NOTES ON GAS FLOW THROUGH A NOZZLE

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ABSTRACT. In Part 1 the oscillations are examined which may develop in supersonic flow through the divergent part of a nozzle. Hooker's theory is found to be confirmed by observations on a steam nozzle.

In Part 2 it is shown that Taylor's approximate theory of flow between circular arcs can also analyse the throat conditions when the velocity of approach to the throat is supersonic. Limiting symmetrical flow occurs when the velocity at the centre of the throat just exceeds the local velocity of sound, no matter what the radii of the arcs and the ratio of the specific heats of the gas may be. The unique asymmetrical case is merely the reverse of Taylor's solution for the flow when the velocity of approach is subsonic.

A numerical comparison has been made of the velocities across the throat as calculated by Taylor's method and by Fox and Southwell's iterative process when the velocity of approach is subsonic. The agreement is good for limiting symmetrical flow but not so satisfactory for the unique asymmetrical case.

Part 1. Oscillations in a straight divergent nozzle

The possibility of oscillations in a compressible fluid passing through a divergent nozzle was predicted by Hooker (7). At the time his analysis was completed no experimental evidence was available by which his theory could be checked; but, since then, a description of a comprehensive series of steam-nozzle tests has been published by Binnie and Woods (4), and, moreover, the necessary tables of Bessel functions have now appeared. The matter being of importance in the design and use of supersonic wind tunnels, a comparison between theory and experiment is made below.

Hooker examined two-dimensional radial flow between straight walls diverging from each other at an angle π/m; and, using coordinates r, θ, he considered a disturbed but steady state of flow in which the stream function ψ is given by

$$\psi = \theta + Af(r)\cos m\theta, \tag{1}$$

where A is a small constant whose square is negligible. An approximate solution for $f(r)$, valid for nozzles of small angle, was found to be

$$f(r) = tJ_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}mt^{\frac{3}{2}}\right), \tag{2}$$

where

$$t = \left(1-\frac{1}{r}\right)^{\frac{1}{2}}. \tag{3}$$

In this result, which is independent of the ratio γ of the specific heats, the unit of r is so chosen that the local velocity of sound is attained at $r = 1$, and it is evident from (3) that disturbances are to be expected only where the velocity is supersonic. Equation (2) holds good for asymmetrical oscillations which are not detectable by an exploration of pressure along the nozzle axis, and to obtain the symmetrical modes it is necessary to double the value of m. In the steam experiments the angle between the walls was 8°; therefore for insertion in (2) m is to be taken as $180/4 = 45$. The first three loops of

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the central stream-line thus obtained are shown in Fig. 1, in the preparation of which the whole range of Karas's tables (8) of $J_{\frac{1}{3}}(x)$ has been employed. The curve consists of oscillations slowly increasing in amplitude and wave-length. Eventually, after a finite number of zeros, the wave-length becomes infinite where the gas reaches its limiting velocity.

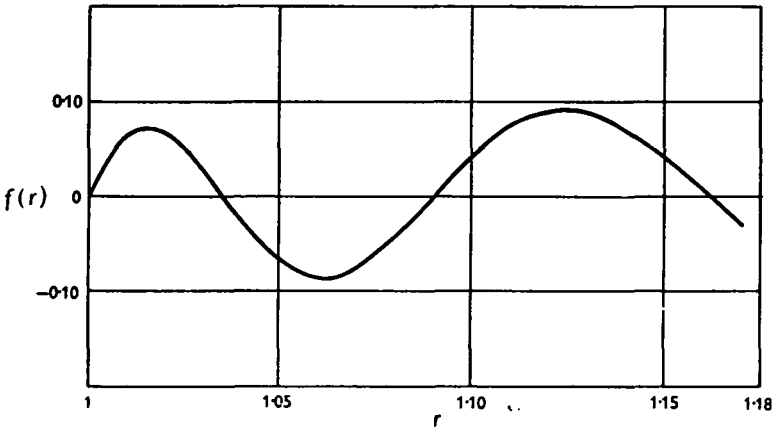


Fig. 1. Theoretical form of oscillation.

The profile of the two-dimensional steam nozzle is shown in the upper part of Fig. 2. The throat was formed by circular arcs of radius 6 in. leading without transition curves into the straight divergent portions. Pressure tappings, thirty-one in number, were drilled along the centre-line of one of the flat walls. In all the tests the pressure curve was of slightly undulating form, and the position, wave-length and amplitude of the waves appeared to be independent of the relatively small changes which could be made in the pressure and temperature of the steam supply. The oscillations existed throughout the divergent part of the nozzle, both before and after the beginning of condensation, but they were less evident close to the throat on account of the steep slope of the expansion curve there. Far downstream there is difficulty in mapping the waves owing to the wide separation of the tappings. Accordingly the experimental results shown by the broken line in Fig. 2 are confined to those observed at tappings situated at moderate distances from the throat. They are expressed as ratios of the observed pressure P to the inlet pressure P_0, and are taken from Binnie and Woods's Fig. 5 and Tables 2 and 3, which give the means of many observations. All refer to experiments with high superheat in the steam supply, so that condensation was delayed to near the nozzle exit and the working fluid may be taken as a gas for which $\gamma = 1{\cdot}3$. The curve clearly exhibits undulations; the difficulty is to add a mean undisturbed curve, and here the usual nozzle theory affords little assistance because in such small apparatus friction effects are appreciable.

Evidently it is useless to attempt a detailed verification of the theoretical amplitudes of the waves, and we must be content to check the positions of the nodes. For this purpose use was made of the table of roots of $J_{\frac{1}{3}}(x) = 0$ due to Abramovitz (1), which extends over a greater range than Karas's tables. By this means the theoretical positions of the 2nd, 3rd, ..., 6th nodes have been added to Fig. 2. First, the origin of r was

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determined by the point of intersection of the divergent walls. An almost negligible correction was then made for the fact that the local velocity of sound was attained on the axis not at the geometrical throat, but at about 0·06 in. downstream from it. With the positions of the nodes in mind a plausible undisturbed curve was drawn on Fig. 2

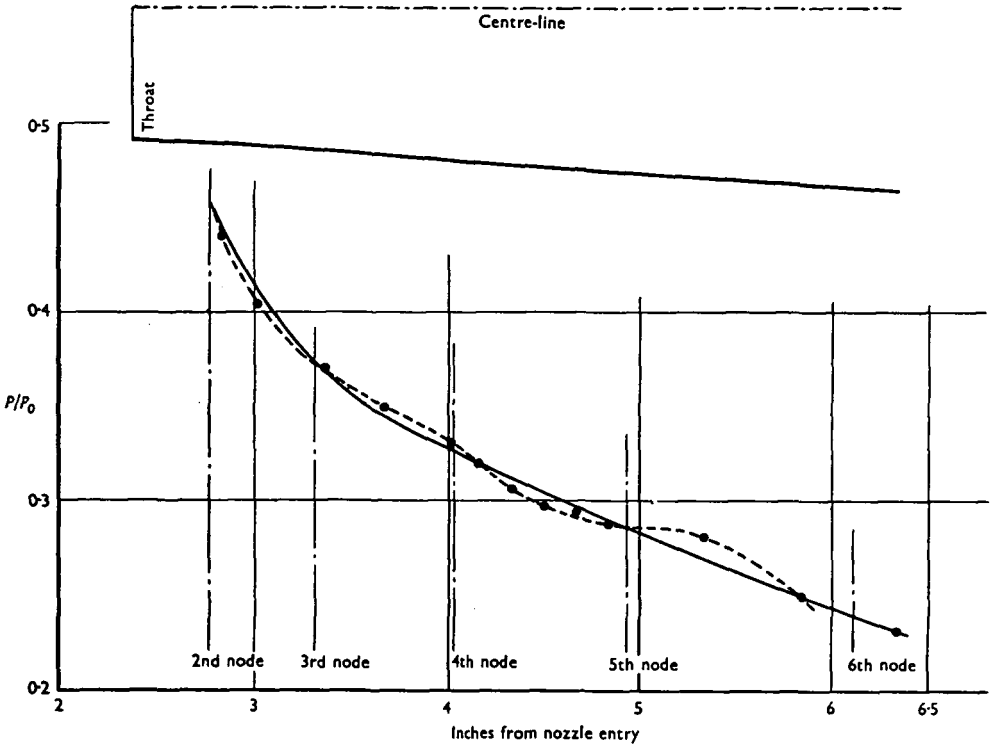


Fig. 2. Comparison of theoretical and observed oscillations, ● experimental points.

and checked as follows. If it be assumed that the pressure-volume relation in the expansion from the supply reservoir to a tapping point is PV^n = constant, the effective value of n can be obtained from the equation

$$\frac{(P_t/P_0)^{2/n} - (P_t/P_0)^{(n+1)/n}}{(P/P_0)^{2/n} - (P/P_0)^{(n+1)/n}} = \left(\frac{X}{X_t}\right)^2, \tag{4}$$

X being the area of the cross-section where P and V are the pressure and specific volume. The suffix t refers to the throat. The ratio P_t/P_0 was taken to have its observed value $0{\cdot}563_5$, and the method, when applied to the cross-sections distant 2·84, 4·34 and 5·84 in. from the entry, yielded $n = 1{\cdot}18$, 1·17 and 1·16. This fall in n is in agreement with the general trend of the experiments, and its steadiness indicates that the suggested undisturbed curve is reasonable.

The result shows the undulations as loops of slowly increasing amplitude, and reveals a fairly convincing agreement between the experiments and the approximate theory, but obviously more evidence is required to complete the verification. Similar though less marked disturbances can be seen in the results of Binnie and Green (2), who

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employed the same apparatus fitted with a nozzle diverging at only 4°; the waves are, however, too minute to be of use for the present purpose.

At the time the first series of observations was made it was thought that the disturbances were due to the absence of transition curves between the circular arcs at the throat and the divergent straight portions. A more basic probable cause is the fact that the divergent flow emanated, not from a point source, but from a throat; and across this the pressure distribution was symmetrical but not uniform. Turbulence in the steam supply seems an unlikely origin of these steady oscillations: when the apparatus was used for some preliminary tests the mercury pressure gauges attached to the first two tappings showed signs of unsteadiness, but this trouble was cured completely by the insertion of smoothing devices into the steam reservoir.

The results set out above throw light on the difficulty encountered by Fox & Southwell (5) in their theoretical treatment of flow through a nozzle. Whereas their method of approximate calculation converged rapidly to a definite result when applied to a subsonic regime, yet when used in the supersonic part of the nozzle signs of instability appeared and the high accuracy previously reached was not attainable. In this region it appears that a computer may find his method yielding undulations of the kind investigated by Hooker.

Part 2. The calculation of throat velocities

An approximate method of calculating the velocity distribution in the neighbourhood of the throat of a two-dimensional nozzle was put forward by Taylor (9). Assuming the throat to be formed by circular arcs he expressed the velocity potential ϕ as a polynomial with a limited number of terms involving the coordinates x and y. Relatively brief computations sufficed to yield the required velocities. No check was available on the accuracy of the method, which was applied to the limiting case of asymmetrical subsonic flow and to the unique case of asymmetrical flow which is partly subsonic and partly supersonic. More recently, Fox and Southwell (5) described an iterative process of solving both problems. As a demonstration of their method, which maps the velocities throughout the whole of the nozzle and which is consequently somewhat lengthy, they worked out the distribution in a nozzle of the same form as Taylor's. The opportunity has been taken of comparing the numerical results of the two methods, particularly because Taylor's procedure was used by Binnie and Hooker (3) and by Binnie and Woods (4) as a check on their experiments on the flow of water and of steam. An improvement in Taylor's method of finding the limiting symmetrical velocity distribution is also explained below, and an account is given of the types of flow which are possible when the velocity of approach to the throat is supersonic.

(i) *Subsonic symmetrical flow.* Taking the origin at the centre of the throat and using coordinates x measured positively downstream along the nozzle axis and y across the throat, Taylor used the expansion

$$\phi = a_1 x + a_3 x^3 + a_5 x^5 + c_3 xy^2 + c_5 x^3 y^2 + e_5 xy^4. \tag{5}$$

The combination of the equation of continuity with the pressure-density relation yields

$$\{2-(\gamma-1)(\theta^2-1)\}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=\frac{\partial\phi}{\partial x}\frac{\partial\theta^2}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\theta^2}{\partial y}, \tag{6}$$

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where
$$\theta = \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{\frac{1}{2}} \tag{7}$$
is the ratio of the velocity to the local velocity of sound; in this form all the velocities will be expressed. On inserting (5) into (6) and equating in turn the coefficients of x, x^3 and xy^2 he obtained
$$-3(\gamma+1)a_3(a_1^2-1)+c_3L=0, \tag{8}$$
$$-10(\gamma+1)a_5(a_1^2-1)+c_5L-18(\gamma+1)a_1a_3^2-6(\gamma-1)a_1a_3c_3=0, \tag{9}$$
$$-6(\gamma+1)a_1a_3c_3-2(\gamma+3)a_1c_3^2-3(\gamma+1)c_5(a_1^2-1)+6e_5L=0, \tag{10}$$
where
$$L=2-(\gamma-1)(a_1^2-1). \tag{11}$$
The boundary conditions reduce to
$$\frac{2c_3h^2+4e_5h^4}{a_1+c_3h^2+e_5h^4}=K, \tag{12}$$
and
$$F_1+F_2=K^2, \tag{13}$$
where $K=h/R$, h is half the throat width, R is the radius of the circular arcs, and
$$F_1=\frac{2c_3h^2+12e_5h^4+4c_5h^4/K}{a_1+c_3h^2+e_5h^4}, \tag{14}$$
$$F_2=\frac{-(2c_3h^2+4e_5h^4)\{2c_3h^2+4e_5h^4+6(a_3h^2+c_5h^4)/K\}}{(a_1+c_3h^2+e_5h^4)^2}. \tag{15}$$
To obtain a solution in a given nozzle for a particular value of $-a_1$, which is the velocity at the origin, a series of values of a_3h^2 is taken and from (8), (12) and (10) the values of c_3h^2, e_5h^4 and c_5h^4 are calculated. Then F_1 and F_2 are found from (14) and (15), and the sum F_1+F_2 plotted on a base of a_3h^2. The intersection of this curve with the straight line given by (13) determines the value of a_3h^2, and the solution can then be quickly completed. On repeating these calculations for various values of $-a_1$, Taylor found that when it exceeded a certain value no intersection occurred, and thus he was able to deduce the limiting velocity distribution.

This procedure has been applied to the case $K=0{\cdot}116417$, $\gamma=1{\cdot}40$, investigated by Fox and Southwell, and the results are shown in Fig. 3. It appears that the second intersection of a curve with the straight line is irrelevant. For example, with $-a_1=0{\cdot}9465$ the roots are $a_3h^2=0{\cdot}271$ and $0{\cdot}322$, to which the corresponding velocities at the throat wall, given by $-a_1-c_3h^2-e_5h^4$, are $1{\cdot}026$ and $1{\cdot}036$: with $-a_1=0{\cdot}94$ the roots are $0{\cdot}189$ and $0{\cdot}430$ and the throat wall velocities $1{\cdot}008$ and $1{\cdot}059$. Thus, if the upper root be used, the unreasonable conclusion is reached that the throat wall velocity rises as the velocity at the centre of the throat is diminished. The results shown in Fig. 3 indicate that the critical value of $-a_1$ is about $0{\cdot}947$. It will be seen that for a specified value of a_1 the method of solution is good, but when repeated in order to find the limiting case it is less satisfactory; interpolation is then difficult because the rate of change in a_3h^2 is very great as the critical value is approached. For the latter purpose it is better to use the procedure employed by Hooker (6) in his numerical exploration of the case $K=0{\cdot}25$, $\gamma=1{\cdot}408$. The method may readily be expressed in a general form.

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The four equations (8), (10), (12) and (13) involve a_3, c_3, c_5 and e_5, and from them three of these quantities can be eliminated. Writing for brevity

$$C_3 = \frac{c_3 h^2}{a_1}, \quad E_5 = \frac{e_5 h^4}{a_1}, \quad \text{etc.,} \tag{16}$$

we find in this way that C_3 is given by the quadratic equation

$$aC_3^2 + 2bC_3 + c = 0, \tag{17}$$

where

$$a = \frac{2(2-3K)a_1^2}{3(\gamma+1)(a_1^2-1)}\left(\frac{L}{a_1^2-1} + \gamma + 3\right), \tag{18}$$

$$2b = \frac{K(K+6)(2-K)^2}{2(4-K)} - \frac{K}{2}(2-2K-K^2) + \frac{KL}{(\gamma+1)(a_1^2-1)} + \frac{2L(2-3K)(2-K)}{(4-K)(\gamma+1)(a_1^2-1)}, \tag{19}$$

$$c = \frac{K^3}{2} - \frac{K^2(6+K)(2-K)}{2(4-K)} - \frac{2KL(2-3K)}{(4-K)(\gamma+1)(a_1^2-1)}. \tag{20}$$

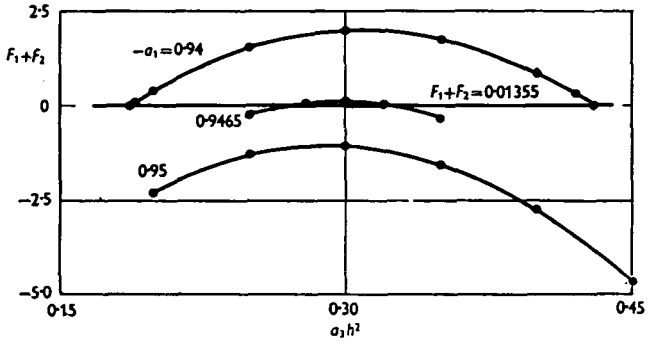


Fig. 3. Subsonic symmetrical flow; determination of limiting velocity.

Now C_3 must be real, and its limiting value is reached when $b^2 = ac$. Thus from (18), (19) and (20) an equation for a_1^2 is derived which can be solved by trial to any required degree of accuracy. To obtain the throat wall velocity we calculate first the limiting value $-b/a$ of C_3, and then e_5 is obtained from (12). By these means the velocities at the centre and the wall of the throat were found to be 0·9468 and 1·0309 for the case under consideration.

The corresponding results obtained by Fox and Southwell were 0·948 and 1·043. Thus the agreement is excellent at the centre of the throat but, as anticipated by Taylor, the approximate method loses accuracy at outlying parts of the field. The shape of the nozzle on each side of the circular arcs is ignored in the approximate theory, yet it appears to have little influence on the throat conditions.

(ii) *Asymmetrical subsonic-supersonic flow.* For this type of motion Taylor employed the expression

$$\phi = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + c_2 y^2 + c_3 xy^2 + c_4 x^2 y^2 + e_4 y^4, \tag{21}$$

the axes being unchanged. With a procedure similar to that employed in the previous case eight relations can be obtained to determine the eight coefficients in (21), and thus the solution is unique. As explained in the references previously mentioned, a complicated equation for a_1^2 is derived, the relevant root of which can be found graphically.

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On applying the method to Fox and Southwell's case it yielded $a_1^2 = 0{\cdot}964784$, $a_1 = -0{\cdot}982234$; and thus it was found that the velocities at the centre and the wall of the throat were 0·982 and 1·043. This result accords with those of Binnie and Woods who, using an almost identical nozzle for which $K = 0{\cdot}1161$, found $a_1 = -0{\cdot}98226$ for $\gamma = 1{\cdot}30$ and $a_1 = -0{\cdot}98225$ for $\gamma = 1{\cdot}29$. The magnitude of the velocities is therefore little dependent on the value of γ.

The corresponding figures calculated by Fox and Southwell were $0{\cdot}96_3$ and $1{\cdot}03_8$, thus here the agreement is not nearly so good at the centre of the throat although satisfactory at the wall. It seems possible that the approximate theory loses accuracy because the help of symmetry is not available.

(iii) *Supersonic symmetrical flow.* The method explained above for dealing with subsonic symmetrical motion throws light on the possibility of irrotational flow when the velocities are supersonic. It is valid for this case also, and in its modified form it was used by Hooker, who found that for his numerical case the discriminant $b^2 - ac$ passes from a positive to a negative value as $-a_1$ increases through 0·917, and alters from $-\infty$ to $+\infty$ as a_1 increases through 1·00. He concluded that symmetrical flow could not exist when a_1 lies in the range $0{\cdot}917 < -a_1 < 1{\cdot}00$.

It may easily be shown that this upper limit is a general result, independent of the values of γ and K. When we search for a value of a_1^2, very close to unity, which will satisfy the conditions of the problem, we may neglect in (18), (19) and (20) those terms which lack the factor $(a_1^2 - 1)$ in the denominator. We then find that

$$b^2 - ac = \frac{L^2}{(\gamma+1)^2 (a_1^2-1)^2} \left\{ \frac{1}{4} \frac{(8 - 12K + 5K^2)^2}{(4-K)^2} + \frac{4}{3} \frac{K(2-3K)^2 a_1^2}{(4-K)(a_1^2-1)} \right\}. \tag{22}$$

Dismissing the case $K \geqslant 4$ as unusual and therefore not worth attention, we see that the discriminant changes from negative to positive as a_1^2 increases in the region $a_1^2 = 1$, and thus flow is possible when $-a_1$ exceeds unity. Hooker found that $c_3 = 0$ when $a_1^2 = 1$, and this result can similarly be proved to be generally true, as may also the fact that a_3 is then infinite.

It appears that when a_1 is exactly unity a finite velocity field cannot exist. For $a_1 = -1$, $c_3 = 0$, it is seen from (12) that $e_5 h^4 = -K/(4-K)$, thus the velocities over the throat cross-section are calculable and are independent of γ. But (5) shows that the velocities elsewhere involve a term containing a_3, and they are therefore infinite. Only when $-a_1$ exceeds unity is the flow possible, and Hooker's conclusion must be modified accordingly. If under these conditions c_3 is determined with the aid of (17), we require the smaller numerical value, and this is found to be negative. From (8) we see that a_3 is also negative, and it is then clear from (5) that we are dealing with a field in which the velocity increases on each side of the central plane $x = 0$.

(iv) *Asymmetrical supersonic-subsonic flow.* When we examine the solution of (21) given in § (ii) we see that $a_1 = +0{\cdot}982234$ is a possible root. Its use involves a change of sign in all the coefficients, and implies reversing the direction of flow at every point in the field. Thus the theory indicates that a unique asymmetrical flow can exist when the velocity of approach to the throat is supersonic; and as with subsonic entry conditions, the limiting symmetrical value of a_1 differs from the asymmetrical value.

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