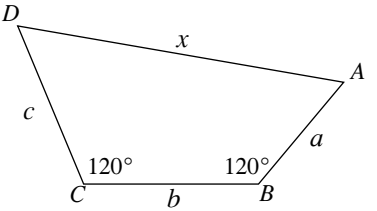
## 109.11 Integer polygons containing 120°

As a teacher and amateur puzzle setter, I am frequently on the lookout for interesting results that may be adapted for my usage. Whilst looking through GCSE problems involving trigonometry, it became obvious that finding the 4th side of a quadrilateral can be quite involved, and this inspired the problem below:

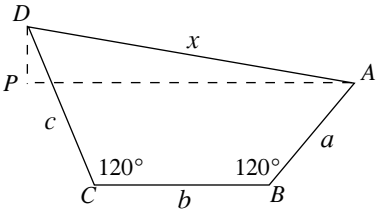



For the quadrilateral $ABCD$ above, is it possible to find integer values for $a$, $b$, $c$ and $x$?

The corresponding solution for a triangle containing 120° has been investigated (see for example [1]), and a rudimentary search on Wikipaedia [2] turns up the parametric solution $a = m^2 + mn + n^2$, $b = 2mn + n^2$, $c = m^2 - n^2$, where $a$ is opposite the 120° angle. The same rudimentary search does not however seem to return solutions to the above problem.

*Towards a solution*

Firstly, if $AD$ is parallel to $BC$, we may find infinitely many integer solutions, as this is similar to drawing isosceles trapezia on isometric paper, or the CAME activity "Roofs" (see [3]). As an example, the sides 1,1,1,2 will make a trapezium which solves the problem, and a general solution is given by $a$, $b$, $a$, $a + b$. This for me is not in the spirit of the problem however, so I will impose the condition that $AD$ is *not* parallel to $BC$.




If we construct the right-angled triangle above, we can use elementary trigonometry to show that $AP = a\cos 60° = \frac{1}{2}a + b + \frac{1}{2}c$ and $PD = |c\sin 60° - a\sin 60°| = \frac{\sqrt{3}}{2}|a - c|$. We may then use Pythagoras' Theorem to obtain:

$$x^2 = \tfrac{3}{4}(a - c)^2 + (\tfrac{1}{2}a + b + \tfrac{1}{2}c)^2,$$

$$(2x)^2 - 3(a - c)^2 = (a + 2b + c)^2.$$

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This now takes the form of a Pell equation which is much beloved of puzzle-setters (see [4], and [5] for a glorious quote about the attachment of Pell's name to this type of equation). Usually, we would start with the simpler equation and then 'scale up' by multiplying each side by $(a + 2b + c)^2$, but this keeps leading to negative values for the side lengths. However, a solution can be found as follows.

If we let $X = a + b, Y = b + c$ then the equation may be rewritten as $(2x)^2 - 3(X - Y)^2 = (X + Y)^2$, which simplifies to $x^2 = X^2 - XY + Y^2$. This is the equation that results when $x$, $X$, $Y$ form a triangle with an angle of 60° opposite the side $x$, and has integer solutions given by $x = m^2 - mn + n^2, Y = m^2 - n^2$ (see [2]).

Since $X = a + b$, $Y = m^2 - n^2$ we can set a value of $b$, and find values for the sides of the quadrilateral thus:

$$AB = 2mn - n^2 - b$$

$$BC = b$$

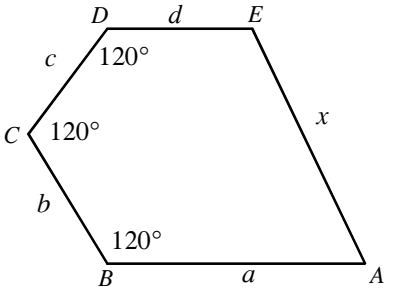
$$CD = m^2 - n^2 - b$$

$$AD = m^2 - mn + n^2.$$

For example, the values $m = 3$, $n = 2$, $b = 2$ return the values $AB = 6$, $BC = 2$, $CD = 7$, $AD = 7$. So, the answer to the original problem is "yes", and we have a method for generating integer solutions. What we have not done is worked out if this generates *all* the solutions, but that is when the amateur moves aside and lets the professional take over!

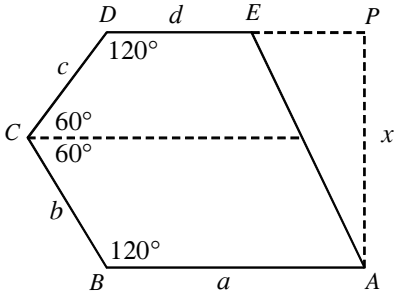
*Pentagons*

It is natural to develop this idea to polygons with greater numbers of sides. Consider first the pentagon below:




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We ask again whether integer values may be found for $a, b, c, d, x$. We proceed in the same manner as for the quadrilateral. First construct the right-angled triangle $AEP$ thus, then use trigonometry and Pythagoras' as before:




$$\begin{aligned} AP &= c \sin 60° + b \sin 60° = \tfrac{\sqrt{3}}{2}(b + c) \\ EP &= \left|a + b \cos 60° - \left(c \cos 60° + d\right)\right| \\ &= \left|a + \tfrac{1}{2}b - \tfrac{1}{2}c - d\right| \\ x^2 &= \tfrac{3}{4}(b + c)^2 + \left(a + \tfrac{1}{2}b - \tfrac{1}{2}c - d\right)^2 \\ (2x)^2 - 3(b + c)^2 &= (2a + b - c - 2d)^2 . \end{aligned}$$

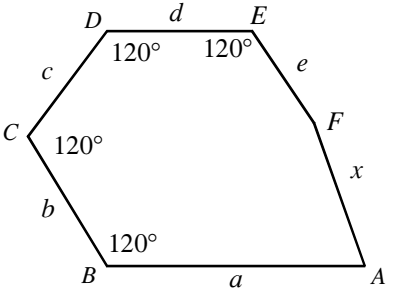
This is once again a Pell equation, and solutions can be found by using the substitution $X = a + b - d$, $Y = a - c - d$, to once again express the equation in the form $(2x)^2 - 3(X - Y)^2 = (X + Y)^2$, giving $x^2 = X^2 - XY + Y^2$ as in the example with the quadrilateral. Following a similar process, this time we can set $a$ and $d$ to obtain the following solution:

$$\begin{aligned} AB &= a \\ BC &= 2mn - n^2 - a + d \\ CD &= a - d - m^2 + n^2 \\ DE &= d \\ AE &= m^2 - mn + n^2 \end{aligned}$$

In this case, we must take care to ensure that $BC$ and $DE$ are both positive. We require both $2mn - n^2 - a + d$ and $a - d - m^2 + n^2$. Combining the two inequalities gives the requirement that $m < 2n$. We also need to check carefully that the value of $a - d$ is chosen so that neither $BC$ or $DE$ are negative. One such solution is given by $m = 3, n = 2, a = 9, d = 3$ to give $AB = 9, BC = 2, CD = 1, DE = 3$ and $AE = 7$.

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*Hexagons*




The resulting equation in this case is given by

$$x^2 = |b\sin 60° + c\sin 60° - e\sin 60°|^2$$

$$+|(a + b\cos 60°) - (c\cos 60° + d + e\cos 60°)|^2.$$

After simplification this gives:

$$(2x)^2 - 3(b + c - e)^2 = (2a + b - c - 2d - e)^2.$$

This time, in order to once again give an equation of the form $x^2 = X^2 - XY + Y^2$ we substitute $X = a + b - d - e$, $Y = a - c - d$. Integer solutions can be obtained by setting values of $a$. $d$, $e$ to get the following formulas:

$$AB = a$$

$$BC = 2mn - n^2 - a + d + e$$

$$CD = a - d - m^2 + n^2$$

$$DE = d$$

$$EF = e$$

$$AF = m^2 - mn + n^2.$$

Care must once again be taken to ensure that no values are negative, but if $m = 2$, $n = 3$, $a = 7$, $d = 6$, $e = 1$ we obtain $AB = 7$, $BC = 3$, $CD = 6, DE = 6, EF = 1, AF = 7$.

*Polygons with a greater number of sides*

Similar analysis can find solutions for greater numbers of sides, and I leave this to the interested reader. Care must be taken to ensure that the polygon can be constructed however, and it raises the question: is it possible to do this for any number of sides?

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## 109.12 A natural occurrence of the 3:4:5 triangle in the truncated regular octahedron

The central angle of any Platonic or Archimedean solid can be defined as the angle subtended at its centre by an edge. We will show that the central angle of the truncated octahedron is the most acute angle of the 3:4:5 triangle.

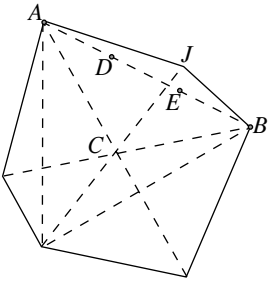



FIGURE 1

The regular octahedron has eight faces and six vertices. Its faces are equilateral triangles, four of which meet at each vertex. The octahedron can be seen as eight congruent triangular pyramids whose apexes come together at its centre. In Figure 1, $ABCJ$ is one such example. As $\angle ACB$ is a right angle and given a unit radius, the ratio of the lengths of the sides, $AC : BC : AB$ is $1 : 1 : \sqrt{2}$.

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