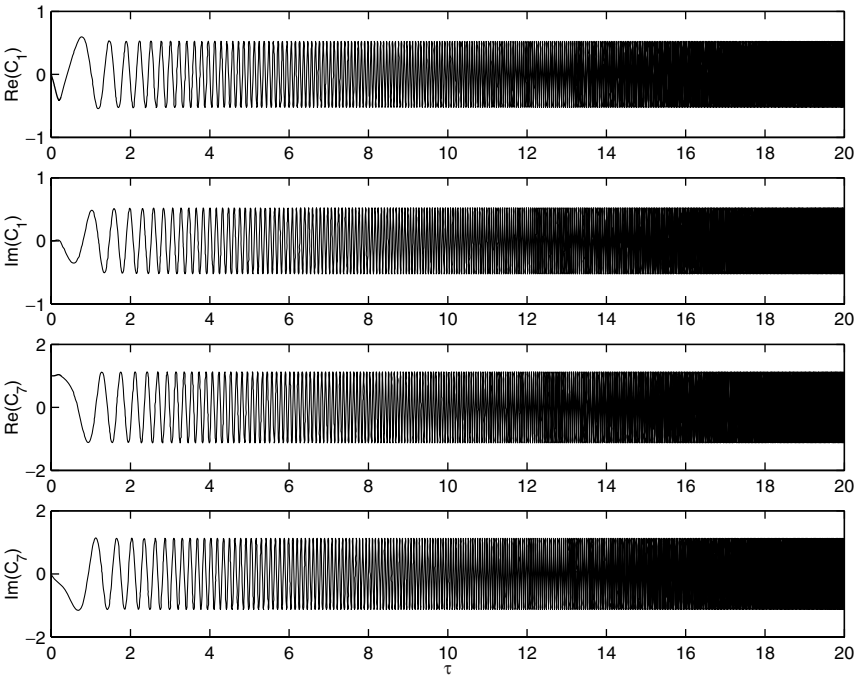


Figure 7. Real and imaginary parts of $c_j'(t)$ $(j = 1, 7)$ in the R-system versus $\tau = k_x bt$ for the same parameters as in Fig. 1, except that $\tilde{A} = A/(k_x b) = 10$.

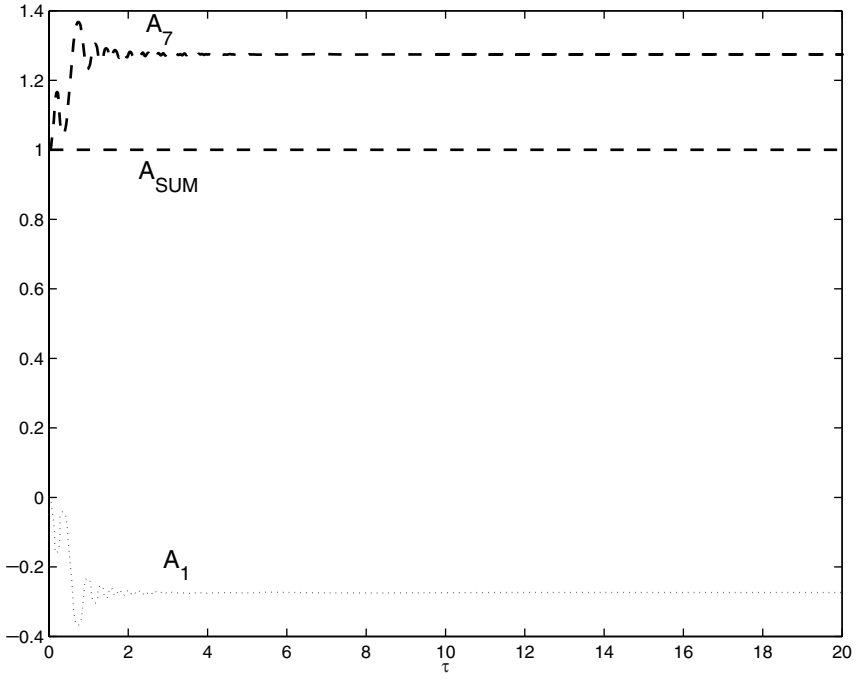


Figure 8. Wave action densities A_j' $(j = 1, 7)$ in the R-system versus $\tau = k_x bt$ and their sum $A_{\rm sum} = A_1' + A_7'$ for the same parameter values as in Fig. 7 (i.e. $\tilde{A} = 10$).

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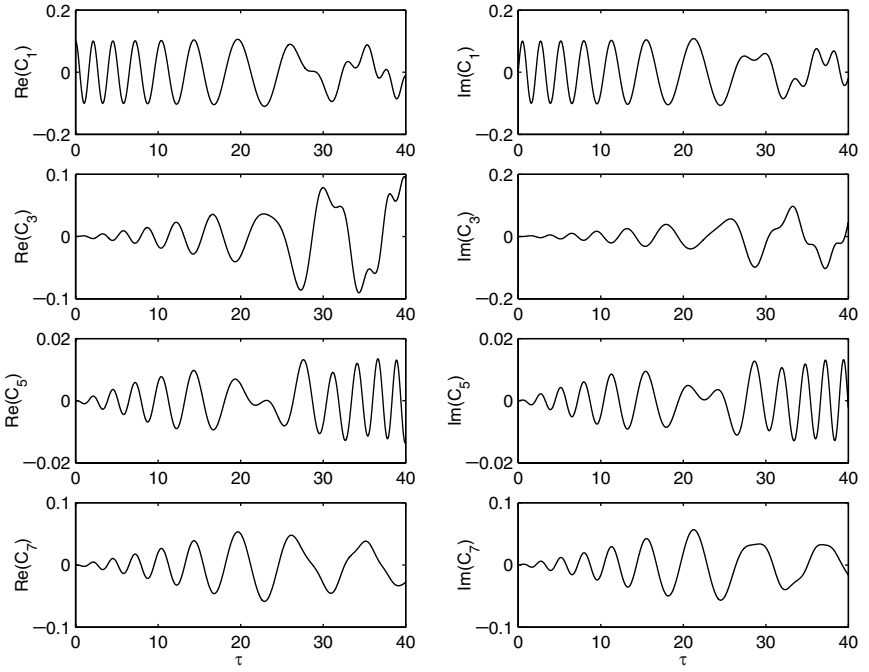


Figure 9. Real and imaginary parts of $c_j(t)$ $(j = 1, 3, 5, 7)$ in the K-system, versus $\tau = k_x bt$. Initial data: $c_1(0) = 0.1$; $c_j(0) = 0$ for $j = 3, 5, 7$. Parameters: $(a/b)^2 = 0.99$; $\mathbf{k} = k_x(1, 2, 0)^{\mathrm{T}}$; $\tilde{A} = 0.1$.

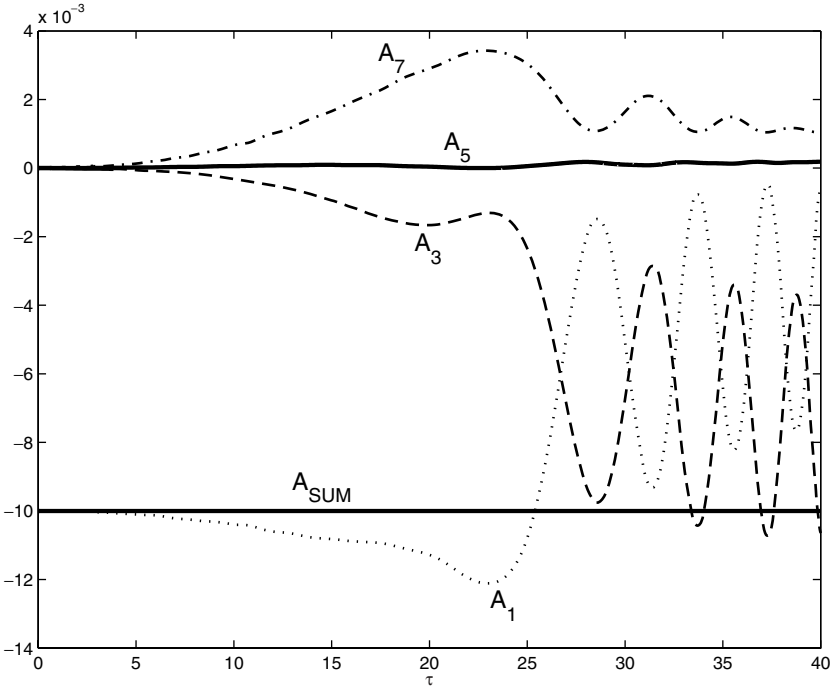


Figure 10. The wave action densities A_1,A_3, A_5 and A_7 in the K-system versus $\tau = k_x bt$ and their sum A_{sum} for the same parameters as in Fig. 9.

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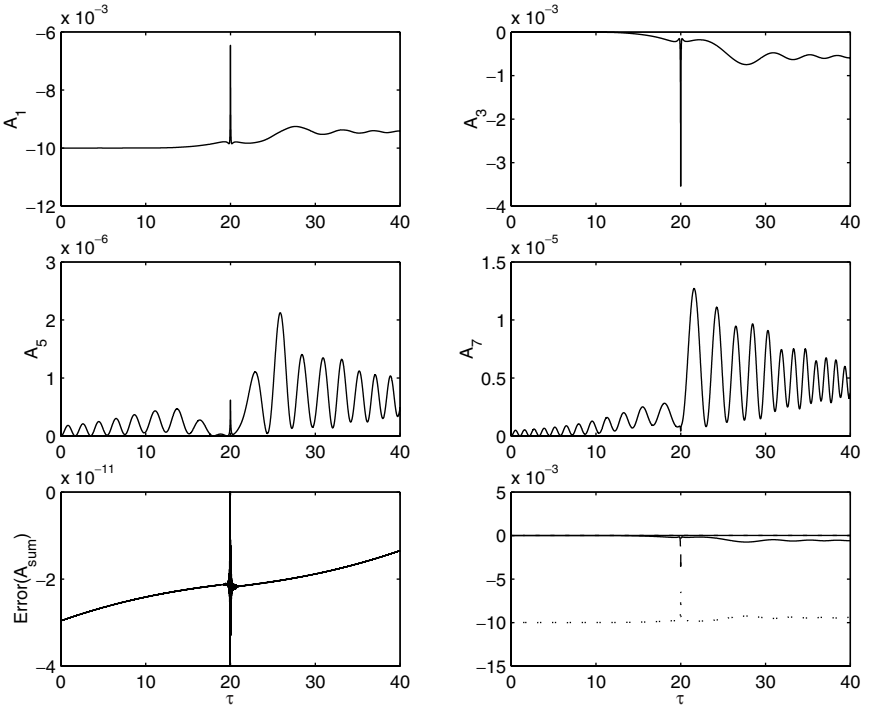


Figure 11. Wave action densities A'_j $(j = 1, 3, 5, 7)$ in the R-system versus $\tau = k_x bt$ and the numerical error in their sum $A_{\rm sum}$ for the same parameter values as in Fig. 9.

between the backward fast mode (A'_1) and backward slow mode (A'_3) (the action densities $A'_5 \sim O(10^{-6})$ and $A'_7 \sim O(10^{-5})$ are at least two orders smaller than A'_1 and A'_3). At $\tau = 20$ ($k'_y = 0$) there is a very strong interaction between A'_1 and A'_3, with a dramatic spike-like decrease in A'_3 (i.e. $|A'_3|$ increases) coupled with a corresponding spike-like increase in A'_1 (i.e. $|A'_1|$ decreases). At later times $\tau > 20$, A'_1 and A'_3 resume a lower level of interaction. Both A'_1 and A'_3 exhibit oscillatory profiles that approach their constant mean values as $\tau \to \infty$. The total wave action $A'_{\rm sum}$ remains constant throughout the evolution. The bottom left panel shows that $A_{\rm sum}$ is essentially constant to $O(10^{-11})$. The bottom right panel shows the evolution of all the action densities $\{A'_j : j = 1, 3, 5, 7\}$, but only A'_1 (dotted curve) and A'_3 (dashed curve) are discernible. A strong localized interaction between A'_1 and A'_3 is clearly visible at $\tau = 20$.

Figures 12 and 13 show solutions of the wave interaction equations (4.15) for a case where all the wave modes (the backward and forward, magnetoacoustic modes and the Alfvén modes) are generated. Figure 12 shows the wave amplitudes $\{c'_j(t) : j = 1, 2, 3, 5, 6, 7\}$ in the R-system for a case where $\mathbf{k} = k_x(1, 2, 2)$ and $\tilde{A} = A/(k_x b) = 0.1$. The plasma beta is such that $(a/b)^2 = 0.99$. Initially $c'_1(0) = 0.1$ and $c'_j = 0$ for $j = 2, 3, 5, 6, 7$ (i.e. a backward fast mode is initially present). Due to the small shear parameter, the excited wave amplitudes are approximately two orders of magnitude smaller than that of the initial wave amplitude. The wave action densities in the R-system are displayed in Fig. 13. Note that the oscillations in the generated magnetoacoustic waves A'_3, A'_5, A'_7 first decrease in amplitude for

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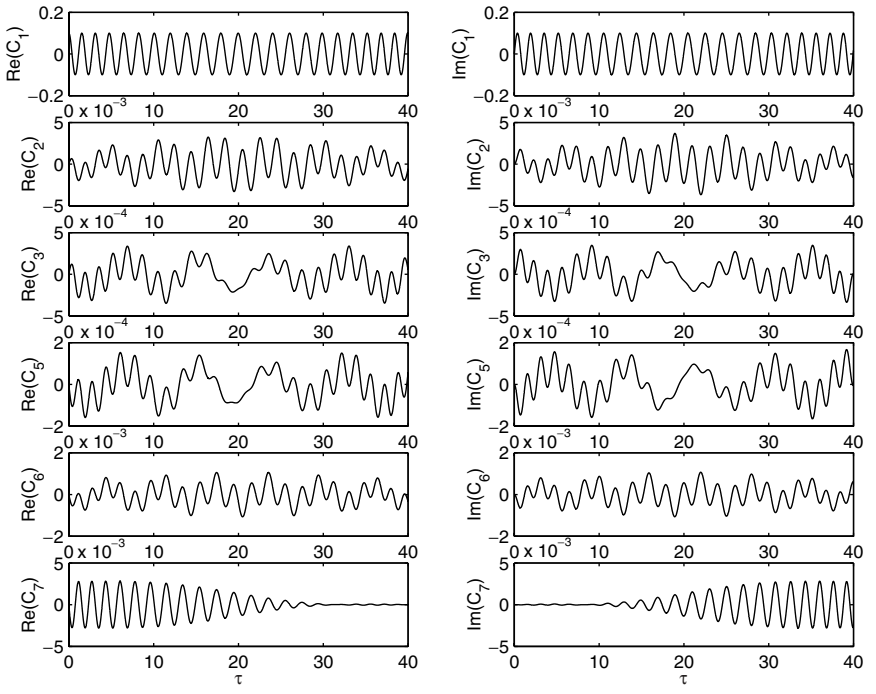


Figure 12. Real and imaginary parts of $c_j'(t)$ $(j = 1, 2, 3, 5, 6, 7)$ in the R-system versus $\tau = k_x bt$ for the initial data $c_1'(0) = 0.1$ and $c_j' = 0$ otherwise. Parameters: $\mathbf{k} = k_x(1, 2, 2)^{\mathrm{T}}$; $\tilde{A} = A/(k_x b) = 0.1$; $(a/b)^2 = 0.99$.

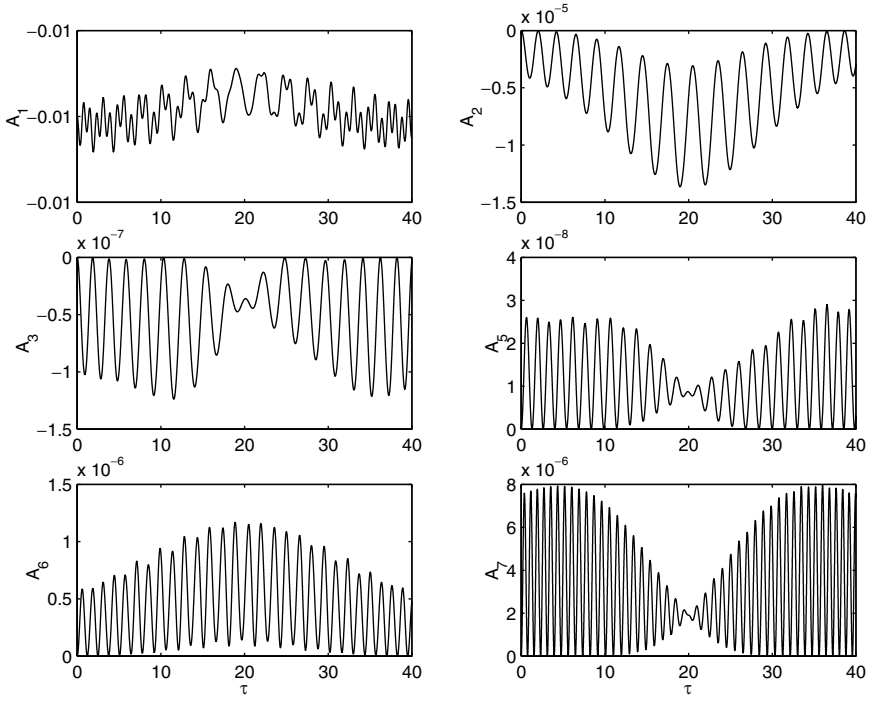


Figure 13. Wave action densities A_j' $(j = 1, 2, 3, 5, 6, 7)$ in the R-system versus $\tau = k_x bt$ for the same parameter values as in Fig. 12.

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$0 < t < t_c$ where $t_c = k_y/(Ak_x)$, and increase in amplitude for $t > t_c$. In contrast, the Alfvén modes A_2 and A_6 have maximum amplitudes at $t = t_c$. This concludes our examples of wave interactions in MHD shear flows.

8. Summary and concluding remarks

In this paper, we have investigated linear, MHD wave interactions in a linear shear flow, using the Lagrangian fluid displacement ξ and the Lagrangian entropy perturbation ΔS to describe the waves, in which a SFH solution is obtained in the frame of the background shear flow using Kelvin's method (Kelvin 1887). This problem has been studied previously by a number of authors, based on Eulerian perturbation variables (e.g., Chagelishvili et al. 1997; Poedts et al. 1998; Kaghashvili 1999, 2002; Bodo et al. 2001; Rogava et al. 2003; Gogoberidze et al. 2004). In particular, Gogoberidze et al. (2004) noticed that the Eulerian perturbation equations admitted a wave action integral.

Our analysis shows that the SFH equations admit two different variational formulations that can be reduced to Hamiltonian form. The wave action integral arises as a consequence of Noether's first theorem. The SFHs for the Lagrangian displacement in the fluid frame satisfy three coupled oscillator equations, with time-dependent coefficients, which depend on the shear of the background flow, the $\mathbf{k}$ vector of the waves, the gas sound speed a and the Alfvén speed b characteristic of the background medium. The normal mode analysis of the coupled oscillator system (3.8) can be carried out in two different ways. The coupled oscillator system was first reduced to an equivalent first-order differential equation system for $\mathbf{w} = (\xi^x, \xi^y, \xi^z, \dot{\xi}^x, \dot{\xi}^y, \dot{\xi}^z)$, where $\dot{\xi} = d\xi/dt$ is the first-order time derivative of ξ, in which the wave interactions are described by a time-dependent matrix, $\mathbf{C}$, and in which the entropy wave perturbation appears as a source term.

In the first solution method (Sec. 3), the matrix $\mathbf{C}$ is split up into a time-independent matrix $\bar{\mathbf{C}}$ describing the background state, but not including the effects of the shear, plus a time-dependent matrix $\delta\mathbf{C}$ describing the effects of the shear (i.e. we write $\mathbf{C} = \bar{\mathbf{C}} + \delta\mathbf{C}$). The solution is expanded in terms of the eigenvectors, or normal modes of the matrix $\bar{\mathbf{C}}$, which correspond to the usual MHD eigenmodes (i.e. the backward and forward Alfvén waves, and the backward and forward fast and slow magnetoacoustic waves). Compatibility conditions on the solution ansatz yields a set of six coupled, first-order differential equations for the wave amplitudes $\{c_j(t)\}$. The system of equations is denoted as the K-system since the $\mathbf{k}$ vector of the waves and the eigenfrequencies ω_j used in the analysis are constant. By using wave amplitudes $\{c_j(t)\}$ related to the wave action, A_j (i.e. $A_j = E_{wj}/\omega_j$ where E_{wj} is the physical energy density of the wave, $A_j = \sigma_j |c_j|^2$ where $\sigma_j = \mathrm{sgn}(\omega_j)$, $\omega_j = k\lambda_j$ and the λ_j are the usual MHD eigenvelocities), results in variational and Hamiltonian formulations of the equations. The wave action integral arises as a consequence of Noether's theorem and only applies if there are zero entropy perturbations, ΔS.

In the second solution method (Sec. 4), the solution for $\mathbf{w}$ is expanded in terms of the eigensystem of the matrix $\mathbf{C}$ including the effects of the shear. The eigenvalues $\omega'_j = k'\lambda'_j$ of $\mathbf{C}$ correspond to the usual MHD waves (the backward and forward fast and slow mode waves, and the Alfvén waves), in which the $\mathbf{k}$ vector of the waves is replaced by the wavevector $\mathbf{k}' = (k_x, k_y - Atk_x, k_z)$ where A describes the shear of the background fluid velocity (i.e. $u_x = u_0 + Ay$). The wavevector $\mathbf{k}'$ gives the variation of the wavevector of each of the MHD eigenmodes along the rays of

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geometrical MHD optics (Appendix B). Six first-order differential equations for the wave amplitudes $\{c'_j(t)\}$ are obtained as compatibility conditions on the solution ansatz. Since $\mathbf{k}'$ evolves according to the ray equations, the differential equation system is called the R-system. The natural Hamiltonian variables $\{c'_j(t)\}$ used to describe the waves are related to the wave action densities $\{A'_j\}$ (i.e. $A'_j = \sigma_j |c'_j|^2 = E'_{wj}/\omega'_j$, $\sigma_j = \mathrm{sgn}(\omega'_j)$). Variational and Hamiltonian forms of the R-system are obtained.

Several special limits of the wave interaction equations are studied in detail (Sec. 3.3). For the case of wave propagation along the magnetic field, two of the MHD wave speeds coincide. Despite the degeneracy in two of the MHD phase speeds (eigenvalues), the eigenvectors are distinct and form a complete set, although special care is needed near the triple point for the case $a = b$ where three of the MHD wave speeds coincide (Brio and Wu 1988; Roe and Balsara 1996). The explicit form of the wave interaction coefficients are obtained in the limit as $\theta \to 0$, where $\cos\theta = \mathbf{n}\cdot\mathbf{B}/B$ and $\mathbf{n} = \mathbf{k}/k$ is the wave normal. For wave propagation perpendicular to $\mathbf{B}$ ($k_x = 0$), all of the wave interaction coefficients Λ_{js} are zero. Thus, no wave transformation processes can occur for wave propagation in this case.

A further irregular case is the cold gas limit in which the gas sound speed $a = 0$. Roe and Balsara (1996) point out that the slow mode eigenvectors in the limit $a \to 0$ collapse onto a single eigenvector, implying that the linear MHD initial value problem is ill-posed as $a \to 0$ (i.e. the solution does not depend continuously on the initial data). Falle and Hartquist (2002), who restrict themselves to the case where there are no entropy waves, show that a complete set of right eigenvectors for the slow mode eigenspace can be constructed by using the eigenvectors of the matrices $\mathbf{A}$ and $\mathbf{A}^2$ corresponding to the slow mode eigenspeed $\lambda = 0$, where $\mathbf{A}$ is the basic matrix describing Eulerian one-dimensional MHD. In this case, the basic matrix $\mathbf{A}$ is similar to a Jordan canonical matrix, when the eigenvectors are used as base vectors. The net upshot of their analysis for $a = 0$ is that the initial value problem for linearized MHD in one Cartesian space dimension admits a secular Jordan slow mode that grows linearly with time. Our analysis of the SFH equations for MHD waves in a linear shear flow for $a = 0$ also shows there is a secular Jordan slow mode, but the slow modes are decoupled from the fast and slow mode equations.

The Hamiltonian wave amplitudes $\{c_j(t)\}$ in the K-system and the corresponding Hamiltonian variables $\{c'_j(t)\}$ in the R-system are related by matrix equations of the form $c'_j = P_{js}c_s$ and $c_j = Q_{js}c'_s$ where $P_{js} = \mathbf{L}'_j \cdot \mathbf{R}_s$ and $Q_{js} = \mathbf{L}_j \cdot \mathbf{R}'_s$ are inverse matrices. Here $\{\mathbf{L}'_j\}$ and $\{\mathbf{R}'_s\}$ are the left and right eigenvectors of the matrix $\mathbf{C}$ used to describe the R-system, and $\{\mathbf{L}_j\}$ and $\{\mathbf{R}_s\}$ are the left and right eigenvectors of the matrix $\bar{\mathbf{C}}$ used in the K-system. These relations can be used to prove that the wave action in both the R- and K-systems are exactly the same integral (Sec. 5). The solutions of both systems can be described by S-matrix theory. The transformations also imply a direct connection between the Lagrangian and Hamiltonian densities in the two systems.

Gogoberidze et al. (2004) analyzed the wave interactions in the R-system, using normal mode, second-order coupled oscillator equations, and determined transmission, reflection and wave generation due to mode coupling by using the wave action conservation equation and S-matrix ideas. The detailed connection between their normal modes variables $\{\psi_j : j = 1, 2, 3\}$ and the Hamiltonian normal variables $\{c'_j(t)\}$ for the R-system is described in Sec. 6 and Appendix E.

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One can also obtain coupled oscillator equations in terms of normal variables in the K-system, which are analogous to the variables $\{\psi_j\}$ used by Gogoberidze et al. (2004) (Sec. 6). In the cold gas limit, and for the case $k_z = 0$, this latter equation system reduces to a single second-order differential equation for the normal variable $u = c_1 + c_7$ describing the backward and forward fast mode waves. This latter differential equation has the form of a harmonic oscillator equation with a time-dependent frequency $\Omega(t) = k'b$ where $k' = [k_x^2 + (k_y - Atk_x)^2]^{1/2}$ is the wavenumber in the R-system. The equation admits exact solutions in terms of confluent hypergeometric functions. Time-dependent oscillator equations of this form are known as Ermakov systems (Ermakov 1880; Lutzsky 1978; Leach 1991), and admit a first integral known as the Lewis invariant, which is equivalent to the wave action integral. If one uses the complex amplitude and phase form for u, $u = r\exp(i\theta) \equiv c_1 + c_7$, the system splits up into radial (r) and polar (θ) components, reminiscent of particle motion in a central force field. The θ equation of motion has the integral $r^2\dot{\theta} = h = kb(|c_7|^2 - |c_1|^2) =$ constant, which is analogous to the angular momentum conservation in a central force field. In other words, the wave action integral in this formulation is analogous to the angular momentum conservation equation in the central force field problem. These ideas can be extended to more general cases. The radial force equation is that expected for particle motion in a central force field $-\Omega^2(t)\mathbf{r}$.

The examples of numerical solutions of wave interactions in shear flows (Sec. 7.3) show that the R-system is more useful than the K-system for physical interpretation, but the mathematical form of the K-system has simpler wave interaction coefficients. For example, the scattering matrix analysis of Appendix F of the interaction of the backward and forward fast magnetoacoustic waves in a cold gas shows that the wave action densities A_1 and A_7 in the K-system grow linearly with t at late times ($A_1, A_7 \propto k'/k$), but are bounded in the R-system. Both systems conserve the total wave action. Furthermore, the wave action densities in the K-system show fast oscillations at late times, whereas the action densities in the R-system are essentially constant at late times. Thus, the action densities in the R-system behave more like wave envelopes and are more useful for calculating reflection, transmission and mode conversion coefficients. The numerical examples were used to illustrate the phenomenon of wave amplification of magnetoacoustic waves (Figs 6 and 8), sharp resonant spikes in the wave action profiles, between the backward fast and slow magnetoacoustic modes in $\beta \approx 1$ plasmas (Fig. 11) and examples of interaction between all the MHD eigenmodes, for cases where the initial $\mathbf{k}$ vector has no zero components (Fig. 13). WKB-type methods to calculate the wave reflection, transmission and mode conversion coefficients for the case of small shear and other special cases have been developed by Gogoberidze et al. (2004). Mode conversion is effective when two dispersion equation branches $\omega_j'(\mathbf{k}')$ and $\omega_p'(\mathbf{k}')$ evolve in such a way as to almost overlap at some time $t = t_{jp}$ (e.g., Gogoberidze et al. 2004; Tracy and Kaufman 1993; Swanson 1998). Strong wave interaction also occurs at the time when $k_y' = 0$, i.e. at the time $t = t_c = k_y/(Ak_x)$.

Using the observations of McComas et al. (1998), we estimate the shear parameter A for the transition from slow to fast solar wind near 1 AU as $A = \Delta u_{sw}/L = 4 \times 10^{-6}$ s^{-1} corresponding to a change of $\Delta u_{sw} = 400$ km s^{-1} of the flow speed over a scale of $L \sim 10^8$ km. The choice of $A/(k_x b) = 0.1$ assuming an Alfvén velocity of $b \sim 50$ km s^{-1} gives $k_x \sim 8\times10^{-7}$ km^{-1} or wavelength $L_x \sim 8\times10^6$ km. The time for significant wave interaction occurs when $k_y' = k_y - At_c k_x = 0$ (e.g., Chagelishvili

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et al. 1997). Thus, $k_x b t_c = 20$, or $t_c \sim 5 \times 10^5$ s. This time scale is comparable to the flow time $t_{sw} = L/\Delta u_{sw} = 2.5 \times 10^5$ s. Thus, $t_c \sim t_{sw}$, indicating that the wave transformation time scale is of the same order as the flow time scale. We conclude, that significant wave transformation can occur on the dynamical time scale t_{sw} for the wind.

It is unclear how the present analysis relates to MHD wave transmission, reflection and mode conversion at a shear layer or tangential discontinuity (e.g., McKenzie et al. 1970; Walker 2000). Wave amplification at a tangential discontinuity requires the canonical wave energy to change sign as one crosses the boundary. For this to occur the difference of the flow speed across the discontinuity must be sufficiently 'supersonic' (McKenzie 1970, Appendix III).

The present paper has been principally concerned with providing a Hamiltonian and variational formulation of MHD wave interactions in shear flows, using Kelvin's method. It is clear that reflection, transmission and mode conversion processes are important in models of turbulent, shearing flows, such as that observed between fast and slow solar wind (e.g., McComas et al. 1998). Shear driving of solar wind turbulence (Zank et al. 1996; Matthaeus et al. 2004; Breech et al. 2005; Milano et al. 2004) has been invoked as an important element in the evolution of solar wind turbulence and also for models of the cosmic-ray diffusion tensor in the solar wind. The significance of the above idealized, MHD wave interaction processes for solar wind turbulence remains as an open problem.

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Appendix A

In this appendix we list the relationships between the Eulerian and Lagrangian perturbations of the physical quantities ρ, $\mathbf{u}$, $\mathbf{B}$, p and S and the Lagrangian displacement ξ and ΔS (see, e.g., Lundgren 1963; Webb et al. 2005b). The Lagrangian perturbation $\Delta\psi$ and the corresponding Eulerian perturbation $\delta\psi$ of a physical quantity ψ are related by the equation

$$\delta\psi = \Delta\psi - \xi \cdot \nabla\psi. \tag{A 1}$$

The Lagrangian perturbations of ρ, $\mathbf{u}$, $\mathbf{B}$, p and S are given by

$$\Delta\rho = -\rho\nabla\cdot\xi, \quad \Delta\mathbf{u} = \dot{\xi},$$

$$\Delta\mathbf{B} = \mathbf{B}\cdot\nabla\xi - \mathbf{B}\nabla\cdot\xi, \quad \Delta p = p_S\Delta S - \gamma p\nabla\cdot\xi, \tag{A 2}$$

where $\dot{\xi} = \xi_t + \mathbf{u}\cdot\nabla\xi$ is the Lagrangian time derivative of ξ. We assume an adiabatic gas with adiabatic index γ and $p_S = p/C_v$ where C_v is the specific heat of the gas at constant entropy. The Eulerian perturbations corresponding to (A 2) are

$$\delta\rho = -\nabla\cdot(\rho\xi), \quad \delta\mathbf{u} = \dot{\xi} - \xi\cdot\nabla\mathbf{u},$$

$$\delta\mathbf{B} = \nabla\times(\xi\times\mathbf{B}), \quad \delta p = p_S\Delta S - a^2\rho\nabla\cdot\xi - \xi\cdot\nabla p, \tag{A 3}$$

where $a = (\gamma p/\rho)^{1/2}$ is the adiabatic sound speed of the gas.

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Appendix B

The characteristic manifolds of the MHD wave equations (2.1) and (2.2) are manifolds on which the initial value problem for the waves (the Cauchy problem) does not have a unique solution (e.g., Webb et al. 2005b). The characteristic manifolds $\phi(\mathbf{x},t) =$ constant are related to the local dispersion relations for the MHD wave modes (the fast and slow magnetoacoustic waves, the Alfvén waves and the entropy wave) by the equations

$$\omega = -\phi_t, \quad \mathbf{k} = \nabla\phi, \quad \omega = \Omega(\mathbf{k},\mathbf{x},t), \tag{B 1}$$

where $\omega = -\phi_t$ and $\mathbf{k} = \nabla\phi$ are the local frequency and wavenumber, and $\omega = \Omega(\mathbf{k},\mathbf{x},t)$ is the dispersion equation for the mode of interest. From (B 1) the dispersion equation for a given wave mode can be written as a first-order partial differential equation for the wave phase ϕ in the form

$$F \equiv \phi_t + \Omega(\nabla\phi,\mathbf{x},t) = 0. \tag{B 2}$$

Equation (B 2) is the Hamilton–Jacobi equation for the wave eikonal function $\phi(\mathbf{x},t)$ with Hamiltonian Ω.

The characteristics or ray equations for (B 2) may be written in the form (Webb et al. 2005b)

$$\frac{dx^i}{d\tau} = \frac{\partial\Omega}{\partial k^i}, \quad \frac{dk^i}{d\tau} = -\frac{\partial\Omega}{\partial x^i}, \quad 1 \le i \le 3, \tag{B 3}$$

$$\frac{dt}{d\tau} = 1, \quad \frac{d\phi}{d\tau} = 0, \quad \frac{d\omega}{d\tau} = \frac{\partial\Omega}{\partial t}. \tag{B 4}$$

The ray equations (B 3) are Hamilton's equations with Hamiltonian $H = \Omega(\mathbf{k},\mathbf{x},t)$.

For the entropy wave,

$$\omega = \mathbf{k}\cdot\mathbf{u}, \quad F = \phi_t + \mathbf{u}\cdot\nabla\phi = 0. \tag{B 5}$$

For the shear flow (2.7) the ray equations for (B 5) have integrals

$$x - (u_0 + Ay)t = x_0, \quad y = y_0, \quad z = z_0, \quad t = \tau, \tag{B 6}$$

$$k_x = k_{x0}, \quad k_y = k_{y0} - k_x A t, \quad k_z = k_{z0}, \quad \omega = \omega_0, \tag{B 7}$$

where the subscript 0 quantities are constant on the characteristics. Note in particular that $\bar{k}_y = k_y - Atk_x$ gives the variation of $\bar{k}_y$ along the characteristics, where k_y and k_x now refer to the constant values of the components of $\mathbf{k}$ at time $t = 0$ and $\bar{k}_y$ is the value of k_y at time $t = \tau$. Note, in particular, that this formula for $\bar{k}_y$ is precisely that given in the Fourier solution for $\tilde{\xi}$ in (3.1)–(3.3). The vector $\tilde{\mathbf{k}} = (k_x, \bar{k}_y, k_z)^{\mathrm{T}}$ describes the variation of $\mathbf{k}$ along the rays for the other MHD wave modes as well (see below). Using (B 6) and (B 7) the characteristic manifolds for the entropy wave have the general solution

$$\phi = \phi_E(x - Ayt, y, z, t), \tag{B 8}$$

where ϕ_E is an arbitrary (differentiable) function of its arguments.

Similarly, for the Alfvén waves

$$\omega = \mathbf{k}\cdot(\mathbf{u} \pm \mathbf{b}), \quad F = \phi_t + (\mathbf{u} \pm \mathbf{b})\cdot\nabla\phi = 0. \tag{B 9}$$

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The ray equations (B 3) and (B 4) have integrals

$$x-(u_0+Ay\pm b)\tau=x_0,\quad y=y_0,\quad z=z_0, \tag{B 10}$$

and the ray equations for ω and $\mathbf{k}$ have the integrals (B 7).

For the magnetoacoustic waves,

$$\omega=\mathbf{k}\cdot\mathbf{u}\pm c,\quad F=\phi_t+\mathbf{u}\cdot\nabla\phi\pm c|\nabla\phi|=0, \tag{B 11}$$

where c denotes either the fast or slow magnetoacoustic speeds ($c=c_f$ or $c=c_s$), which satisfy the magnetoacoustic dispersion equation

$$c^4-(a^2+b^2)c^2+a^2b_n^2=0, \tag{B 12}$$

where a is the gas sound speed and $b_n=\mathbf{b}\cdot\mathbf{n}=bn^x$ is the Alfvén phase speed with $\mathbf{b}$ the Alfvén velocity. The ray equations reduce to

$$\frac{dx}{d\tau}=u^x(y)+V'_{gx},\quad \frac{dy}{d\tau}=V'_{gy},\quad \frac{dz}{d\tau}=V'_{gz},\quad \frac{dt}{d\tau}=1, \tag{B 13}$$

$$\frac{dk_x}{d\tau}=0,\quad \frac{dk_y}{d\tau}=-k_xA,\quad \frac{dk_z}{d\tau}=0,\quad \frac{d\omega}{d\tau}=0, \tag{B 14}$$

where

$$V'_{gx}=n^x\left(c-\frac{a^2b^2[1-(n^x)^2]}{c[2c^2-(a^2+b^2)]}\right),$$

$$V'_{gy}=n^y\left(c+\frac{a^2b^2(n^x)^2}{c[2c^2-(a^2+b^2)]}\right),$$

$$V'_{gz}=n^z\left(c+\frac{a^2b^2(n^x)^2}{c[2c^2-(a^2+b^2)]}\right) \tag{B 15}$$

are the components of the group velocity in the fluid frame and $\mathbf{n}=\mathbf{k}/k$ is the wave normal. Note that the ray equations (B 14) have integrals (B 7). The ray equations (B 13) have formal integrals

$$y-\int_0^\tau V'_{gy}(\tau)\,d\tau=y_0,\quad z-\int_0^\tau V'_{gz}(\tau)\,d\tau=z_0,$$

$$x-\left(u_0\tau+A\int_0^\tau y(\tau)\,d\tau+\int_0^\tau V'_{gx}(\tau)\,d\tau\right)=x_0. \tag{B 16}$$

Note that the ray equations (B 14) for $\mathbf{k}$ and ω have integrals (B 7).

The most important point to arise from the above analysis is that the $\mathbf{k}$ vector $\tilde{\mathbf{k}}$ in (3.3) gives the variation of $\mathbf{k}$ along each of the ray trajectories for all the MHD eigenmodes.

Appendix C

In this appendix we derive the results (3.37) and (3.38) for the wave energy density $E_w^{(j)}$ and action A_j for the MHD eigenmodes, based on the constant background state in which there is no shear. For a constant background state, the Lagrangian and Eulerian perturbations of the generic physical quantity ψ in (A 1) are the same since $\nabla\psi=0$ for a constant background state. In general, the right eigenvectors

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(3.22) give the solution for

$$\mathbf{w} = [\xi, \dot{\xi}]^{\mathrm{T}} = \sum_j a_j \mathbf{R}_j \tag{C 1}$$

for the Lagrangian displacement perturbations in Fourier space.

Consider the physical perturbations for the Alfvén wave case ($j = 2$ or 6). Using (C 1) and the Alfvén wave eigenvectors (3.22) and the results of Appendix A, we obtain

$$\Delta\rho = -\rho_0 k\mathbf{n}\cdot\xi = 0, \quad \Delta p = p\Delta S - \gamma p k\mathbf{n}\cdot\xi = 0,$$

$$\Delta\mathbf{u}_j = -i\omega_j a_j N_r^{(j)}(0, n^z, -n^y), \quad \Delta\mathbf{B}_j = ikB_0 a_j N_r^{(j)} n^x (0, n^z, -n^y), \tag{C 2}$$

for the Lagrangian density, pressure, fluid velocity and magnetic field Fourier space perturbations. From (C 2) we find that the linear Alfvén wave eigenmodes are incompressible, have zero entropy and pressure perturbations, and in general have non-zero field aligned current and vorticity. Using (C 2) with $N_r^{(j)} = 1$ we obtain

$$E_w^{(j)} = \frac{1}{2}\rho|\Delta\mathbf{u}_j|^2 + \frac{|\Delta\mathbf{B}_j|^2}{2\mu}, \quad j = 2, 6, \tag{C 3}$$

for the energy density of the Alfvén eigenmodes. Using the physical perturbations (C 2) we obtain

$$E_{\mathrm{KE}} = \frac{1}{2}\rho|\Delta\mathbf{u}_j|^2 = \frac{1}{2}\rho|a_j|^2\omega_j^2\Delta_j, \quad E_B = \frac{|\Delta\mathbf{B}_j|^2}{2\mu} = \frac{1}{2}\rho|a_j|^2\omega_j^2\Delta_j, \quad j = 2, 6, \tag{C 4}$$

for the kinetic (E_{KE}) and magnetic (E_B) energy densities for the Alfvén waves where $\Delta_j = 1 - (n^x)^2$ for $j = 2, 6$. Thus, the total energy density for the Alfvén modes is

$$E_w^{(j)} = \rho|a_j|^2\omega_j^2\Delta_j \quad \text{where } \Delta_j = 1 - (n^x)^2, \ j = 2, 6. \tag{C 5}$$

Equations (C 4) show equipartition of the kinetic and magnetic energy densities for the Alfvén modes.

For the magnetoacoustic modes ($j = 1, 3, 5, 7$), we obtain

$$\Delta\rho_j = -ik\rho_0 a_j N_r^{(j)}(\lambda_j^2 - b_n^2), \quad \Delta p = a^2\Delta\rho_j,$$

$$\Delta\mathbf{u}_j = -i\omega_j a_j N_r^{(j)}[(\lambda_j^2 - b^2)n^x, \lambda_j^2 n^y, \lambda_j^2 n^z],$$

$$\Delta\mathbf{B}_j = ikB_0 N_r^{(j)}\lambda_j^2 a_j[-[1 - (n^x)^2], n^x n^y, n^x n^z] \tag{C 6}$$

for the Lagrangian density, pressure, fluid velocity and magnetic field perturbations in Fourier space. The energy density for the magnetoacoustic waves has the form

$$E_w^{(j)} = \frac{1}{2}\rho|\Delta\mathbf{u}_j|^2 + \frac{p_{\rho\rho}|\Delta\rho_j|^2}{2(\gamma - 1)} + \frac{|\Delta\mathbf{B}_j|^2}{2\mu} \equiv E_{\mathrm{KE}} + E_{\mathrm{int}} + E_B \tag{C 7}$$

corresponding to the kinetic energy (E_{KE}), internal energy density (E_{int}) and magnetic energy density (E_B) contributions, respectively. Using the perturbations (C 6) with $N_r^{(j)} = 1$ we obtain

$$E_{\mathrm{KE}} = \tfrac{1}{2}\rho|a_j|^2\omega_j^2\Delta_j, \quad E_{\mathrm{int}} = \tfrac{1}{2}\rho a^2 k^2|a_j|^2(\lambda_j^2 - b_n^2)^2,$$

$$E_B = \tfrac{1}{2}\rho k^2 b^2|a_j|^2\lambda_j^4[1 - (n^x)^2], \tag{C 8}$$

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where Δ_j is given by (3.25). Since the $\{\lambda_j\}$ satisfy the magnetoacoustic dispersion equation (3.21), we obtain the useful identities

$$(\lambda_j^2 - b_n^2)(\lambda_j^2 - a^2) = \lambda_j^2 b_\perp^2,$$

$$\Delta_j = (\lambda_j^2 - b_n^2)^2 + b_n^2 b_\perp^2 = \frac{(\lambda_j^2 - b_n^2)(\lambda_j^4 - a^2 b_n^2)}{\lambda_j^2}, \tag{C 9}$$

where $b_\perp^2 = b^2 - b_n^2$. From (C 8) and (C 9) we find

$$E_{int} + E_B = E_{\rm KE} = \tfrac{1}{2}\rho |a_j|^2 \omega_j^2 \Delta_j. \tag{C 10}$$

Thus, for the magnetoacoustic modes, there is equipartition between the kinetic energy density and the total internal energy density due to the magnetic field and the gas thermal energy density. From (C 7) and (C 10)

$$E_w^{(j)} = \rho |a_j|^2 \omega_j^2 \Delta_j \tag{C 11}$$

gives the total energy density for the magnetoacoustic modes.

The wave action density $A_j = E_w^{(j)}/\omega_j$ by definition.

Appendix D

In this appendix we prove the result (4.12) for the wave mixing coefficient $\bar{\Lambda}'_{jj}$. We also discuss other formulae for the Λ'_{js}.

To derive (4.12) first note from (3.22)–(3.25) that, for the normalization $N_r^{(j)} = 1$,

$$\mathbf{R}_j = \mathbf{U}_j - i\omega_j \mathbf{V}_j, \quad \mathbf{L}_j = \frac{i}{2\omega_j \Delta_j}(-i\omega_j \mathbf{U}_j^{\rm T} + \mathbf{V}_j^{\rm T}), \tag{D 1}$$

where

$$V_j^k = 0, \quad V_j^{k+3} = U_j^k, \quad k = 1, 2, 3, \tag{D 2}$$

where k_y is replaced by $\bar{k}_y = k_y - Atk_x$. Using (D 1) and (D 2) in (4.5) we obtain

$$\begin{aligned}\bar{\Lambda}'_{js} &= \frac{i}{2\omega_j \Delta_j}(-i\omega_j \mathbf{U}_j^{\rm T} + \mathbf{V}_j^{\rm T})\frac{d}{dt}(\mathbf{U}_s - i\omega_s \mathbf{V}_s) \\ &= \frac{1}{2\omega_j \Delta_j}\left(\omega_j \mathbf{U}_j^{\rm T}\frac{d\mathbf{U}_s}{dt} + \mathbf{U}_j^{\rm T}\frac{d}{dt}(\omega_s \mathbf{U}_s)\right).\end{aligned} \tag{D 3}$$

Setting $s = j$ in (D 3) we obtain

$$\bar{\Lambda}'_{jj} = \frac{1}{2\omega_j \Delta_j}\frac{d}{dt}(\omega_j \mathbf{U}_j^{\rm T}\mathbf{U}_j). \tag{D 4}$$

Since $\mathbf{L}_j \cdot \mathbf{R}_s = \delta_{js}$ and $\mathbf{L}_j \cdot \mathbf{R}_j = 1$, this implies

$$\mathbf{U}_j^{\rm T}\mathbf{U}_j = \Delta_j. \tag{D 5}$$

Thus, from (D 4) and (D 5) we obtain

$$\bar{\Lambda}'_{jj} = \frac{1}{2}\frac{d}{dt}\ln(|\omega_j|\Delta_j), \tag{D 6}$$

which is the result (4.12).

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Formulae (4.6) and (4.7) for $\bar{\Lambda}'_{js}$ and $d\omega_s/dt$ follow from differentiating the eigenvalue equation

$$(\mathbf{C} - i\omega_s \mathbf{l}_6) \cdot \mathbf{R}_s = 0 \tag{D 7}$$

with respect to t, and then taking the scalar product of the resultant equation with $\mathbf{L}'_j$ to obtain

$$(\omega'_j - \omega'_s)\bar{\Lambda}'_{js} - i\mathbf{L}'_j \cdot \frac{d}{dt}(\mathbf{C} - i\omega'_s \mathbf{l}_6) \cdot \mathbf{R}'_s = 0. \tag{D 8}$$

Equation (D 8) gives (4.6) if $j \neq s$ and $\omega'_j \neq \omega'_s$ whereas if $j = s$ we obtain (4.7) for $d\omega'_s/dt$. Equation (D 8) can also be written in the form

$$\bar{\Lambda}'_{js} = \frac{i}{\omega'_j - \omega'_s} \mathbf{L}'_j \cdot \left[\frac{d}{dt}(\mathbf{C} - i\omega'_s \mathbf{l}_6) \cdot \mathbf{R}'_s \right], \tag{D 9}$$

which suggests that one can also obtain $\bar{\Lambda}'_{jj}$ by letting $\omega'_s \to \omega'_j$ and using L'Hopital's rule in (D 9).

From (D 9), it follows that the magnetoacoustic–magnetoacoustic wave interaction coefficients are given by

$$\bar{\Lambda}'_{js} = \frac{An^x n^y}{2\lambda_j \Delta_j (\lambda_j - \lambda_s)} (2(a^2 + b^2)\lambda_j^2 \lambda_s^2 - a^2 b_n^2 (\lambda_s^2 + \lambda_j^2)), \tag{D 10}$$

where $j, s = 1, 3, 5, 7$. Similarly, the Alfvén–magnetoacoustic interaction coefficients are

$$\bar{\Lambda}'_{js} = \frac{An^x n^z}{2\lambda_j \Delta_j (\lambda_j - \lambda_s)} ((a^2 + b^2)\lambda_s^2 - b_n^2 (\lambda_s^2 + a^2)) \tag{D 11}$$

($j = 2, 6$; $s = 1, 3, 5, 7$) and the magnetoacoustic–Alfvén wave interaction coefficients are given by

$$\bar{\Lambda}'_{js} = \frac{An^x n^z}{2\lambda_j \Delta_j (\lambda_j - \lambda_s)} ((a^2 + b^2)\lambda_j^2 - b_n^2 (\lambda_j^2 + a^2)) \tag{D 12}$$

($s = 2, 6$; $j = 1, 3, 5, 7$). Also $\bar{\Lambda}'_{26} = \bar{\Lambda}'_{62} = 0$.

Appendix E

In this appendix we show that the normal variables $\{\psi_j : \ j = 1, 2, 3\}$ in (6.9) are essentially the same as the normal variables used by Gogoberidze et al. (2004). The normal variables (6.9) are defined by the equation

$$\psi_j = \mathbf{l}_j \cdot \tilde{\xi}, \tag{E 1}$$

where $\mathbf{l}_j$ and $\mathbf{r}_j$ are the left and right eigenvectors of the matrix $\mathbf{A}$ in (3.4). For the magnetoacoustic modes

$$\mathbf{l}_j = \mathbf{r}_j^{\mathrm{T}} = \frac{1}{\sqrt{\Delta_j}} [(\lambda_j^2 - b_n^2) n^x, \lambda_j^2 n^y, \lambda_j^2 n^z], \tag{E 2}$$

where $\Delta_j = (\lambda_j^2 - b_n^2)^2 + b_n^2 b_\perp^2$ ($j = 1, 3$). For the Alfvén modes,

$$\mathbf{l}_2 = \mathbf{r}_j^{\mathrm{T}} = \frac{1}{\sqrt{\Delta_2}} (0, n^z, -n^y) \quad \text{where } \Delta_2 = 1 - (n^x)^2. \tag{E 3}$$

All variables in (E 1)–(E 3) use the ray wavevector $\mathbf{k}' = (k_x, k_y - Atk_x, k_z)$, but the primed superscripts on the variables have been suppressed. Using (E 1)–(E 3)

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we obtain

$$\psi_1 \equiv \psi_f = \frac{1}{\sqrt{\Delta_1}}(c_f^2\tilde{\xi}\cdot\mathbf{n} - b^2 n^x \tilde{\xi}^x), \tag{E 4}$$

$$\psi_2 \equiv \psi_{\rm A} = \frac{1}{\sqrt{\Delta_2}}(n^z\tilde{\xi}^y - n^y\tilde{\xi}^z), \tag{E 5}$$

$$\psi_3 \equiv \psi_s = \frac{1}{\sqrt{\Delta_3}}(c_s^2\tilde{\xi}\cdot\mathbf{n} - b^2 n^x \tilde{\xi}^x) \tag{E 6}$$

for the normal mode variables ψ_j.

From the spatial Fourier transforms of $\delta\rho$ and $\delta\mathbf{B}$ obtained from (A 1)–(A 3) we find

$$\delta\bar{B}_x = -i(k^y\tilde{\xi}^y + k^z\tilde{\xi}^z), \quad \delta\mathbf{B}^y = ik^x\tilde{\xi}^y, \quad \delta\bar{B}^z = ik^x\tilde{\xi}^z, \quad \delta\bar{\rho} = -i\mathbf{k}\cdot\tilde{\xi}, \tag{E 7}$$

where $\delta\bar{\rho} = \delta\tilde{\rho}/\rho$, $\delta\bar{\mathbf{B}} = \delta\tilde{\mathbf{B}}/B_0$. Using (E 7) we obtain

$$\mathbf{n}\cdot\tilde{\xi} = -\frac{\delta\bar{\rho}}{ik}, \quad n^x\tilde{\xi}^x = -\frac{1}{ik}(\delta\bar{\rho} + K^y\delta\bar{B}^y + K^z\bar{B}^z), \tag{E 8}$$

where $\mathbf{K} = \mathbf{k}/k_x$.

Using (E 8) in (E 4)–(E 6) gives

$$\psi_f = -\frac{i}{k_x}\left\{\frac{\Omega_s^2(K^y\delta\bar{B}^y + K^z\delta\bar{B}^z) + (\Omega_s^2 - \beta)\delta\bar{\rho}}{K[(\Omega_s^2-\beta)^2 + \beta^2K_\perp^2]^{1/2}}\right\}, \tag{E 9}$$

$$\psi_{\rm A} = -\frac{i}{k_x}\left\{\frac{K^z\delta\mathbf{B}^y - K^y\delta\bar{B}^z}{K_\perp}\right\}, \tag{E 10}$$

$$\psi_s = -\frac{i}{k_x}\left\{\frac{\beta K_\perp^2\delta\bar{\rho} + (\beta K^2 - \Omega_s^2)[K^y\delta\bar{B}^y + K^z\delta\bar{B}^z]}{K_\perp[(\Omega_s^2-\beta)^2 + \beta^2K_\perp^2]^{1/2}}\right\}, \tag{E 11}$$

where

$$\Omega_s^2 = \frac{c_s^2}{b_n^2}, \quad \Omega_f^2 = \frac{c_f^2}{b_n^2}, \quad \Omega_{\rm A}^2 = 1, \quad \beta = \frac{a^2}{b^2} \tag{E 12}$$

and $\mathbf{K}_\perp = (0, k_y', k_z)/k_x$. The normal variables (E 9)–(E 11) are essentially the normal variables used by Gogoberidze et al. (2004) except for the normalization factor $-i/k_x$ and some other minor differences. In particular, the factor of K in the denominator in (E 9) for ψ_f is absent in the corresponding equation (29) of Gogoberidze et al. (2004). Similarly, the numerator for ψ_s in (E 11) has a term $(\beta K^2 - \Omega_s^2)$ that is given as $(\Omega_s^2 - \beta K^2)$ in (30) of Gogoberidze et al. (2004).

Appendix F

In this appendix we use the WKB solutions of (7.19) for $u(t)$ to generate solutions of the K-system wave interaction equations (7.10) and (7.11) for $c_1(t)$ and $c_7(t)$, and the corresponding R-system solutions of (7.13) and (7.14) for $c_1'(t)$ and $c_7'(t)$ for the case where the shear parameter A is sufficiently small. For small $A/(kb)$, (7.19) for $u(t)$ can be written in the form

$$\epsilon^2\frac{d^2u}{d\tau^2} = Q(\tau)u, \tag{F 1}$$

where

$$\epsilon = \frac{A}{kb}, \quad \tau = \epsilon t, \quad Q = -b^2k'^2, \quad k' = k[(n^x)^2 + (n^y - bk_x\tau)^2], \tag{F 2}$$

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Equation (F 1) for small ϵ has approximate WKB solutions of the form

$$u_1 = \frac{\exp(i\theta)}{\sqrt{k'}}, \quad u_2 = \frac{\exp(-i\theta)}{\sqrt{k'}}, \tag{F 3}$$

where

$$\theta = \frac{b}{\epsilon}\int_0^\tau k'(\tau')\,d\tau' \equiv \int_0^t bk'(t')\,dt' \tag{F 4}$$

(e.g., Bender and Orszag 1978, ch. 10, p. 484). The general, approximate WKB solution of (F 1) is given by a linear combination of the independent solutions u_1 and u_2.

Approximate solutions of (7.10) and (7.11) for small shear parameter $A/(kb)$ using the WKB solutions (F 3), with initial data $c_1(t) = c_1(0)$ and $c_7(t) = c_7(0)$ at time $t = 0$, are given by the matrix equation

$$\begin{pmatrix} c_1(t) \\ c_7(t) \end{pmatrix} = \begin{pmatrix} T_{11}(t) & T_{17}(t) \\ T_{17}^*(t) & T_{11}^*(t) \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_7(0) \end{pmatrix} \equiv \mathbf{T} \begin{pmatrix} c_1(0) \\ c_7(0) \end{pmatrix}, \tag{F 5}$$

where

$$T_{11}(t) = \sqrt{k/k'}[\alpha\alpha_0^* \exp(i\theta) + \beta\beta_0 \exp(-i\theta)],$$

$$T_{17}(t) = \sqrt{k/k'}[\alpha\beta_0^* \exp(i\theta) + \beta\alpha_0 \exp(-i\theta)],$$

$$\alpha = \frac{1}{2}\left(1 + \frac{k'}{k} - i\frac{Ak_xk_y'}{2(kb)k'^2}\right), \quad \beta = \frac{1}{2}\left(1 - \frac{k'}{k} - i\frac{Ak_xk_y'}{2(kb)k'^2}\right),$$

$$\alpha_0 \equiv \alpha(0) = 1 - i\frac{Ak_xk_y}{4k^3b}, \quad \beta_0 \equiv \beta(0) = -i\frac{Ak_xk_y}{4k^3b}. \tag{F 6}$$

To derive (F 5) and (F 6) first note from (7.17) and (7.18) that

$$c_1 = \frac{1}{2}\left(u + \frac{iu_t}{\omega_1}\right), \quad c_7 = \left(u - \frac{iu_t}{\omega_1}\right), \tag{F 7}$$

where $\omega_1 = -kb$. As $u = a_1u_1 + a_2u_2$ is the general solution for u, then the general solutions for $c_1(t)$ and $c_7(t)$ are given by the matrix equation

$$\begin{pmatrix} c_1(t) \\ c_7(t) \end{pmatrix} = \begin{pmatrix} \alpha u_1 & \beta u_2 \\ \beta^* u_1 & \alpha^* u_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{F 8}$$

Setting $t = 0$ in (F 8) and solving the resultant matrix equation for a_1 and a_2, we obtain the solution

$$\begin{pmatrix} c_1(t) \\ c_7(t) \end{pmatrix} = \sqrt{k} \begin{pmatrix} \alpha u_1 & \beta u_2 \\ \beta^* u_1 & \alpha^* u_2 \end{pmatrix} \begin{pmatrix} \alpha_0^* & \beta_0^* \\ \beta_0 & \alpha_0 \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_7(0) \end{pmatrix} \tag{F 9}$$

for $c_1(t)$ and $c_7(t)$. Multiplying out the matrices in (F 9), we obtain the scattering or transfer matrix $\mathbf{T}$ in (F 5) and (F 6).

Note that the total wave action $|c_7(t)|^2 - |c_1(t)|^2 = |c_7(0)|^2 - |c_1(0)|^2 = \text{constant}$ for the WKB solutions is conserved. As $t \to \infty$, we find to lowest order in A that

$$T_{11}(t) \simeq \frac{1}{2}\sqrt{k'/k}\left(\exp(i\theta) + \frac{iAk_xk_y}{4k^3b}\exp(-i\theta)\right),$$

$$T_{17}(t) \simeq \frac{1}{2}\sqrt{k'/k}\left[\frac{iAk_xk_y}{4k^3b}\exp(i\theta) - \left(1 - \frac{iAk_xk_y}{4k^3b}\right)\exp(-i\theta)\right]. \tag{F 10}$$

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Thus, both $T_{11}(t)$ and $T_{17}(t)$ scale as $\sqrt{k'/k} \sim \sqrt{Atn^x}$ as $t \to \infty$. From (F 10),

$$|T_{11}|^2 \sim |T_{17}|^2 \sim \frac{k'}{4k}\left[1 + \frac{Ak_xk_y}{2k^3b}\sin(2\theta)\right], \tag{F 11}$$

as $t \to \infty$. Thus, both $|T_{11}|^2$ and $|T_{17}|^2$ have an oscillatory component proportional to $\sin(2\theta)$ as $t \to \infty$.

The approximate WKB solution for $(c_1'(t), c_7'(t))$ of (7.13) and (7.14) satisfying given initial data $(c_1'(0), c_7'(0)) = (c_1(0), c_7(0))$ is given by

$$\begin{pmatrix} c_1'(t) \\ c_7'(t) \end{pmatrix} = \begin{pmatrix} T_{11}'(t) & T_{17}'(t) \\ T_{17}'^*(t) & T_{11}'^*(t) \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_7(0) \end{pmatrix} \equiv \mathbf{T}' \begin{pmatrix} c_1(0) \\ c_7(0) \end{pmatrix}, \tag{F 12}$$

where

$$\begin{aligned} T_{11}'(t) &= -\frac{Ak_xk_y}{4k^3b}\frac{Ak_xk_y'}{4k'^3b}\exp(-i\theta) + \left(1 + \frac{iAk_xk_y}{4k^3b}\right)\left(1 - \frac{iAk_xk_y'}{4k'^3b}\right)\exp(i\theta), \\ T_{17}'(t) &= -\frac{iAk_xk_y'}{4k'^3b}\left(1 - \frac{iAk_xk_y}{4k^3b}\right)\exp(-i\theta) + \frac{iAk_xk_y}{4k^3b}\left(1 - \frac{iAk_xk_y'}{4k'^3b}\right)\exp(i\theta). \end{aligned} \tag{F 13}$$

The proof is a straightforward consequence of (7.16) and (F 5).

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