A CERTAIN TRIPLE WHITEHEAD PRODUCT

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1. *Introduction.* The triple Whitehead product we consider in this note is $[[\iota_n, \iota_n], \iota_n] \in \pi_{3n-2}(S^n)$, where ι_n generates $\pi_n(S^n)$. It follows from the Jacobi identity for Whitehead products†

$$(-1)^{pr}[[\alpha, \beta], \gamma] + (-1)^{qp}[[\beta, \gamma], \alpha] + (-1)^{rq}[[\gamma, \alpha], \beta] = 0, \tag{1·1}$$

$\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\gamma \in \pi_r(X)$, that $3[[\iota_n, \iota_n], \iota_n] = 0$. Now, if n is odd, $2[\iota_n, \iota_n] = 0$, so that $2[[\iota_n, \iota_n], \iota_n] = 0$, whence

$$[[\iota_n, \iota_n], \iota_n] = 0, \quad \text{if } n \text{ is odd.} \tag{1·2}$$

Also, if $n = 2$, $[[\iota_n, \iota_n], \iota_n] = 0$, because the only non-trivial products‡ in S^2 are multiples of $[\iota_2, \iota_2] \in \pi_3(S^2)$. Thus we study $[[\iota_n, \iota_n], \iota_n]$ $(n > 2)$, n even. We prove in § 2 that

$$[[\iota_n, \iota_n], \iota_n] \in E\pi_{3n-3}(S^{n-1}), \tag{1·3}$$

where E is the suspension operator. The proof requires a lemma which may be described as a weak converse of the fact that the Hopf invariant (in any sense) of a suspension element is zero. The next two sections are devoted to proving that $[[\iota_4, \iota_4], \iota_4]$ is not zero; we require to know the structure of the homotopy groups $\pi_{n+6}(S^n)$, and this information is presented in § 3, with an outline of the proof, due to J.-P. Serre, that $\pi_9(S^3) = Z_3$. The entire 6-stem is then deduced, and the non-triviality of $[[\iota_4, \iota_4], \iota_4]$ is established in the following section.

§ 5 consists of a study of Whitehead products in S^4 up to $\pi_{13}(S^4)$; in particular, it is shown that all products in $\pi_{11}(S^4)$, $\pi_{12}(S^4)$, $\pi_{13}(S^4)$ are trivial. The final section generalizes a formula§ proved in (1) and used in § 4.

2. *Proof of* (1·3). We take from (7) the following generalization‖ of (4·1) of (1).

THEOREM 2·1. *Let $\beta \in \pi_p(S^n)$, $\gamma \in \pi_r(S^p)$, and let H^* be the Hopf invariant in the sense of* (5). *Then*

$$H^*(\beta \circ \gamma) = H^*(\beta) \circ E\gamma + E^n\beta \circ E^p\beta \circ H^*(\gamma).$$

We now prove

THEOREM 2·2. *Let $\alpha \in \pi_r(S^n)$, n even, let $H^*(\alpha) = 0$, and let α be of odd order; then α is a suspension element.*

If $r = 2n-1$, then H^* coincides with the classical Hopf invariant, so that α is certainly a suspension element; for convenience of statement we exclude this case from the subsequent discussion.

† This formula seems to be well known; see a forthcoming paper by W. Hurewicz for an 'elementary' proof. Here we use only the simple case $p = q = r$.

‡ See (2·4) of (8). Alternatively, we note that $\pi_4(S^2)$ contains no element of order 3.

§ See (3·24), (3·25) of (1).

‖ Actually, (4·1) of (1) itself would be sufficient to establish (1·3).

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Let C be a class of Abelian groups in the sense of Serre (11). We recall that a homomorphism ϕ of the additive Abelian group G into the additive Abelian group H is *C-univalent* if $\phi^{-1}(0) \in C$ and *C-onto* if $H - \phi G \in C$. It is a *C-isomorphism* if it is C-univalent and C-onto. Let C be the class of torsion groups whose elements are of order a power of 2 and let $u : S^{2n-1} \to S^n$ be a map whose Hopf invariant is a power of 2. Then Corollary 2 to Proposition 5 of Chapter IV of (11) asserts that if

$$\phi : \pi_r(S^{2n-1}) + \pi_{r-1}(S^{n-1}) \to \pi_r(S^n)$$

is the homomorphism which coincides on $\pi_r(S^{2n-1})$ with the homomorphism u_* induced by u and on $\pi_{r-1}(S^{n-1})$ with E, then ϕ is a C-isomorphism†.

We take $u \in [\iota_n, \iota_n]$, so that $\alpha = [\iota_n, \iota_n] \circ \beta + E\gamma + \delta$,

where $\beta \in \pi_r(S^{2n-1})$, $\gamma \in \pi_{r-1}(S^{n-1})$, δ is of order 2^d. Now α is of odd order p; let β be of order $q \cdot 2^b$, γ of order $r \cdot 2^c$, where q, r are odd and let $m = \max(b, c, d)$. Choose λ, μ so that $\lambda p + \mu \cdot 2^m = 1$. Then

$$\alpha = \mu \cdot 2^m \alpha = [\iota_n, \iota_n] \circ \beta' + E\gamma', \tag{2·3}$$

where β', γ' are of odd order.

By (2·1), $$H^*(\alpha) = H^*([\iota_n, \iota_n] \circ \beta') = 2\iota_{2n} \circ E\beta' = 2E\beta'.$$

Since $H^*(\alpha) = 0$ and β' is of odd order, we have $E\beta' = 0$. Now $E : \pi_r(S^{2n-1}) \to \pi_{r+1}(S^{2n})$ is C-univalent, so that β' is of order a power of 2; but again, since β' is of odd order, this implies $\beta' = 0$, so that $\alpha = E\gamma'$ and the theorem is proved.

COROLLARY 2·4. $[[\iota_n, \iota_n], \iota_n] \in E\pi_{3n-3}(S^{n-1})$.

This is trivial if n is odd. We assume n even. Now

$$(-\iota_n) \circ [[\iota_n, \iota_n], \iota_n] = [[-\iota_n, -\iota_n], -\iota_n] = -[[\iota_n, \iota_n], \iota_n].$$

Also, by (2·1),

$$H^*((-\iota_n) \circ [[\iota_n, \iota_n], \iota_n]) = H^*[[\iota_n, \iota_n], \iota_n], \quad \text{so that} \quad 2H^*([[\iota_n, \iota_n], \iota_n]) = 0.$$

Since $3[[\iota_n, \iota_n], \iota_n] = 0$, it follows that $H^*([[\iota_n, \iota_n], \iota_n]) = 0$. The corollary now follows from Theorem 2·2.

COROLLARY 2·5. $[[[\iota_n, \iota_n], \iota_n], \iota_n] = 0$.

This is trivial if n is odd. We assume n even, so that $n \geqslant 2$. Then $[[\iota_n, \iota_n], \iota_n] = E\beta$, $\beta \in \pi_{3n-3}(S^{n-1})$, and $E^2\beta = 0$. Thus

$$[[[\iota_n, \iota_n], \iota_n], \iota_n] = [E\beta, \iota_n] = [\iota_n, \iota_n] \circ E^n\beta = 0.$$

It also follows, from the Jacobi identity, that $[[\iota_n, \iota_n], [\iota_n, \iota_n]] = 0$.

We may note that, in Theorem 2·2, we cannot allow n to be odd; for $\pi_6(S^3)$ contains an element of order 3, whereas $\pi_5(S^2)$ and $\pi_7(S^6)$ are both of order 2. Also it does not appear that the theorem remains true if α is allowed to have even order. For, unless $\pi_{2m-1}(S^m)$ contains an element of Hopf invariant 1 for all $m = 2^k$, we may choose m such‡ that $\pi_{2m-1}(S^m)$ contains an element of Hopf invariant 1, say γ, but $\pi_{4m-1}(S^{2m})$

† Since, as shown by Serre, all homotopy groups of spheres are finitely generated, the class C may be replaced by the subclass of finite groups whose order is a power of 2. We use the larger class only to stress that Serre's proof is based on Theorem 3 of Chapter III of (11), which requires the larger class.

‡ Recent work of H. Cartan indicates that we may take $m = 8$.

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does not contain such an element. Then $[\iota_{2m-1}, \iota_{2m-1}] \neq 0$ and $\gamma \circ [\iota_{2m-1}, \iota_{2m-1}]$ is not a suspension element, whereas $H^*(\gamma \circ [\iota_{2m-1}, \iota_{2m-1}]) = 0$.

As shown by G. W. Whitehead in (13), if $r \leqslant 3n-4$, then the conclusion of Theorem 2·2 follows with no restriction on n or the order of α.

3. *Calculation of* $\pi_{n+6}(S^n)$. We first fix notation. Thus, let α_n generate $\pi_{n+1}(S^n)$ $(n \geqslant 2)$; let ω be the Blakers-Massey element in $\pi_6(S^3)$; let γ be the Hopf class in $\pi_7(S^4)$ and let $\gamma_n = E^{n-4}\gamma \in \pi_{n+3}(S^n)$ $(n \geqslant 5)$. Then it is shown in (2) that ω generates $\pi_6(S^3)$ and γ_n generates $\pi_{n+3}(S^n)$. We suppose the sign of ω so chosen that†

$$[\iota_4, \iota_4] = 2\gamma + E\omega. \tag{3·1}$$

Now $\pi_8(S^2) = Z_2$ (see (10)). Moreover, it is generated by

$$\alpha_2 \circ \omega \circ \alpha_6 \circ \alpha_7 = \alpha_2 \circ \alpha_3 \circ \gamma \circ \alpha_7,$$

and $E\pi_8(S^2) = 0$. This latter fact may be seen as follows:

$$E(\alpha_2 \circ \omega) = \alpha_3 \circ E\omega = \alpha_3 \circ (2\gamma + E\omega) = \alpha_3 \circ [\iota_4, \iota_4],$$

since

$$\alpha_3 \circ 2\gamma = 2(\alpha_3 \circ \gamma) = 2\alpha_3 \circ \gamma = 0,$$

by (4·4) of (6). Thus $E^2(\alpha_2 \circ \omega) = 0$, but E maps $\pi_7(S^3)$ univalently, so that $E(\alpha_2 \circ \omega) = 0$, whence, of course, $E(\alpha_2 \circ \omega \circ \alpha_6 \circ \alpha_7) = 0$.

We now outline Serre's proof that $\pi_9(S^3) = Z_3$. By his general theorems, $\pi_9(S^3)$ is a finite group, whose order is of the form $2^a.3^b$, $a, b \geqslant 0$. We write $S^n \mid m$ for the space consisting of S^n with an $(n+1)$-cell attached by a map of degree m. Let $f: S^6 \mid 3 \to S^3$ be a map which, restricted to S^6, yields an element in $\pi_6(S^3)$ of order 3. Then ((11), Prop. 8) f induces an isomorphism of $\pi_r(S^6 \mid 3)$ on to the 3-component of $\pi_r(S^3)$ $(r \leqslant 10)$. Now $\pi_9(S^6 \mid 3) = Z_3$, generated by the injection of γ_6. Thus the 3-component of $\pi_9(S^3)$ is Z_3 and the order of $\omega \circ \gamma_6$ is divisible by 3; so, also, is the order of $\omega \circ E^3\omega = -2(\omega \circ \gamma_6)$.

It remains to show that the 2-component of $\pi_9(S^3)$ is zero. Embed S^2 in Ω_3, the space of loops on S^3, so that the injection $i: \pi_r(S^2) \to \pi_r(\Omega_3)$ corresponds to the suspension. Let Q be the space of paths on Ω_3 beginning in S^2 and ending at a fixed point of S^2. Then $\pi_r(Q) \approx \pi_{r+1}(\Omega_3, S^2)$ and we have the exact sequence

$$\ldots \to \pi_r(S^2) \xrightarrow{i} \pi_r(\Omega_3) \to \pi_{r-1}(Q) \to \pi_{r-1}(S^2) \to \ldots. \tag{3·2}$$

Now the space of paths on Ω_3 beginning in S^2 and with free ends is clearly deformable, rel S^2, on to S^2. Thus there is a fibration in which the fibre-space has the homotopy type of S^2, the base-space is Ω_3 and the fibres have the homotopy type of Q. Moreover, (3·2) may be regarded as the homotopy sequence of this fibration.

The cohomology ring of Ω_3 is known; we have $H^{2n}(\Omega_3) = Z_\infty$, generated by e_n, say, $H^{2n+1}(\Omega_3) = 0$. Moreover, $(e_1)^n = n!\,e_n$, the product being taken, of course, in the sense of cup-product. Using the spectral cohomology sequence associated with the given fibration, Serre computes the cohomology groups of Q. For we have a spectral sequence in which $E_2^{p,q} = H^p(\Omega_3) \otimes H^q(Q)$ and E_∞ is the graduated algebra associated with $H(S^2)$.

We know from (3·2) that $\pi_1(Q) = 0$, $\pi_2(Q) = 0$, $\pi_3(Q) = Z_\infty$. Thus $H^1(Q) = 0$, $H^2(Q) = 0$, $H^3(Q) = Z_\infty$, generated by v, say. Identifying v with its natural image in $E_4^{0,3}$ it is clear that $d_4 v = e_2$. We may quickly verify that $H^r(Q)$ is finite if $r > 3$, and

† See (6·1) of (8).

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$H^4(Q) = 0$, $H^5(Q) = Z_3$. From this point, Serre calculates only the 2-components (because that is all he needs) and obtains

$$^2H^6(Q) = 0, \quad ^2H^7(Q) = Z_2, \quad ^2H^8(Q) = 0, \quad ^2H^9(Q) = 0,$$
$$^2H^{10}(Q) = Z_2, \quad ^2H^{11}(Q) = 0, \quad ^2H^{12}(Q) = 0,$$

where ^2G refers to the 2-component of G. Moreover, if $^2H^7(Q)$ is generated by y, then $^2H^{10}(Q)$ is generated by the cup-product $w = vy$.

Let $f: S^3 \to Q$ represent a generator of $\pi_3(Q)$ and let $\tilde{Q}$ be the mapping-cylinder of f. Then $\tilde{Q}$ has the homotopy type of Q and f is replaced by the identity map $S^3 \to \tilde{Q}$. Let Y be the space of paths on $\tilde{Q}$ beginning in S^3 and ending at a fixed point of S^3. Then $\pi_r(Y) \approx \pi_{r+1}(\tilde{Q}, S^3)$, and there is an exact sequence

$$\ldots \to \pi_r(S^3) \to \pi_r(\tilde{Q}) \to \pi_{r-1}(Y) \to \pi_{r-1}(S^3) \to \ldots. \tag{3·3}$$

Now the space of paths on $\tilde{Q}$ beginning in S^3 and with free ends is deformable, rel S^3, on to S^3. Thus there is a fibration in which the fibre-space has the homotopy type of S^3, the base-space is $\tilde{Q}$ and the fibres have the homotopy type of Y. Moreover, (3·3) is the homotopy sequence of this fibration.

From the 2-components of $H^r(Q)$, we obtain the cohomology groups mod 2 of Q and thus of $\tilde{Q}$. The result is

$$H^3(\tilde{Q}, Z_2) = Z_2, \quad \text{generated by } a;$$
$$H^4(\tilde{Q}, Z_2) = 0; \quad H^5(\tilde{Q}, Z_2) = 0; \quad H^6(\tilde{Q}, Z_2) = Z_2, \quad \text{generated by } b;$$
$$H^7(\tilde{Q}, Z_2) = Z_2, \quad \text{generated by } c = Sq^1 b,$$

Sq^1 being the Steenrod square $H^n(X, Z_2) \to H^{n+1}(X, Z_2)$;

$$H^8(\tilde{Q}, Z_2) = 0; \quad H^9(\tilde{Q}, Z_2) = Z_2, \quad \text{generated by } d;$$
$$H^{10}(\tilde{Q}, Z_2) = Z_2, \quad \text{generated by } e = Sq^1 d; \quad H^{11}(\tilde{Q}, Z_2) = 0.$$

Since we had $w = vy$, it follows that $e = ac$. Thus $Sq^1 ab = ac = e = Sq^1 d$, so that $ab = d$.

Associated with the fibration, there is a spectral sequence such that

$$E_2^{p,q} = H^p(\tilde{Q}, Z_2) \otimes H^q(Y, Z_2)$$

and E_∞ is the graduated algebra associated with $H(S^3)$. Serre uses this to compute $H^q(Y, Z_2)$.

It is clear (by the relative Hurewicz theorem) that $\pi_r(\tilde{Q}, S^3) = 0$ $(r < 5)$ and $\pi_5(\tilde{Q}, S^3) = Z_3$. Thus $\pi_r(Y) = 0$ $(r < 4)$, $\pi_4(Y) = Z_3$. Also

$$H_5(\tilde{Q}, S^3; Z_2) = 0, \quad H_6(Q, S^3; Z_2) = Z_2,$$

so that

$$\pi_5(Y) \otimes Z_2 = Z_2, \quad H_5(Y, Z_2) = Z_2.$$

From the fact that $H^p(\tilde{Q}, Z_2) = 0$ $(0 < p < 3)$, $H^q(Y, Z_2) = 0$ $(0 < q < 5)$, it follows that there is an exact sequence

$$H^7(Y, Z_2) \leftarrow H^7(S^3, Z_2) \leftarrow H^7(\tilde{Q}, Z_2) \xleftarrow{t} H^6(Y, Z_2) \leftarrow \ldots,$$

where t is the transgression. Thus $t: H^r(Y, Z_2) \approx H^{r+1}(\tilde{Q}, Z_2)$ $(r = 5, 6)$, so that $H^5(Y, Z_2) = Z_2$, generated by m, $H^6(Y, Z_2) = Z_2$, generated by n, and

$$tSq^1 m = Sq^1 tm = Sq^1 b = c = tn,$$

so that $Sq^1 m = n$. Arguing further from the spectral sequence, Serre shows that $H^r(Y, Z_2) = 0$ $(r = 7, 8, 9)$.

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We may improve the assertion $\pi_5(Y) \otimes Z_2 = Z_2$; for we have $H_3(\tilde{Q}) = Z_\infty$, ${}^2H_4(\tilde{Q}) = 0$, ${}^2H_5(\tilde{Q}) = 0$, ${}^2H_6(\tilde{Q}) = 0$. Thus ${}^2H_r(\tilde{Q}, S^3) = 0$ $(r < 6)$, ${}^2H_6(\tilde{Q}, S^3) = Z_2$; it follows that the 2-components of $\pi_r(Y)$ are zero $(r < 5)$, and that of $\pi_5(Y)$ is Z_2. Thus the 2-components of $H_r(Y)$ are zero $(1 \leqslant r \leqslant 4,\ 6 \leqslant r \leqslant 9)$ and ${}^2H_5(Y) = Z_2$.

Let C be the class of groups of odd order. We may infer from the results above the existence of a map of $S^5 \mid 2$ into Y which induces C-isomorphisms of the homology groups of $S^5 \mid 2$ on to those of Y up to dimension 9. By Theorem 3, Chapter III, of (11), it therefore induces C-isomorphisms of the homotopy groups of $S^5 \mid 2$ on to those of Y up to dimension 8. We may thus calculate the 2-components of $\pi_r(Y)$ $(r \leqslant 8)$.

Let us write $G \equiv H$ if their 2-components are equal; then, writing down the 2-components of (3·2), (3·3), we have

$$\pi_8(S^2) \xrightarrow{0} \pi_8(\Omega_3) \xrightarrow{(1-1)} \pi_7(Q) \xrightarrow{\text{onto}} \pi_7(S^2) \xrightarrow{0} \pi_7(\Omega_3) \xrightarrow{(1-1)} \pi_6(Q) \xrightarrow{\text{onto}} \pi_6(S^2) \xrightarrow{0} \pi_6(\Omega_3),$$

and $$\pi_7(S^3) \to \pi_7(\tilde{Q}) \to \pi_6(Y) \to \pi_6(S^3) \to \pi_6(\tilde{Q}) \to \pi_5(Y) \to \pi_5(S^3) \xrightarrow{\text{onto}} \pi_5(\tilde{Q}) \to 0.$$

Here the symbols '$0, (1-1)$, onto' mean that the relevant homomorphisms are the zero homomorphism, $(1-1)$, onto respectively.

It is clear from the first sequence that the result to be proved $(\pi_9(S^3) \equiv 0)$ will follow if we can show that $\pi_7(Q) \equiv Z_2$. Since $\pi_7(\Omega_3) = Z_2$, $\pi_6(S^2) \equiv Z_4$, it follows that $\pi_6(Q) \equiv Z_8$ or $Z_4 + Z_2$. We depart slightly from Serre's procedure to prove†

LEMMA 3·4. $\pi_6(Q) \equiv Z_8$.

We note first, from (3·2), that, since the kernel of $i : \pi_3(S^2) \to \pi_3(\Omega_3)$ is generated by 2α (writing α for α_2), the image of the generator of $\pi_3(Q)$ represented by the map f is $\pm 2\alpha$.

Let us suppose $\pi_6(Q) \equiv Z_4 + Z_2$; then $\pi_6(\tilde{Q}) \equiv Z_4 + Z_2$. Since $\pi_6(S^3) \equiv Z_4$, $\pi_5(Y) \equiv Z_2$, it follows that we may take as generators elements $\tilde{\eta}$, $\tilde{\beta}$, where $\tilde{\eta}$ is the image under injection of $\omega \in \pi_6(S^3)$. Then the element $\tilde{\eta}$ in $\pi_6(\tilde{Q})$ corresponds to the element η in $\pi_6(Q)$ represented by $S^6 \to S^3 \xrightarrow{f} Q$, where $S^6 \to S^3$ represents ω. Thus the image of η in $\pi_6(S^2)$ is $\pm 2\alpha \circ \omega = \pm 2(\alpha \circ \omega)$; but $2(\alpha \circ \omega)$ is of order 6, so that no element in the image of $\pi_6(Q)$ can have order 4. This contradiction establishes the lemma. Moreover, the 2-component of $\pi_6(\tilde{Q})$ is generated by an element $\tilde{\beta}$ such that $2\tilde{\beta} = \tilde{\eta}$.

Now the image in $\pi_7(\tilde{Q})$ of the generator $\omega \circ \alpha_6$ of $\pi_7(S^3)$ is $\tilde{\eta} \circ \alpha_6$; but

$$\tilde{\eta} \circ \alpha_6 = 2\tilde{\beta} \circ \alpha_6 = \tilde{\beta} \circ 2\alpha_6 = 0.$$

Since $\pi_6(Y) \equiv Z_2$, it follows that $\pi_7(\tilde{Q}) \equiv Z_2$, whence $\pi_7(Q) \equiv Z_2$ and $\pi_9(S^3) \equiv 0$.

We have proved $\pi_9(S^3) = Z_3$, generated by $\omega \circ \gamma_6 = \omega \circ E^3\omega$. It follows now that $\pi_{10}(S^4) = Z_{24} + Z_3$, generated by elements $x = \gamma \circ \gamma_7$, $y = E\omega \circ \gamma_7 = E\omega \circ E^4\omega$. It was proved in (8) that $E\pi_{10}(S^4) = \pi_{11}(S^5)$ and $E : \pi_{n+6}(S^n) \approx \pi_{n+7}(S^{n+1})$ $(n \geqslant 5)$. It follows from Theorems 3·10 and 4·9 of (5) that $E(\gamma \circ \gamma_7) \neq 0$. We will prove that $2Ex = 0$, $Ey = 0$; since y is of order 3, it is sufficient to prove $2Ex = 0$, $2Ey = 0$.

Now, by (3·2) of (1), if $\theta \in \pi_{m+h}(S^m)$, $\eta \in \pi_{n+k}(S^n)$, then

$$E^m\eta \circ E^{n+k}\theta = (-1)^{hk} E^n\theta \circ E^{m+h}\eta. \tag{3·5}$$

Let $\eta = \theta$, $m = n$, $h = k = 2l + 1$; then

$$2(E^n\eta \circ E^{n+k}\eta) = 0. \tag{3·6}$$

† It may be noted that the lemma implies $\pi_6(Q) = Z_{24}$.

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Replacing η by γ, ω, we get

$$2E^4(\gamma \circ E^3\gamma) = 0, \quad 2E^3(\omega \circ E^3\omega) = 0,$$

or

$$2E^3x = 0, \quad 2E^2y = 0.$$

The assertion now follows from the univalence of E in the relevant dimensions. We have proved† that, if $n \geqslant 5$,

$$\pi_{n+6}(S^n) = Z_2, \quad \text{generated by } \gamma_n \circ \gamma_{n+3}. \tag{3·7}$$

4. *Proof of the non-triviality of* $[[\iota_4, \iota_4], \iota_4]$. Let e^{2n} be attached to S^n by a map $f: S^{2n-1} \to S^n$ of Hopf invariant k, and let a, b be the positive generators of $H^n(S^n \cup e^{2n})$, $H^{2n}(S^n \cup e^{2n})$.

LEMMA 4·1. $a^2 = kb$.

We use Steenrod's characterization of the Hopf invariant as a special case of his 'functional cup-product'. Consider the diagram

$$\begin{array}{ccccccc} & & 0 \to H^n(X, S^{2n-1}) & \xrightarrow{j^n} & H^n(X) \to 0, & & \\ & & \downarrow \cup a' & & & & \\ 0 \to H^{2n-1}(S^{2n-1}) & \xrightarrow{\delta^{2n}} & H^{2n}(X, S^{2n-1}) \to 0, & & & & \end{array}$$

where X is the mapping cylinder of f. Identifying a with the corresponding generator of $H^n(X)$ and writing $a' = (j^n)^{-1}(a)$, the vertical arrow stands for the cup-product of an element of $H^n(X, S^{2n-1})$, with a', and, as pointed out in (12), if c is the positive generator of $H^{2n-1}(S^{2n-1})$, then

$$(\delta^{2n})^{-1}(a' \cup a') = kc,$$

or

$$a' \cup a' = k(\delta^{2n}c).$$

The lemma now follows by shrinking S^{2n-1} to a point in X. The identification space is $S^n \cup e^{2n}$, and the shrinking map s induces inverse isomorphisms

$$s^*: H^r(S^n \cup e^{2n}) \approx H^r(X, S^{2n-1})$$

such that $s^*a = a'$, $s^*b = \delta^{2n}c$. Thus $s^*a^2 = s^*a \cup s^*a = a' \cup a' = ks^*b$ and $a^2 = kb$.

Now let e^8 be attached to S^4 by a map in the class $[\iota_4, \iota_4]$. Then the embedding map $S^4 \to \Omega_5$ may be extended to a map $f: S^4 \cup e^8 \to \Omega_5$.

LEMMA 4·2. *f induces isomorphisms* $f_*: \pi_r(S^4 \cup e^8) \approx \pi_r(\Omega_5)$ $(r < 11)$.

Now the cohomology ring of Ω_5 is generated by elements u_n of dimension $4n$ and $u_1^2 = 2u_2$ (see (9), Theorem 18). Using the notation of 4·1, we have $f^*u_1 = a$, $a^2 = 2b$, so that $2f^*u_2 = f^*u_1^2 = (f^*u_1)^2 = a^2 = 2b$, and $f^*u_2 = b$. By duality f induces isomorphisms of the homology groups in dimensions 0, 4 and 8, and therefore in dimensions less than 12. The lemma now follows‡ from Theorem 3 of Chapter III of (11).

Essentially we have proved the following: if $E: \pi_{10}(S^4) \to \pi_{11}(S^5)$ is the suspension and $i: \pi_{10}(S^4) \to \pi_{10}(S^4 \cup e^8)$ the injection, then $E^{-1}(0) = i^{-1}(0)$. Now, by Theorem 1 of (3), $i^{-1}(0)$ is the subgroup generated by $[\iota_4, \iota_4] \circ \gamma_7$ and $[[\iota_4, \iota_4], \iota_4]$. Now

$$[\iota_4, \iota_4] \circ \gamma_7 = 2x + y$$

† This result, and the application of (3·5), are known to J.-P. Serre. (*Added in proof.*) H. Toda has also calculated $\pi_{n+6}(S^n)$; see *Proc. Jap. Acad.* 29 (1953), 299–304.

‡ The proof of (4·2) was suggested to the author by J.-P. Serre. Stronger results in the same direction have been proved by E. Pitcher using critical point theory. See also (13).

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and is of order 12. Since $E^{-1}(0)$ is of order 36, it follows that $[[\iota_4, \iota_4], \iota_4]$ is of order 3, so that, combining this with (1·3), we have

$$[[\iota_4, \iota_4], \iota_4] = \pm y = \pm \omega \circ E^2\gamma. \tag{4·3}$$

Let us compute all Whitehead products in $\pi_{10}(S^4)$; it is only necessary to study $[\gamma, \iota_4]$ and $[E\omega, \iota_4]$. Now $[E\omega, \iota_4] = [\iota_4, \iota_4] \circ E^4\omega = (2\gamma + E\omega) \circ E^4\omega = -4x + y$, so that

$$[E\omega, \iota_4] = -4x + y, \tag{4·4}$$

and is of order 6.

It was proved in (1) that, calculating in $S_1^4 \vee S_2^4$, so that $\gamma^{(i)}, \iota_4^{(i)} (i = 1, 2)$, represent the elements corresponding to γ, ι_4 in $\pi_r(S_i^4)$, we have

$$[\gamma^{(1)}, \iota_4^{(2)}] = [\iota_4^{(1)}, \iota_4^{(2)}] \circ E^3\gamma - [[\iota_4^{(1)}, \iota_4^{(2)}], \iota_4^{(1)}]. \tag{4·5}$$

It follows then that

$$\begin{aligned}[\gamma, \iota_4] &= [\iota_4, \iota_4] \circ E^3\gamma - [[\iota_4, \iota_4], \iota_4] \\ &= 2x + y \mp y.\end{aligned}$$

Note that $(-\iota_4) \circ \gamma = \gamma + E\omega$, so that $(-\iota_4) \circ x = x + y$. Thus

$$[\gamma, \iota_4] = 2x \quad \text{or} \quad (-\iota_4) \circ 2x, \tag{4·6}$$

and is of order 12. We do not resolve the ambiguity of sign in (4·3) and (4·6).

Note finally that $[\theta, \eta] = [\eta, \theta]$ if $[\theta, \eta] \in \pi_{10}(S^4)$ and

$$[\alpha_4 \circ \alpha_5, \alpha_4] = [\alpha_4 \circ \alpha_5 \circ \alpha_6, \iota_4] = [12\gamma, \iota_4] = 0.$$

Thus all products in $\pi_{10}(S^4)$ are computed.

5. *Triviality of products in* $\pi_{11}(S^4)$, $\pi_{12}(S^4)$, $\pi_{13}(S^4)$. All Whitehead products in $\pi_r(S^4)$ $(r < 7)$ are trivial. In $\pi_7(S^4)$, $[\iota_4, \iota_4] = 2\gamma + E\omega$. In $\pi_8(S^4)$, $[\alpha_4, \iota_4] = E(\omega \circ \alpha_6)$ and is the suspension element of order 2. In $\pi_9(S^4)$, $[\alpha_4 \circ \alpha_5, \iota_4] = [\alpha_4, \alpha_4] = E(\omega \circ \alpha_6 \circ \alpha_7)$ and is the suspension element of order 2. Products in $\pi_{10}(S^4)$ were studied in the previous section. In this section we show that all products are trivial in $\pi_{11}(S^4)$ $\pi_{12}(S^4)$, $\pi_{13}(S^4)$.

Now all products in $\pi_{11}(S^4)$ are of the form $[\theta, \iota_4]$, $\theta \in \pi_8(S^4)$, and $\pi_8(S^4) = Z_2 + Z_2$, generated† by $\gamma \circ \alpha_7$, $E\omega \circ \alpha_7$.

But $[\gamma \circ \alpha_7, \iota_4] = [\gamma, \iota_4] \circ \alpha_{10} = (2x \circ \alpha_{10})$ or $(2(x+y) \circ \alpha_{10}) = 0$, since $2\alpha_{10} = 0$,

and $[E\omega \circ \alpha_7, \iota_4] = [E\omega, \iota_4] \circ \alpha_{10} = (-4x + y) \circ \alpha_{10} = 0$, since $3y = 0$.

Again all products in $\pi_{12}(S^4)$ are of the form $[\eta, \iota_4]$, $\eta \in \pi_9(S^4)$ and $\pi_9(S^4) = Z_2 + Z_2$, generated by $\gamma \circ \alpha_7 \circ \alpha_8$, $E\omega \circ \alpha_7 \circ \alpha_8$. Thus

$$[\eta, \iota_4] = [\theta \circ \alpha_8, \iota_4], \quad \theta \in \pi_8(S^4), = [\theta, \iota_4] \circ \alpha_{11} = 0.$$

It remains to prove that all products in $\pi_{13}(S^4)$ are trivial. Such products consist of terms $[\xi, \xi']$, $\xi, \xi' \in \pi_7(S^4)$, and terms $[\rho, \iota_4]$, $\rho \in \pi_{10}(S^4)$. We first prove that $y \circ \gamma_{10} = 0$, $2x \circ \gamma_{10} = 0$. For y is of order 3 and $y \circ \gamma_{10} = E\omega \circ \gamma_7 \circ \gamma_{10}$; but $\gamma_7 \circ \gamma_{10} \in \pi_{13}(S^7)$ is of order 2, so that $y \circ \gamma_{10} = 0$. Similarly, $2x \circ \gamma_{10} = 2\gamma \circ \gamma_7 \circ \gamma_{10} = 0$.

Now $\pi_{10}(S^4)$ is generated by x, y and

$$[x, \iota_4] = [\gamma \circ \gamma_7, \iota_4] = [\gamma, \iota_4] \circ \gamma_{10} = (2x \circ \gamma_{10}) \quad \text{or} \quad (2(x+y) \circ \gamma_{10}) = 0,$$

$$[y, \iota_4] = [E\omega \circ \gamma_7, \iota_4] = [E\omega, \iota_4] \circ \gamma_{10} = (-4x + y) \circ \gamma_{10} = 0.$$

† The triviality of products in $\pi_{11}(S^4)$ follows immediately from the fact that $\pi_{11}(S^4) = Z_{15}$, but we give a direct proof.

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Thus $$[\rho, \iota_4] = 0, \quad \rho \in \pi_{10}(S^4).$$

Finally, consider $[\xi, \xi']$, $\xi, \xi' \in \pi_7(S^4)$. We first note that $[\xi, \xi] = \xi \circ [\iota_7, \iota_7] = 0$, since S^7 carries a multiplication. Thus $[\gamma, \gamma] = 0$, $[E\omega, E\omega] = 0$. It remains to prove that $[\gamma, E\omega] = 0$; but $[\gamma, E\omega] = \pm [\gamma, \iota_4] \circ E^7\omega = \pm 2[\gamma, \iota_4] \circ \gamma_{10} = 0$. This completes the proof.

6. *Generalization of* (4·5). The formula we will prove is

THEOREM 6·1. *Let $\rho \in \pi_{2n-1}(S^n)$ have Hopf invariant k. Then*†, *in $\pi_{3n-2}(S_1^n \vee S_2^n)$,*

$$[\rho^{(1)}, \iota_n^{(2)}] = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho - k[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}].$$

Theorems 3·24, 3·25 of (1) are the cases $n = 2$, $n = 4$ of (6·1).

First, let $k = 0$. Then $\rho = E\sigma$, $\sigma \in \pi_{2n-2}(S^{n-1})$ and

$$[\rho^{(1)}, \iota_n^{(2)}] = [E\sigma^{(1)}, \iota_n^{(2)}] = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^n\sigma = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho.$$

Now take $k \neq 0$. Then n is even. Assume first that k is even, so that $k = 2l$. Then

$$\rho = l[\iota_n, \iota_n] + E\sigma,$$

and $$[\rho^{(1)}, \iota_n^{(2)}] = [E\sigma^{(1)}, \iota_n^{(2)}] + l[[\iota_n^{(1)}, \iota_n^{(1)}], \iota_n^{(2)}].$$

Now $$[E\sigma^{(1)}, \iota_n^{(2)}] = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^n\sigma = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho,$$

and, by the Jacobi identity,

$$\begin{aligned}[[\iota_n^{(1)}, \iota_n^{(1)}], \iota_n^{(2)}] &= -[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}] - [[\iota_n^{(2)}, \iota_n^{(1)}], \iota_n^{(1)}] \\ &= -2[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}], \quad \text{since } n \text{ is even.}\end{aligned}$$

Thus $$[\rho^{(1)}, \iota_n^{(2)}] = [\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho - k[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}],$$

and (6·1) is established if k is even. To obtain (6·1) if k is odd, we use the decomposition

$$\pi_{3n-2}(S_1^n \vee S_2^n) \approx \pi_{3n-2}(S_1^n) + \pi_{3n-2}(S_2^n) + \pi_{3n-2}(S^{2n-1}) + \pi_{3n-2}(S^{3n-2}) + \pi_{3n-2}(S^{3n-2}), \tag{6·2}$$

where the third factor is embedded by composition with $[\iota_n^{(1)}, \iota_n^{(2)}]$, the fourth by composition with $[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}]$, and the fifth by composition with $[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(2)}]$. Thus‡

$$[\rho^{(1)}, \iota_n^{(2)}] = [\iota_n^{(1)}, \iota_n^{(2)}] \circ \beta + k_1[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}] + k_2[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}], \tag{6·3}$$

for some $\beta \in \pi_{3n-2}(S^{2n-1})$, integers k_1, k_2.

Let d be the univalent homomorphism $d : \pi_{3n-1}(S_1^n \times S_2^n, S_1^n \vee S_2^n) \to \pi_{3n-2}(S_1^n \vee S_2^n)$ and let χ be the 'shrinking' homomorphism $\pi_{3n-1}(S_1^n \times S_2^n, S_1^n \vee S_2^n) \to \pi_{3n-1}(S^{2n})$. Then, as shown in (1), we may apply χd^{-1} to both sides of (6·3), getting

$$E^n\rho = E\beta,$$

whence $\beta = E^{n-1}\rho$, since $E : \pi_{3n-2}(S^{2n-1}) \approx \pi_{3n-1}(S^{2n})$.

Now double both sides of (6·3); since 2ρ has even Hopf invariant $2k$, we have

$$2[\rho^{(1)}, \iota_n^{(2)}] = 2[\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho + 2k_1[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}] + 2k_2[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}]$$

and $$2[\rho^{(1)}, \iota_n^{(2)}] = 2[\iota_n^{(1)}, \iota_n^{(2)}] \circ E^{n-1}\rho - 2k[[\iota_n^{(1)}, \iota_n^{(2)}], \iota_n^{(1)}],$$

whence $k_1 = -k$, $k_2 = 0$, and (6·1) is established.

† We adopt the notation of (4·5) and (1).

‡ $[\rho^{(1)}, \iota_*^{(2)}]$ obviously projects to zero in $\pi_{3n-2}(S_1^n)$, $\pi_{3n-2}(S_2^n)$.

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Theorem (6·1) may be further generalized. We may take $\rho \in \pi_{2m-1}(S^m)$ and consider $[\rho, \iota_n] \in \pi_{2m+n-2}(S^m \vee S^n)$. If ρ has even Hopf invariant, k, we get immediately, by the same methods,

$$[\rho, \iota_n] = [\iota_m, \iota_n] \circ E^{n-1}\rho - k[[\iota_m, \iota_n], \iota_m]. \tag{6·4}$$

If ρ has odd Hopf invariant, and $m < n$, then we use the decomposition

$$\pi_{2m+n-2}(S^m \vee S^n) \approx \pi_{2m+n-2}(S^m) + \pi_{2m+n-2}(S^n) + \pi_{2m+n-2}(S^{m+n-1}) + \pi_{2m+n-2}(S^{2m+n-2}) \tag{6·5}$$

to deduce (6·4) again.

If $m > n \geqslant 2$, Chang has recently proved that†

$$\pi_r(S^m \vee S^n) \approx \pi_r(S^n) + \pi_r(S^{2m+n-2}) + \sum_{j=0}^{\infty} \pi_r(S^{m+jn-j}), \tag{6·6}$$

if $r \leqslant 2(m+n)-4$. It thus appears possible that if ρ has odd Hopf invariant and $m > n$, then the right-hand side of (6·4) may only express the projection of $[\rho, \iota_n]$ on $\pi_{2m+n-2}(S^{m+n-1}) + \pi_{2m+n-2}(S^{2m+n-2})$ and the two sides of (6·4) may differ by an element of order 2 in $\sum_{j=2}^{\infty} \pi_{2m+n-2}(S^{m+jn-j})$.

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† See (4).

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